Discussiones Mathematicae
Graph Theory 37 (2017) 443–464
doi:10.7151/dmgt.1940

ONE MORE TURÁN NUMBER AND RAMSEY NUMBER FOR THE LOOSE 3-UNIFORM PATH OF LENGTH THREE

Joanna Polcyn

Adam Mickiewicz University
Poznań, Poland

e-mail: joaska@amu.edu.pl

Abstract

Let P denote a 3-uniform hypergraph consisting of 7 vertices a, b, c, d, e, f, g and 3 edges $\{a, b, c\}, \{c, d, e\}$, and $\{e, f, g\}$. It is known that the r-color Ramsey number for P is $R(P; r) = r + 6$ for $r \leqslant 9$. The proof of this result relies on a careful analysis of the Turán numbers for P. In this paper, we refine this analysis further and compute the fifth order Turán number for P, for all n. Using this number for $n = 16$, we confirm the formula $R(P; 10) = 16$.

Keywords: Ramsey numbers, Turán numbers.

2010 Mathematics Subject Classification: 05D10.

1. Introduction

For the sake of brevity, 3-uniform hypergraphs will be called here 3-*graphs*. Given a family of 3-graphs $\mathcal{F}$, we say that a 3-graph H is $\mathcal{F}$-*free* if for all $F \in \mathcal{F}$ we have $H \not\supseteq F$.

For a family of 3-graphs $\mathcal{F}$ and an integer $n \geqslant 1$, the *Turán number of the* 1*st order*, that is, the ordinary Turán number, is defined as

$$\mathrm{ex}(n; \mathcal{F}) = \mathrm{ex}^{(1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every n-vertex $\mathcal{F}$-free 3-graph with $\mathrm{ex}^{(1)}(n; \mathcal{F})$ edges is called 1-*extremal for* $\mathcal{F}$. We denote by $\mathrm{Ex}^{(1)}(n; \mathcal{F})$ the family of all, pairwise non-isomorphic, n-vertex 3-graphs which are 1-extremal for $\mathcal{F}$. Further, for an integer $s \geqslant 1$, the *Turán number of the* $(s + 1)$-*st order* is defined as

$$\begin{aligned}\mathrm{ex}^{(s+1)}(n; \mathcal{F}) = \max\{&|E(H)| : |V(H)| = n,\ H \text{ is } \mathcal{F}\text{-free, and} \\ &\forall H' \in \mathrm{Ex}^{(1)}(n; \mathcal{F}) \cup \cdots \cup \mathrm{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'\},\end{aligned}$$

if such a 3-graph H exists. Note that if $\mathrm{ex}^{(s+1)}(n;\mathcal{F})$ exists then, by definition,

$$\mathrm{ex}^{(s+1)}(n;\mathcal{F}) < \mathrm{ex}^{(s)}(n;\mathcal{F}). \tag{1}$$

An n-vertex $\mathcal{F}$-free 3-graph H is called $(s+1)$*-extremal for* $\mathcal{F}$ if $|E(H)| = \mathrm{ex}^{(s+1)}(n;\mathcal{F})$ and for every $H' \in \mathrm{Ex}^{(1)}(n;\mathcal{F}) \cup \cdots \cup \mathrm{Ex}^{(s)}(n;\mathcal{F})$, $H \nsubseteq H'$; we denote by $\mathrm{Ex}^{(s+1)}(n;\mathcal{F})$ the family of n-vertex 3-graphs which are $(s+1)$-extremal for $\mathcal{F}$. In the case when $\mathcal{F} = \{F\}$, we will write F instead of $\{F\}$.

A *loose* 3*-uniform path of length* 3 is a 3-graph P consisting of 7 vertices, say, a,b,c,d,e,f,g, and 3 edges $\{a,b,c\},\{c,d,e\}$, and $\{e,f,g\}$. The *Ramsey number* $R(P;r)$ is the least integer n such that every r-coloring of the edges of the complete 3-graph K_n results in a monochromatic copy of P. Gyárfás and Raeisi [6] proved, among many other results, that $R(P;2) = 8$. (This result was later extended to loose paths of arbitrary lengths, but still $r = 2$, in [13].) Then Jackowska [9] showed that $R(P;3) = 9$ and $r + 6 \leqslant R(P;r)$ for all $r \geqslant 3$. In turn, in [10, 11] and [15], the Turán numbers of the first four orders, $\mathrm{ex}^{(i)}(n;P)$, $i = 1,2,3,4$, have been determined for all feasible values of n. Using these numbers, in [11] and [15], we were able to compute the Ramsey numbers $R(P;r)$ for $4 \leqslant r \leqslant 9$.

Theorem 1 [6, 9, 11, 15]**.** *For all $r \leqslant 9$, $R(P;r) = r + 6$.*

In this paper we determine, for all $n \geqslant 7$, the Turán numbers for P of the fifth order, $\mathrm{ex}^{(5)}(n;P)$. This allows us to compute one more Ramsey number.

Theorem 2. *$R(P;10) = 16$.*

It seems that in order to make a further progress in computing the Ramsey numbers $R(P;r)$, $r \geqslant 11$, one would need to determine still higher order Turán numbers $\mathrm{ex}^{(s)}(n;P)$, at least for some small values of n.

Throughout, we denote by S_n the 3-graph on n vertices and with $\binom{n-1}{2}$ edges, in which one vertex, referred to as the *center*, forms edges with all pairs of the remaining vertices. Every sub-3-graph of S_n without isolated vertices is called a *star*, while S_n itself is called the *full star*. We denote by C the *triangle*, that is, a 3-graph with six vertices a,b,c,d,e,f and three edges $\{a,b,c\}$, $\{c,d,e\}$, and $\{e,f,a\}$. Finally, M stands for a pair of disjoint edges. For a given 3-graph H and a vertex $v \in V(G)$ we denote by $\deg_H(v)$ the number of edges in H containing v.

In the next section we state some known and new results on Turán numbers for P, including Theorem 11 which provides a complete formula for $\mathrm{ex}^{(5)}(n;P)$. We also define conditional Turán numbers and quote from [11] and [14] some useful lemmas about the conditional Turán numbers with respect to P, C, M. Then, in Section 3, we prove Theorem 2, while the remaining sections are devoted to proving Theorem 11.

2. Turán Numbers

We restrict ourselves exclusively to the case $k = 3$ only. A celebrated result of Erdős, Ko, and Rado [2] asserts, in the case of $k = 3$, that for $n \geqslant 6$, $\mathrm{ex}^{(1)}(n; M) = \binom{n-1}{2}$. Moreover, for $n \geqslant 7$, $\mathrm{Ex}^{(1)}(n; M) = \{S_n\}$. We will need the higher order versions of this Turán number, together with its extremal families. The second of these numbers has been found by Hilton and Milner, [8] (see [4] and [14] for a simple proof). For a given set of vertices V, with $|V| = n \geqslant 7$, let us define two special 3-graphs. Let $x, y, z, v \in V$ be four different vertices of V. We set

$$G_1(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : v \in h, h \cap \{x, y, z\} \neq \emptyset \right\},$$

$$G_2(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : |h \cap \{x, y, z\}| = 2 \right\}.$$

Note that for $i \in \{1, 2\}$, $M \not\subset G_i(n)$ and $|G_i(n)| = 3n - 8$.

Theorem 3 [8]**.** *For $n \geqslant 7$, $\mathrm{ex}^{(2)}(n; M) = 3n - 8$ and $\mathrm{Ex}^{(2)}(n; M) = \{G_1(n), G_2(n)\}$.*

Later, we will use the fact that $C \subset G_i(n) \not\supset P$, $i = 1, 2$.

Recently, the third order Turán number for M has been established for general k by Han and Kohayakawa in [7]. Let $G_3(n)$ be the 3-graph on n vertices, with distinguished vertices x, y_1, y_2, z_1, z_2 whose edge set consists of all edges spanned by x, y_1, y_2, z_1, z_2 except for $\{y_1, y_2, z_i\}$, $i = 1, 2$, and all edges of the form $\{x, z_i, v\}$, $i = 1, 2$, where $v \notin \{x, y_1, y_2, z_1, z_2\}$.

Theorem 4 [7]**.** *For $n \geqslant 7$, $\mathrm{ex}^{(3)}(n; M) = 2n - 2$ and $\mathrm{Ex}^{(3)}(n; M) = \{G_3(n)\}$.*

For $k = 3$ we were able to take the next step and determine the next Turán number for M.

Theorem 5 [14]**.** *For $n \geqslant 7$, $\mathrm{ex}^{(4)}(n; M) = n + 4$.*

The number $\binom{n-1}{2}$ serves as the Turán number for two other 3-graphs, C and P. The Turán number $\mathrm{ex}^{(1)}(n; C)$ has been determined in [3] for $n \geqslant 75$ and later for all n in [1].

Theorem 6 [1]**.** *For $n \geqslant 6$, $\mathrm{ex}^{(1)}(n; C) = \binom{n-1}{2}$. Moreover, for $n \geqslant 8$, it holds $\mathrm{Ex}^{(1)}(n; C) = \{S_n\}$.*

In [10], we filled an omission of [5] and [12] and calculated $\mathrm{ex}^{(1)}(n; P)$ for all n. Given two 3-graphs F_1 and F_2, by $F_1 \cup F_2$ denote a vertex-disjoint union of F_1 and F_2. If $F_1 = F_2 = F$ we will sometimes write $2F$ instead of $F \cup F$.

Theorem 7 [10]. $\mathrm{ex}^{(1)}(n;P) =$

$$\begin{cases} \binom{n}{3} & \textit{and} \quad Ex^{(1)}(n;P) = \{K_n\} \qquad \textit{for } n \leqslant 6, \\ 20 & \textit{and} \quad Ex^{(1)}(n;P) = \{K_6 \cup K_1\} \qquad \textit{for } n = 7, \\ \binom{n-1}{2} & \textit{and} \quad Ex^{(1)}(n;P) = \{S_n\} \qquad \textit{for } n \geqslant 8. \end{cases}$$

In [11] we have completely determined the second order Turán number $\mathrm{ex}^{(2)}(n;P)$, together with the corresponding 2-extremal 3-graphs. A *comet* $\mathrm{Co}(n)$ is an n-vertex 3-graph consisting of the complete 3-graph K_4 and the full star S_{n-3}, sharing exactly one vertex which is the center of the star (see Figure 1). This vertex is called the *center* of the comet, while the set of the remaining three vertices of the K_4 is called its *head*.

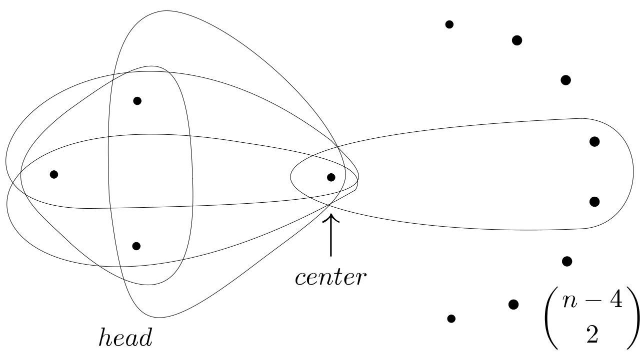

Figure 1. The comet $\mathrm{Co}(n)$.

Theorem 8 [11]. $\mathrm{ex}^{(2)}(n;P) =$

$$\begin{cases} 15 & \textit{and} \quad \mathrm{Ex}^{(2)}(n;P) = \{S_7\} & \textit{for } n = 7, \\ 20 + \binom{n-6}{3} & \textit{and} \quad \mathrm{Ex}^{(2)}(n;P) = \{K_6 \cup K_{n-6}\} & \textit{for } 8 \leqslant n \leqslant 12, \\ 40 & \textit{and} \quad \mathrm{Ex}^{(2)}(n;P) = \{2K_6 \cup K_1, \mathrm{Co}(13)\} & \textit{for } n = 13, \\ 4 + \binom{n-4}{2} & \textit{and} \quad \mathrm{Ex}^{(2)}(n;P) = \{\mathrm{Co}(n)\} & \textit{for } n \geqslant 14. \end{cases}$$

In [11] ($n = 12$) and in [15] (for all n), we calculated the third order Turán number for P.

Theorem 9 [11, 15]. $\mathrm{ex}^{(3)}(n;P) =$

$$\begin{cases} 3n - 8 & \textit{and} \quad \mathrm{Ex}^{(3)}(n;P) = \{G_1(n), G_2(n)\} & \textit{for } 7 \leqslant n \leqslant 10, \\ 25 & \textit{and} \quad \mathrm{Ex}^{(3)}(n;P) = \{G_1(n), G_2(n), \mathrm{Co}(n)\} & \textit{for } n = 11, \\ 32 & \textit{and} \quad \mathrm{Ex}^{(3)}(n;P) = \{\mathrm{Co}(n)\} & \textit{for } n = 12, \\ 20 + \binom{n-7}{2} & \textit{and} \quad \mathrm{Ex}^{(3)}(n;P) = \{K_6 \cup S_{n-6}\} & \textit{for } 13 \leqslant n \leqslant 14, \\ 4 + \binom{n-5}{2} & \textit{and} \quad \mathrm{Ex}^{(3)}(n;P) = \{K_4 \cup S_{n-4}\} & \textit{for } n \geqslant 15. \end{cases}$$

Surprisingly, as an immediate consequence we obtained also an exact formula for the 4th Turán number for P. We define a *rocket* $\mathrm{Ro}(n)$ to be the 3-graph obtained from the star S_{n-4} with center x by adding to it 4 more vertices, say, a, b, c, d, and three edges: $\{x, a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$. Let K_5^{+t} be the 3-graph obtained from K_5 by fixing two of its vertices, say a, b, and adding t more vertices $v_1, v_2, \ldots, v_t$ and t edges $\{a, b, v_i\}$, $i = 1, 2, \ldots, t$.

Theorem 10 [15]**.** $\mathrm{ex}^{(4)}(n; P) =$

$$\begin{cases} 12 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{G_3(n), K_5^{+2}\} & \textit{for } n = 7, \\ 2n-2 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{G_3(n)\} & \textit{for } 8 \leqslant n \leqslant 9, \\ 20 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{K_5 \cup K_5\} & \textit{for } n = 10, \\ 20 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{G_3(n)\} & \textit{for } n = 11, \\ 28 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{G_1(n), G_2(n)\} & \textit{for } n = 12, \\ 33 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{K_6 \cup G_1(n), K_6 \cup G_2(n)\} & \textit{for } n = 13, \\ 40 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{2K_6 \cup 2K_1, K_4 \cup S_{10}\} & \textit{for } n = 14, \\ 48 & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{\mathrm{Ro}(n), K_6 \cup S_9\} & \textit{for } n = 15, \\ 3 + \binom{n-5}{2} & \textit{and} \quad \mathrm{Ex}^{(4)}(n;P) = \{\mathrm{Ro}(n)\} & \textit{for } n \geqslant 16. \end{cases}$$

The main Turán-type result of this paper provides a complete formula for the fifth order Turán number for P.

Theorem 11. $\mathrm{ex}^{(5)}(n; P) =$

$$\begin{cases} 11 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \mathrm{Ex}^{(4)}(7;M) & \textit{for } n = 7, \\ 13 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_5^{+3}\} & \textit{for } n = 8, \\ 14 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_5^{+4}, K_5 \cup K_4\} \cup \mathrm{Ex}(9; \{P, C\} | M) & \textit{for } n = 9, \\ 19 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{\mathrm{Co}(10)\} & \textit{for } n = 10, \\ 19 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_4 \cup S_7\} & \textit{for } n = 11, \\ 25 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_5 \cup S_7, K_4 \cup S_8\} & \textit{for } n = 12, \\ 32 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_4 \cup S_9, K_6 \cup K_5^{+2}, K_6 \cup G_3(7)\} & \textit{for } n = 13, \\ 39 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{\mathrm{Ro}(14)\} & \textit{for } n = 14, \\ 46 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_5 \cup S_{10}\} & \textit{for } n = 15, \\ 56 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_6 \cup S_{10}\} & \textit{for } n = 16, \\ 65 & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_5 \cup S_{12}, K_6 \cup S_{11}\} & \textit{for } n = 17, \\ 10 + \binom{n-6}{2} & \textit{and} \quad \mathrm{Ex}^{(5)}(n;P) = \{K_5 \cup S_{n-5}\} & \textit{for } n \geqslant 18. \end{cases}$$

To determine Turán numbers, it is sometimes useful to rely on Theorem 3 and divide all 3-graphs into those which contain M and those which do not. To this end, it is convenient to define conditional Turán numbers (see [10, 11]). For a family of 3-graphs $\mathcal{F}$, an $\mathcal{F}$-free 3-graph G, and an integer $n \geqslant |V(G)|$, the *conditional Turán number* is defined as

$$\mathrm{ex}(n; \mathcal{F}|G) = \max\{|E(H)| : |V(H)| = n, \ H \text{ is } \mathcal{F}\text{-free, and } H \supseteq G\}.$$

Every n-vertex $\mathcal{F}$-free 3-graph with $\mathrm{ex}(n;\mathcal{F}|G)$ edges and such that $H \supseteq G$ is called *G-extremal for $\mathcal{F}$*. We denote by $\mathrm{Ex}(n;\mathcal{F}|G)$ the family of all n-vertex 3-graphs which are G-extremal for $\mathcal{F}$. (If $\mathcal{F}=\{F\}$, we simply write F instead of $\{F\}$.)

To illustrate the above mentioned technique, observe that for $n \geqslant 7$

$$\begin{aligned}\mathrm{ex}^{(2)}(n;P) &= \max\{\mathrm{ex}(n;P|M),\mathrm{ex}^{(2)}(n;M)\} \overset{\text{Thm.3}}{=} \max\{\mathrm{ex}(n;P|M),3n-8\}\\ &= \mathrm{ex}(n;P|M),\end{aligned}$$

the last equality holds for sufficiently large n (see [11] for details).

In the proof of Theorem 11 we will use the following five lemmas, all proved in [11] and [14]. For the first two we need one more piece of notation. If, in the above definition, we restrict ourselves to connected 3-graphs only (connected in the weakest, obvious sense), then the corresponding conditional Turán number and the extremal family are denoted by $\mathrm{ex}_{conn}(n;\mathcal{F}|G)$ and $\mathrm{Ex}_{conn}(n;\mathcal{F}|G)$, respectively.

Lemma 12 [11]**.** *For $n \geqslant 7$,*

$$\mathrm{ex}_{conn}(n;P|C) = 3n-8 \ \text{ and } \ \mathrm{Ex}_{conn}(n;P|C) = \{G_1(n),G_2(n)\}.$$

Lemma 12 as stated in [11] does not provide the family $\mathrm{Ex}_{conn}(n;P|C)$. However, it is clear from its proof that the extremal 3-graphs are the same as in Theorem 3. We will also need another lemma, which is not stated explicitly in [11], but it immediate results from the proof of the previous one.

Lemma 13 [11]**.** *For $n \geqslant 7$,*

$$\mathrm{ex}_{conn}(n;P|\{C,M\}) = n+5 \ \text{ and } \ \mathrm{Ex}_{conn}(n;P|\{C,M\}) = \left\{K_5^{+(n-5)}\right\}.$$

Moreover, if H is an n-vertex connected P-free 3-graph such that $C \subset H$ and $M \subset H$, then $H \subseteq K_5^{+(n-5)}$.

Lemma 14 [11]**.**

$$\mathrm{ex}(n;\{P,C\}|M) = \begin{cases} 2n-4 & \text{for } 6 \leqslant n \leqslant 9,\\ 20 & \text{for } n=10,\\ 4+\binom{n-4}{2} \ \text{ and } \ \mathrm{Ex}(n;\{P,C\}|M)=\{\mathrm{Co}(n)\} & \text{for } n \geqslant 11.\end{cases}$$

Lemma 15 [11]**.** *For $n \geqslant 6$,*

$$\mathrm{ex}(n;\{P,C,P_2 \cup K_3\}|M) = 2n-4,$$

where P_2 is a pair of edges sharing one vertex.

Lemma 16 [14]**.** *For $n \geqslant 6$,*

$$\mathrm{ex}^{(2)}(n;\{M,C\}) = \max\{10,n\}.$$

3. Proof of Theorem 2

As mentioned in the Introduction, Jackowska has shown in [9] that $R(P;r) \geqslant r+6$ for all $r \geqslant 1$. We are going to show that $R(P;10) \leqslant 16$.

We will show that every 10-coloring of K_{16} yields a monochromatic copy of P. The idea of the proof is to gradually reduce the number of vertices and colors (by one in each step), until we reach a coloring which yields a monochromatic copy of P.

Let us consider an arbitrary 10-coloring of K_{16}, $K_{16} = \bigcup_{i=1}^{10} G_i$, and assume that for each $i \in [10]$, $P \nsubseteq G_i$. Since $|K_{16}| = 560$, the average number of edges per color is 56, and therefore, by Theorems 7–11, either for each $i \in [10]$, $G_i = K_6 \cup S_{10}$, or there exists a color, say G_{10}, contained in one of the 3-graphs: $S_{16}, \mathrm{Co}(16), K_4 \cup S_{12}, \mathrm{Ro}(16)$. We will show that the latter case must occur. Indeed, for each vertex $v \in V(K_{16})$ we have $\deg_{K_{16}}(v) = \binom{15}{2} = 105$ whereas for $v \in V(K_6 \cup S_{10})$, $\deg_{K_6 \cup S_{10}}(v) \in \{10, 36, 8\}$ depending on whether v is a vertex of K_6, the center of the star S_{10} or another vertex of the star. Since we are not able to obtain an odd number as a sum of even numbers, we can not decompose K_{16} into edge-disjoint copies of $K_6 \cup S_{10}$. Let us turn back to G_{10}. No matter in which of the four 3-graph G_{10} is contained, we remove the center of the star (or comet, or rocket) together with up to four more edges of G_{10}, so that we get rid of color 10 completely (note that some other colors can also be affected by this deletion).

As a result, we obtain a 3-graph H_{15} on 15 vertices, colored with 9 colors, $H_{15} = \bigcup_{i=1}^{9} G_i$, with $|H(15)| \geqslant 451$ (with some abuse of notation we will keep denoting the subgraphs of G_i obtained in each step again by G_i). The average number of edges per color is at least 50.1, and therefore there exists a color, say G_9, with $|G_9| \geqslant 51$. This time we use Theorems 7–9 to conclude that either $G_9 \subset S_{15}$ or $G_9 \subset \mathrm{Co}(15)$. In either case we remove the center and, in case of the comet, one more edge being its head.

We get a 3-graph $H(14)$ on 14 vertices with $|H(14)| \geqslant 359$, colored by 8 colors, $H(14) = \bigcup_{i=1}^{8} G_i$. The average number of edges per color is at least 44.9, and hence there exists a color, say G_8, with $|G_8| \geqslant 45$. Similarly as in the previous step we reduce the picture to a 3-graph $H(13)$ on 13 vertices with $|H(13)| \geqslant 280$, colored by 7 colors, $H(13) = \bigcup_{i=1}^{7} G_i$.

This time the average number of edges per color is at least 40, and therefore, by Theorems 7 and 8, either each color is a copy of $\mathrm{Co}(13)$ or $K_6 \cup K_6 \cup K_1$, or there exists a color, say G_7, contained in the full star S_{13}. We will show in the similar way as before that $H(13)$ can not by decomposed into edge-disjoint copies of $\mathrm{Co}(13)$ and $K_6 \cup K_6 \cup K_1$, and therefore the latter case must occur. Indeed, let us assume that no color is contained in the full star S_{13}. First notice that there is not enough space for two edge-disjoint copies of $K_6 \cup K_6 \cup K_1$ in K_{13} and therefore also in $H(13)$. Fix one copy of $K_6 \cup K_6 \cup K_1$ in K_{13}. By the pigeon-hole principle, any other copy of K_6 must share at least three vertices with one of the fixed copies of K_6 and therefore they are not edge-disjoint. Now observe that since during our procedure we have lost at most 6 edges of K_{13}, for each vertex $v \in V(H(13))$ we have $\deg_{H(13)}(v) \geqslant \binom{12}{2} - 6 = 60$ and also for each vertex of a comet Co(13) which is not its center we have $\deg_{\mathrm{Co}(13)}(v) \leqslant 8$. If $H(13)$ is decomposed into seven copies of Co(13) or six copies of Co(13) and one copy of $2K_6 \cup K_1$, then there must exist a vertex $v \in V(H(13))$ which is not a center of any of these comets and therefore $\deg_{H(13)}(v) \leqslant 10 + 6 \cdot 8 = 58 < 60$, a contradiction. Consequently, we have $G_7 \subseteq S_{13}$ and, by removing the center of this star, we obtain a 6-coloring of a 3-graph $H(12)$ on 12 vertices with $|H(12)| \geqslant 214$.

To proceed, let us assume for a while, that none of the colors G_i, $i \in [6]$, is a star. Then, by Theorems 7–9, each color with more than 32 edges is a subset of $K_6 \cup K_6$. The average number of edges per color is at least 35.6, and hence there exists a color, say G_6, with $G_6 \subset K_6 \cup K_6$. We remove all edges of this copy of $K_6 \cup K_6$, getting a bipartite 3-graph $H'(12)$ with a bipartition $V(H'(12)) = W \cup U$, $|W| = |U| = 6$, and with $|H'(12)| \geqslant 174$ edges colored by 5 colors, $H'(12) = \bigcup_{i=1}^{5} G_i$. Note that every subgraph of $K_6 \cup K_6$ contained in $H'(12)$ (and consequently each color class of $H'(12)$) has at most 36 edges. Since $2 \cdot 36 + 3 \cdot 33 = 171 < 174$, at least 3 color classes have at least 34 edges, and thus each of them must be subsets of $K_6 \cup K_6$. Now observe that if two color classes, say G_1 and G_2, have at least 34 edges each, then they are disjoint unions of two copies of K_6, one on the vertex set $U_i' \cup W_i'$, the other one on $U_i'' \cup W_i''$, with four missing edges U_i', U_i'', W_i', W_i'', where $U = U_i' \cup U_i''$, $W = W_i' \cup W_i''$, $i = 1, 2$, and $\{U_1', U_1''\} = \{U_2', U_2''\}$ (see Figure 2).

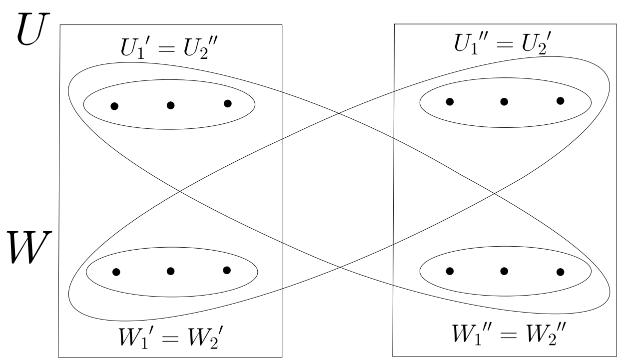

Figure 2. The partition of the set of vertices of $H'(12)$, G_1 and G_2.

Indeed, otherwise, if $1 \leqslant |U_1' \cap U_2'| \leqslant 2$, then G_1 and G_2 would share at least six edges, and thus $|G_1| + |G_2| \leqslant 36 + 36 - 6 < 2 \cdot 34$. This simply means that one of the partitions, of U or of W, must be swapped. But this is impossible for three color classes. Consequently, at least one color, say G_6, is a star. We remove the center of this star to get a 5-coloring of a 3-graph $H(11)$ on 11 vertices with $|H(11)| \geqslant 159$.

By repeating this argument three more times, we finally arrive at a 2-coloring of a 3-graph $H(8) = G_1 \cup G_2$, with $|H(8)| \geqslant 50$ which, by Theorem 7, should contain a copy of P, a contradiction.

4. Proof of Theorem 11

Let us define $\mathcal{H}_n = \mathrm{Ex}^{(1)}(n;P) \cup \mathrm{Ex}^{(2)}(n;P) \cup \mathrm{Ex}^{(3)}(n;P) \cup \mathrm{Ex}^{(4)}(n;P)$. To prove Theorem 11 we need to find, for each $n \geqslant 7$, a P-free n-vertex 3-graph H with the biggest possible number of edges such that, whenever $G \in \mathcal{H}_n$, then $H \nsubseteq G$. Moreover, we will show that $|H| = h_n$, where h_n is the number of edges, given by the formula to be proved.

First note that for each $n \geqslant 7$, all candidates for being 5-extremal 3-graphs do qualify, that is, are P-free, are not contained in any of the 3-graphs from $\mathcal{H}_n$, and have h_n edges. To finish the proof, we will show that each P-free n-vertex 3-graph H, not contained in any of 3-graph from $\mathcal{H}_n$, satisfy $|H| < h_n$ unless it is one of the candidates for being 5-extremal 3-graph itself.

For the latter task, we distinguish two cases: when H is connected and disconnected. The entire proof is inductive, in the sense that here and there we apply the very Theorem 11 for smaller instances of n, once they have been confirmed.

For all $n \geqslant 7$, let H be P-free n-vertex 3-graph such that for each $G \in \mathcal{H}_n$, $H \nsubseteq G$. Moreover, let H be different from all candidates for being 5-extremal 3-graphs with the same number of vertices. We will show that $|H| < h_n$.

4.1. Connected case

We start with the connected case. First let us assume that $M \nsubseteq H$ and consider consecutive intersecting families. Recall that for all $n \geqslant 7$, $H \nsubseteq S_n$, for $7 \leqslant n \leqslant 12$, $H \nsubseteq G_1(n)$ and $H \nsubseteq G_2(n)$, for $7 \leqslant n \leqslant 9$ and $n = 11$, $H \nsubseteq G_3(n)$, and finally, for $n = 7$, H is not equal to any of 4-extremal 3-graphs for M. Therefore, by Theorems 3, 4 and 5, we get that for all $n \geqslant 7$,

$$|H| < h_n.$$

Consequently, we will be assuming by the end of the proof that $M \subset H$. If additionally $C \subset H$, then by Lemma 13, $H \subseteq K_5^{+(n-5)}$ and hence $|H| \leqslant |K_5^{+(n-5)}| = n+5$. Therefore, for $n \geqslant 10$, $|H| < h_n$. If $n = 7$, as $K_5^{+2} \in \mathcal{H}_7$, we have $H \nsubseteq K_5^{+2}$ and thus we may exclude this case. Lastly, for $8 \leqslant n \leqslant 9$, by the definition of H,

$H \neq K_5^{+(n-5)}$ and hence $|H| < h_n$. Therefore, in the rest of the proof we will be assuming that $C \nsubseteq H$.

Finally, let H be connected $\{P, C\}$-free 3-graph containing M. Then by Lemma 14, for $7 \leqslant n \leqslant 8$, $|H| \leqslant 2n - 4 < h_n$ and for $n = 9$, since $H \notin \mathrm{Ex}(9, \{P, C\}|M)$, we have $|H| < 14 = h_9$.

For $10 \leqslant n \leqslant 11$ we need two more facts, which we state here without the proof. Namely, $\mathrm{ex}_{conn}(10; \{P, C\}|M) = 19$ and $\mathrm{Ex}_{conn}(10; \{P, C\}|M) = \{\mathrm{Co}(10)\}$. Since, by the definition of H, $H \neq \mathrm{Co}(10)$, this implies that $|H| < 19 = h_{10}$. Whereas for $n = 11$ we have $\mathrm{ex}^{(2)}_{conn}(11; \{P, C\}|M) = 18$, and therefore, as $H \nsubseteq \mathrm{Co}(11)$, we get $|H| \leqslant \mathrm{ex}^{(2)}_{conn}(11; \{P, C\}|M) = 18 < 19 = h_{11}$.

Recall that for all $n \geqslant 11$, $H \nsubseteq \mathrm{Co}(n)$. Moreover, for $12 \leqslant n \leqslant 13$, since $|\mathrm{Ro}(n)| < h_n$, we may assume that $H \nsubseteq \mathrm{Ro}(n)$. Further, for $n = 14$, by the definition of H we have $H \neq \mathrm{Ro}(14)$ and thus, if $H \subset \mathrm{Ro}(14)$, then $|H| < |\mathrm{Ro}(14)| = h_n$. Finally, for all $n \geqslant 15$ we have $H \nsubseteq \mathrm{Ro}(n)$. Therefore, since for all $n \geqslant 12$ we have

$$h_n \geqslant \binom{n-6}{2} + 10,$$

to complete the proof of the connected case it is enough to prove the following Lemma.

Lemma 17. *If H is a connected, n-vertex, $n \geqslant 12$, $\{P, C\}$-free 3-graph containing M such that $H \nsubseteq \mathrm{Co}(n)$ and $H \nsubseteq \mathrm{Ro}(n)$, then $|H| < \binom{n-6}{2} + 10$.*

We devote an entire Section 5 to prove Lemma 17.

4.2. Disconnected case

Now let H be disconnected and let $m = m(H)$ be the number of vertices in the smallest componet of H. We have $m \neq 2$, since no component of a 3-graph may have two vertices. We now break the proof into several cases.

Let us express H as a vertex disjoint union of two 3-graphs:

$$H = H' \cup H'', \quad |V(H')| = m, \quad |V(H'')| = n - m.$$

Then, clearly, both H' and H'' are P-free, and thus

$$|H| \leqslant \mathrm{ex}^{(1)}(m; P) + \mathrm{ex}^{(1)}(n - m; P). \tag{2}$$

Below, to bound $|H|$, we use the Turán numbers for P of the 1st, 2nd, 3rd, 4th and 5th order and utilize, respectively, Theorems 7, 8, 9, 10 and 11 (by induction).

Let v be an isolated vertex ($m = 1$). Since for $n = 7$ and any 3-graph H'', $K_1 \cup H'' \subseteq K_1 \cup K_6 \in \mathcal{H}_7$, we may assume that $n \geqslant 8$. For $8 \leqslant n \leqslant 11$, as H cannot be a sub-3-graph of S_n, $K_6 \cup K_{n-6}$, $G_1(n)$ or $G_2(n)$, H'' is not a sub-3-graph of S_{n-1}, $K_6 \cup K_{n-7}$, $G_1(n-1)$ and $G_2(n-1)$. Consequently, for $n = 8, 10$,

$$|H| = |H''| \leqslant \mathrm{ex}^{(4)}(n-1; P) < h_n.$$

For $n = 9$ additionally we have $H'' \nsubseteq G_3(8)$ and therefore

$$|H| \leqslant \mathrm{ex}^{(5)}(8; P) = 13 < 14 = h_9,$$

whereas for $n = 11$, $H'' \nsubseteq K_5 \cup K_5$ and $H'' \nsubseteq \mathrm{Co}(10)$. Consequently,

$$|H| = |H''| < \mathrm{ex}^{(5)}(10; P) = 19 = h_{11}.$$

For $n \geqslant 12$, since $H = K_1 \cup H''$ is not a sub-3-graph of any of the 3-graphs in $\mathcal{H}_n$, we have $H'' \nsubseteq S_{n-1}$ and $H'' \nsubseteq \mathrm{Co}(n-1)$. Moreover, for $n = 12, 13$, $H'' \nsubseteq K_6 \cup K_{n-7}$, for $n = 12$, $H'' \nsubseteq G_1(n-1)$ and $H'' \nsubseteq G_2(n-1)$, for $n = 14$, $H'' \nsubseteq 2K_6 \cup K_1$, for $n = 14, 15$, $H'' \nsubseteq K_6 \cup S_{n-7}$ and finally, for $n \geqslant 15$, $H'' \nsubseteq K_4 \cup S_{n-5}$. Consequently,

$$|H| = |H''| \leqslant \mathrm{ex}^{(4)}(n-1; P) < h_n.$$

For $m = 3$ and $n = 7, 8$, by (2) we get

$$|H| \leqslant \mathrm{ex}^{(1)}(3; P) + \mathrm{ex}^{(1)}(n-3; P) = 1 + \mathrm{ex}^{(1)}(n-3; P) < h_n.$$

Since each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 3$ and $|V(H'')| = 6$ is a sub-3-graph of $K_3 \cup K_6 \in \mathcal{H}_9$, we may assume that $n \neq 9$. For $n = 10$ we have $K_3 \cup K_6 \cup K_1 \subset K_4 \cup K_6 \in \mathcal{H}_{10}$. Consequently, $H'' \nsubseteq K_6 \cup K_1$ and thus $|H''| \leqslant \mathrm{ex}^{(2)}(7; P) = 15$. Hence $|H| \leqslant 1 + 15 = 16 < 19 = h_{10}$.

Further, for all $n \geqslant 11$, since $K_3 \cup S_{n-3} \subseteq \mathrm{Co}(n) \in \mathcal{H}_n$, we have $H'' \nsubseteq S_{n-3}$. Therefore for $n \geqslant 12$,

$$|H| \leqslant 1 + \mathrm{ex}^{(2)}(n-3; P) < h_n,$$

whereas, for $n = 11$ additionally we have $H \nsubseteq K_3 \cup K_6 \cup K_2 \subset K_6 \cup K_5 \in \mathcal{H}_{11}$. Thus $H'' \nsubseteq K_6 \cup K_2$ and consequently,

$$|H| \leqslant 1 + \mathrm{ex}^{(3)}(8; P) = 17 < 19 = h_{11}.$$

For $m = 4$ and $n = 8$ by (2) we have

$$|H| \leqslant \mathrm{ex}^{(1)}(4; P) + \mathrm{ex}^{(1)}(4; P) = 4 + 4 = 8 < h_8.$$

For $n = 9$, by the definition of H, $H \neq K_4 \cup K_5$ and therefore $|H| < |K_4 \cup K_5| = 14 = h_9$. Similarly like before, we may skip the case $n = 10$, because each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 4$ and $|V(H'')| = 6$ is a sub-3-graph of $K_4 \cup K_6 \in \mathcal{H}_{10}$. For $n = 11$, since $K_4 \cup K_6 \cup K_1 \subset K_5 \cup K_6 \in \mathcal{H}_{11}$, we have $H'' \not\subseteq K_6 \cup K_1$ and therefore $|H''| \leqslant \mathrm{ex}^{(2)}(7; P) = 15$ with the equality only for $H'' = S_7$. But, by the definition of H, $H \neq K_4 \cup S_7$, and hence

$$|H| < |K_4 \cup S_7| = 19 = h_{11}.$$

Further, for $n = 12, 13$, since $\mathrm{Ex}^{(1)}(n-4; P) = \{S_{n-4}\}$ and $H \neq H_4 \cup S_{n-4}$, we have $|H| < |H_4 \cup S_{n-4}| = h_n$. Finally, for $n \geqslant 14$, since $K_4 \cup S_{n-4} \in \mathcal{H}_n$ we get $H'' \not\subseteq S_{n-4}$ and consequently,

$$|H| \leqslant \mathrm{ex}^{(1)}(4; P) + \mathrm{ex}^{(2)}(n-4; P) < h_n.$$

Now let $m = 5$. Notice that each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 5$ and $5 \leqslant |V(H'')| \leqslant 6$ is a sub-3-graph of $K_5 \cup K_5 \in \mathcal{H}_{10}$ and $K_5 \cup K_6 \in \mathcal{H}_{11}$, respectively. Therefore we may consider only $n \geqslant 12$. For $n = 12$, since $K_5 \cup K_6 \cup K_1 \subset K_6 \cup K_6 \in \mathcal{H}_{12}$, we have $|H''| \leqslant \mathrm{ex}^{(2)}(7; P) = 15$ with the equality only for $H'' = S_7$. But, by the definition of H, $H \neq K_5 \cup S_7$ and hence $|H| < |K_5 \cup S_7| = 25 = h_{12}$. Finally, for $n \geqslant 13$, by (2),

$$|H| \leqslant \mathrm{ex}^{(1)}(5; P) + \mathrm{ex}^{(1)}(n-5; P) = 10 + \binom{n-6}{2} \leqslant h_n,$$

where the equality is achieved only by the candidates for 5-extremal 3-graphs with the proper number of vertices.

For $m = 6$ we have $n \geqslant 12$, but as each disconnected 3-graph $H' \cup H''$ with $|V(H')| = |V(H'')| = 6$ is a sub-3-graph of $K_6 \cup K_6 \in \mathcal{H}_{12}$, we may consider only $n \geqslant 13$. Recall that $\{2K_6 \cup K_1, K_6 \cup S_7, K_6 \cup G_1(7), K_6 \cup G_2(7)\} \subset \mathcal{H}_{13}$ and therefore, for $n = 13$, H'' is not contained in any of the 3-graphs $K_6 \cup K_1, S_7, G_1(7), G_2(7)$. Consequently, $|H''| \leqslant \mathrm{ex}^{(4)}(7; P) = 12$ with the equality only for $H'' = G_3(7)$ and $H'' = K_5^{+2}$. But, by the definition of H, $H \neq K_6 \cup K_5^{+2}$ and $H \neq K_6 \cup G_3(7)$ and thus

$$|H| < |K_6 \cup K_5^{+2}| = |K_6 \cup G_3(7)| = h_{13}.$$

For the same reason, if $n = 14$, then $H'' \not\subseteq S_8$ and $H'' \not\subseteq K_6 \cup K_2$. Consequently,

$$|H| = |H'| + |H''| \leqslant \mathrm{ex}^{(1)}(6; P) + \mathrm{ex}^{(3)}(8; P) = 20 + 16 < 39 = h_{14},$$

whereas for $n = 15$, we have $H'' \not\subseteq S_9$ and hence

$$|H| \leqslant \mathrm{ex}^{(1)}(6; P) + \mathrm{ex}^{(2)}(9; P) = 20 + 21 < 46 = h_{15}.$$

Further, for $n = 16, 17$, by the definition of H, $H \neq K_6 \cup S_{n-6}$. Consequently, as $\mathrm{Ex}(n-6; P) = \{S_{n-6}\}$, we get

$$|H| < |K_6 \cup S_{n-6}| = h_n.$$

Finally, for $n \geqslant 18$, by (2),

$$|H| \leqslant \mathrm{ex}^{(1)}(6; P) + \mathrm{ex}^{(1)}(n-6; P) = 20 + \binom{n-7}{2} < \binom{n-6}{2} + 10 = h_n.$$

If $m = 7$, then $n \geqslant 14$. For $n = 14$, since $H \not\subseteq 2K_6 \cup 2K_1 \in \mathcal{H}_{14}$, at least one of the components of H is not a sub-3-graph of $K_6 \cup K_1$ and therefore has at most $\mathrm{ex}^{(2)}(7; P) = 15$ edges. Consequently,

$$|H| \leqslant \mathrm{ex}^{(1)}(7; P) + \mathrm{ex}^{(2)}(7; P) = 20 + 15 < 39 = h_{14}.$$

To bound the number of edges of H for $n \geqslant 15$ we use (2) to get

$$|H| \leqslant \mathrm{ex}^{(1)}(7; P) + \mathrm{ex}^{(1)}(n-7; P) = 20 + \binom{n-8}{2} < \binom{n-6}{2} + 10 \leqslant h_n.$$

Finally, for $m \geqslant 8$ we have $n \geqslant 16$ and, by (2),

$$\begin{aligned} |H| &\leqslant \mathrm{ex}^{(1)}(m; P) + \mathrm{ex}^{(1)}(n-m; P) = \binom{m-1}{2} + \binom{n-m-1}{2} \\ &\leqslant \binom{7}{2} + \binom{n-9}{2} < \binom{n-6}{2} + 10 \leqslant h_n. \end{aligned}$$

5. The Proof of Lemma 17

Recall that H is a connected, n-vertex, $n \geqslant 12$, $\{P, C\}$-free 3-graph such that $M \subset H$, $H \not\subseteq \mathrm{Co}(n)$ and $H \not\subseteq \mathrm{Ro}(n)$. We need to show that

$$|H| < \binom{n-6}{2} + 10.$$

Since for $n \geqslant 11$, by Lemma 15

$$\mathrm{ex}(\{n; P, C, P_2 \cup K_3\} | M) = 2n - 4 < \binom{n-6}{2} + 10,$$

we may assume that $P_2 \cup K_3 \subset H$. Let us denote a copy of P_2 from $P_2 \cup K_3$ in H by Q and the vertex of degree two in Q by x. We let $U = V(Q)$, $V = V(H)$ and $W = V \setminus U$. Moreover, let W_0 be the set of vertices of degree zero in $H[W]$ and $W_1 = W \setminus W_0$ (see Figure 3). Note that, by definition, $H[W] = H[W_1]$ and $|W_1| \geqslant 3$.

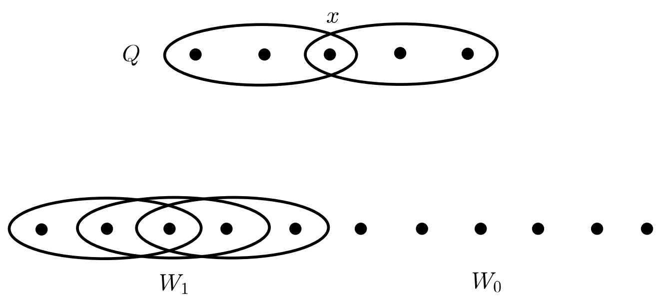

Figure 3. Set-up for the proof of Lemma 17.

We also split the set of edges of H. First, notice that since H is P-free, there is no edge with one vertex in each U, W_0, and W_1. We define $H_i = \{h \in H : h \cap U \neq \emptyset, h \cap W_i \neq \emptyset\}$, where $i = 0, 1$. Then, clearly,

$$H = H[U] \cup H[W] \cup H_0 \cup H_1, \tag{3}$$

with all four parts edge-disjoint. Since by definition $H[U] \cup H_0 = H[U \cup W_0]$, sometimes we will use the following equality

$$H = H[U \cup W_0] \cup H_1 \cup H[W]. \tag{4}$$

Recall that H is C-free, and therefore one can use Theorem 6 to get the bounds, for $|W_0| \geqslant 1$

$$|H[U \cup W_0]| \leqslant \binom{|U \cup W_0| - 1}{2} = \binom{|W_0| + 4}{2} \tag{5}$$

and for $|W_1| \geqslant 6$,

$$|H[W]| \leqslant \binom{|W_1| - 1}{2}. \tag{6}$$

Notice that for each edge $h \in H_0 \cup H_1$ with $|h \cap U| = 1$ we have $h \cap U = \{x\}$, because otherwise h together with Q would form a copy of P in H. We let

$$F^0 = \{h \in H_0 \cup H_1 : h \cap U = \{x\}\}.$$

Also, to avoid a copy of C in H, if for $h \in H_0 \cup H_1$ we have $|h \cap U| = 2$, then the pair $h \cap U$ is contained in an edge of Q. For $k = 1, 2$, we define

$$F^k = \{h \in H_0 \cup H_1 : |h \cap U \setminus \{x\}| = k\}.$$

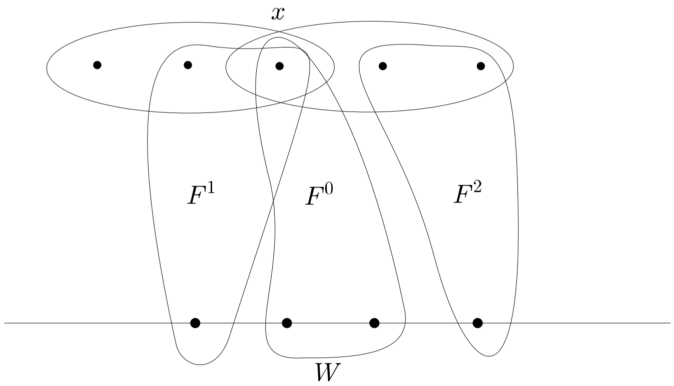

Figure 4. Three types of edges in $H_0 \cup H_1$.

Clearly, $H_0 \cup H_1 = F^0 \cup F^1 \cup F^2$ (see Figure 4). Further, for $i = 0, 1$ and $k = 0, 1, 2$, we set

$$F_i^k = F^k \cap H_i.$$

It is easy to see that, as H is P-free, $F_1^1 = \emptyset$ and therefore,

$$H_1 = F_1^0 \cup F_1^2. \tag{7}$$

Moreover, for all $v \in W$ we have

$$F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset, \tag{8}$$

and, by the definition of F^1 and F^2,

$$|F^1(v)| \leqslant 4 \quad \text{and} \quad |F^2(v)| \leqslant 2, \tag{9}$$

where for a given subset of edges $G \subseteq H$ and for a vertex $v \in V(H)$ we set $G(v) = \{h \in G : v \in h\}$.

In the whole proof we will be using the fact that for all edges $e \in F^0$, the pair $e \cap W_1$ is *nonseparable* in $H[W]$, that is, every edge of $H[W]$ must contain both these vertices or none. Consequently, for each $v \in W_0$, $|F^0(v)| \leqslant |W_0| - 1$ and thus, by (8) and (9),

$$|H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leqslant 4 + \max\{2, |W_0| - 1\}. \tag{10}$$

Moreover, if $F_1^0 \neq \emptyset$, then there exists at least one nonseparable pair in W_1, and therefore one can show the following fact.

Fact 1. If $F_1^0 \neq \emptyset$, then $|H[W]| \leqslant \binom{|W_1|-2}{3} + |W_1| - 2$. Moreover, if in addition $H[W_1] \subseteq S_{|W_1|}$, then $|H[W]| \leqslant \binom{|W_1|-3}{2} + 1$.

To prove another fact let us define an auxiliary graph G for nonseparable pairs on the set of vertices W_1, $G = \{e \setminus \{x\} : e \in F_1^0\}$. Then each component of G has size at most 3. This gives the proof of the following inequality. For any $W_1' \subseteq W_1$ we let $F_1^0[W_1'] \subseteq F_1^0$ to be the set of edges $h \in F_1^0$ such that $h \cap W_1 \subseteq W_1'$. Then,

$$|F_1^0[W_1']| \leqslant |W_1'|. \tag{11}$$

Observe also that, because H is connected, $H_1 \neq \emptyset$. Consequently, since the presence of any edge of H_1 forbids at least 4 edges of $H[U]$,

$$|H[U]| \leqslant 6. \tag{12}$$

Moreover, in [11] the authors have proved the following bounds on the number of edges in H_1.

$$\text{For} \quad |W_1| \geqslant 4, \quad |F_1^2| \leqslant 2|W_1| - 4. \tag{13}$$

$$\text{For} \quad |W_1| \geqslant 3, \quad |H_1| \leqslant 2|W_1| - 3. \tag{14}$$

As a consequence of these inequalities one can prove the following.

$$\text{For} \quad |W_1| \geqslant 7, \quad |H[U]| + |H_1| \leqslant 2|W_1| - 1. \tag{15}$$

Indeed, if $|H_1| \leqslant |W_1|$, then (15) results from (12) and the inequality $|W_1| - 1 \geqslant 7 - 1 = 6$. Otherwise, by (11), (7) and (8), there exists a vertex $v \in W_1$ such that $|F_1^2(v)| = 2$. As expected, assume $|H_1| > |W_1|$, and there does not exist the desired vertex v, i.e., for any vertex $v \in W_1$, $|F_1^2(v)| \leqslant 1$. Further, let $W_1' \subseteq W_1$ be the set of vertices v such that $F_1^2(v) = \emptyset$, and let $W_1'' = W_1 \setminus W_1'$. Then by (7), (8), (11) and the definition of W_1', we have

$$|H_1| = |F_1^0| + |F_1^2| = |F_1^0[W_1']| + \sum_{v \in W_1''} |F_1^2(v)| \leqslant |W_1'| + |W_1''| = |W_1|,$$

a contradiction. As H is $\{P, C\}$-free, by the definition of $F_1^2(v)$, this implies that $|H[U]| = 2$ and (15) follows from (14).

We also need the following fact proven in [15].

Fact 2 [15]**.** If $F_1^2 \neq \emptyset$, then

$$|H[U \cup W_0]| \leqslant \begin{cases} 8 & \text{for } |W_0| = 1, \\ 3|W_0| + 7 & \text{for } 2 \leqslant |W_0| \leqslant 4, \\ \binom{|W_0|+2}{2} + 1 & \text{for } |W_0| \geqslant 5. \end{cases} \tag{16}$$

We split the whole proof of Lemma 17 into a few short parts, Facts 3–7.

Fact 3. For $n \geqslant 13$, if $W_0 = \emptyset$ and $H_1 \neq \emptyset$, then $|H| < 10 + \binom{n-6}{2}$.

Proof. Let us consider two cases, whether or not $H[W] \subseteq S_{n-5}$. First assume that $H[W]$ is a subset of the star S_{n-5} with the center $y \in W_1$. Further assume also that $F^2 \backslash F^2(y) \neq \emptyset$, say, there exists $v \in W_1$, $v \neq y$, such that $F^2(v) \neq \emptyset$. Let $h \in H[W_1]$ be the edge containing y and v (because $H[W] \subseteq S_{n-5}$). Since H is P-free, for any $h' \in H[W_1]$, $h' \neq h$, it holds that $|h' \cap h| \geqslant 2$ and $y \in h' \cap h$. Thus $|H[W_1]| \leqslant 1 + 2(|W_1| - 3)$. By (15), we have $|H| \leqslant 2|W_1| - 1 + (1 + 2(|W_1| - 3)) = 4n - 26 < \binom{n-6}{2} + 10$. Otherwise, $|F^2| = |F^2(y)| \leqslant 2$. Additionally, if $F_1^0 = \emptyset$, then by (3), (12), (7) and (6),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 6 + 2 + \binom{n-6}{2} = \binom{n-6}{2} + 8 < \binom{n-6}{2} + 10.$$

Otherwise, $F_1^0 \neq \emptyset$ and therefore by Fact 1, $|H[W]| \leqslant \binom{n-8}{2} + 1$. By (11), $|F_1^0| \leqslant |W_1| = n - 5$ and hence by (7), $|H_1| \leqslant n - 5 + 2 = n - 3$. Consequently, by (3) and (12),

$$\begin{aligned} |H| = |H[U]| + |H_1| + |H[W]| &\leqslant 6 + n - 3 + \binom{n-8}{2} + 1 \\ &= \binom{n-7}{2} + 12 < \binom{n-6}{2} + 10. \end{aligned}$$

Now we move to the case $H[W] \nsubseteq S_{n-5}$. Since for $n \geqslant 13$, by Theorem 7, $\mathrm{Ex}^{(1)}(n-5; P) = \{S_{n-5}\}$, we may bound the number of edges in $H[W]$ by $\mathrm{ex}^{(2)}(n-5; P)$. Moreover, by (15), $|H[U]| + |H_1| \leqslant 2(n-5) - 1 = 2n - 11$. Consequently, by (3) and Theorem 8,

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 2 \cdot n - 11 + \mathrm{ex}^{(2)}(n-5; P) < \binom{n-6}{2} + 10,$$

where the last inequality is valid for $n \geqslant 15$. For $13 \leqslant n \leqslant 14$ we have to strengthen the bound of $H[W]$. Since W does not contain isolated vertices, we have $H[W] \nsubseteq K_6 \cup K_2$. Therefore by Theorems 7 and 8 we get $|H[W]| < \mathrm{ex}^{(2)}(n-5; P)$ and consequently, for $n = 14$, $|H| < \binom{n-6}{2} + 10$. In addition, for $n = 13$, we use the fact that for $i = 1, 2$, $C \subset G_i(8)$ and H is C-free, hence $H[W] \neq G_i(8)$. Then by Theorems 7, 8 and 9 we have $|H[W]| < \mathrm{ex}^{(3)}(8; P) = 16$ and therefore

$$|H| < 2 \cdot 13 - 11 + 16 = 31 = \binom{13-6}{2} + 10.$$

■

Fact 4. For $n \geqslant 13$, if $H_1 \neq \emptyset$, $H \nsubseteq \mathrm{Co}(n)$ and $|W_1| = 3$, then $|H| < 10 + \binom{n-6}{2}$.

Proof. We have $|H[W]| = 1$, $|U \cup W_0| = n-3$ and by (14), $|H_1| \leqslant 3$. Therefore, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$

Consequently, all we need to do is to bound the number of edges in $H[U \cup W_0]$. Since $H \nsubseteq \mathrm{Co}(n)$, either $F_1^2 \neq \emptyset$ or $H[U \cup W_0] \nsubseteq S_{n-3}$. In the former case we use Fact 2 to get $|H[U \cup W_0]| \leqslant \binom{n-6}{2} + 1$ and therefore

$$|H| \leqslant \binom{n-6}{2} + 1 + 4 = \binom{n-6}{2} + 5 < \binom{n-6}{2} + 10.$$

Otherwise, $H[U \cup W_0] \nsubseteq S_{n-3}$, so by Theorem 7, $|H[U \cup W_0]| \leqslant ex^{(2)}(n-3; P)$. Consequently, by Theorem 8, for $13 \leqslant n \leqslant 15$, $|H[U \cup W_0]| \leqslant 20 + \binom{n-3-6}{3}$ and therefore,

$$|H| \leqslant 20 + \binom{n-9}{3} + 4 = \binom{n-9}{3} + 24 < \binom{n-6}{2} + 10.$$

Whereas for $n \geqslant 16$ we get $|H[U \cup W_0]| \leqslant 4 + \binom{n-3-4}{2}$, and hence

$$|H| \leqslant \binom{n-7}{2} + 4 + 4 = \binom{n-7}{2} + 8 < \binom{n-6}{2} + 10.$$

■

Fact 5. For $n \geqslant 13$, if $H_1 \neq \emptyset$, $H \nsubseteq \mathrm{Ro}(n)$ and $|W_1| = 4$, then $|H| < 10 + \binom{n-6}{2}$.

Proof. The proof goes along the lines of the previous one. We have $|H[W]| \leqslant \binom{4}{3} = 4$, $|U \cup W_0| = n-4$ and by (14), $|H_1| \leqslant 5$. Therefore, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant |H[U \cup W_0]| + 5 + 4 = |H[U \cup W_0]| + 9.$$

Consequently, to finish the proof we need to bound $|H[U \cup W_0]|$. Assume that $F_1^2 = \emptyset$ and $H[U \cup W_0] \subseteq S_{n-4}$. Then, since $F_1^0 \neq \emptyset$, $|W_1| = 4$ and for $e \in F_1^0$ the pair $e \cap W_1$ is nonseparable in $H[W]$, we get $H \subseteq \mathrm{Ro}(n)$, a contradiction. Therefore $F_1^2 \neq \emptyset$ or $H[U \cup W_0] \nsubseteq S_{n-4}$. In the former case we use Fact 2 to get for $n = 13$, $|H[U \cup W_0]| \leqslant 19$ and consequently,

$$|H| \leqslant 19 + 9 = 28 < 31 = 10 + \binom{13-6}{2}.$$

Whereas for $n \geqslant 14$, $|H[U \cup W_0]| \leqslant \binom{n-7}{2} + 1$ and hence,

$$|H| \leqslant \binom{n-7}{2} + 1 + 9 = \binom{n-7}{2} + 10 < \binom{n-6}{2} + 10.$$

Otherwise, $H[U \cup W_0] \nsubseteq S_{n-4}$ so we use Theorem 7 to get $|H[U \cup W_0]| \leqslant \mathrm{ex}^{(2)}(n-4; P)$. Consequently, by Theorem 8, for $13 \leqslant n \leqslant 16$, $H[U \cup W_0]| \leqslant 20 + \binom{n-4-6}{3}$ and hence

$$|H| \leqslant 20 + \binom{n-10}{3} + 9 = \binom{n-10}{3} + 29 < \binom{n-6}{2} + 10.$$

Whereas for $n \geqslant 17$ we have $|H[U \cup W_0]| \leqslant 4 + \binom{n-4-4}{2}$ and therefore,

$$|H| \leqslant 4 + \binom{n-8}{2} + 9 = \binom{n-8}{2} + 13 < \binom{n-6}{2} + 10.$$

■

Fact 6. If $n = 12$, $H_1 \neq \emptyset$ and $H \nsubseteq \mathrm{Co}(12)$, then $|H| < 10 + \binom{12-6}{2} = 25$.

Proof. Let us split the proof into five parts according to the size of the set W_1. We start with $|W_1| = 3$. Then $|W_0| = 4$, $|U \cup W_0| = 9$, $|H[W]| = 1$ and by (14), $|H_1| \leqslant 3$. Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$

Further, as $H \nsubseteq \mathrm{Co}(12)$, either $F_1^2 \neq \emptyset$ or $H[U \cup W_0] \nsubseteq S_{n-3}$. In the former case we use Fact 2 to get $|H[U \cup W_0]| \leqslant 19$. Otherwise, $H[U \cup W_0] \nsubseteq S_{n-3}$, and since $H[U \cup W_0] \neq K_6 \cup K_3$, by Theorems 7 and 8, $|H[U \cup W_0]| < 21$. In both cases $|H[U \cup W_0]| \leqslant 20$ and therefore

$$|H| \leqslant |H[U \cup W_0]| + 4 \leqslant 20 + 4 = 24 < 25.$$

For $|W_1| = 4$ we have $|W_0| = 3$, $|U \cup W_0| = 8$ and $|H[W]| \leqslant \binom{4}{3} = 4$. If $F_1^2 = \emptyset$, then $H_1 = F_1^0 \neq \emptyset$ and as for each $h \in F_1^0$ the pair $h \cap W_1$ is nonseparable, $|H_1| = 1$ and by Fact 1, $|H[W]| \leqslant 2$. Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant \binom{7}{2} + 1 + 2 = 24 < 25.$$

Otherwise, $F_1^2 \neq \emptyset$ and we can use Fact 2 to get $|H[U \cup W_0]| \leqslant 16$. For $F_1^0 \neq \emptyset$, $|H[W]| = 2$ and consequently, by (4) and (14),

$$|H| = |H[U \cup W_0] + |H_1| + |H[W]| \leqslant 16 + 5 + 2 = 23 < 25.$$

Whereas for $F_1^0 = \emptyset$ we use (4), (7) and (13) to get

$$|H| = |H[U \cup W_0] + |H_1| + |H[W]| \leqslant 16 + 4 + 4 = 24 < 25.$$

Now let $|W_1| = 5$, $|W_0| = 2$, $|U \cup W_0| = 7$ and $|H[W]| \leqslant \binom{5}{3} = 10$. For $F_1^2 \neq \emptyset$, by Fact 2 we get $|H[U \cup W_0]| \leqslant 13$ and moreover $|H[W]| \leqslant 6$, because otherwise we would not be able to avoid a path P in H. If additionally $P_2 \subseteq H[W]$, then again by $P \nsubseteq H$, $|H_1| = |F_1^0| + |F_1^2| \leqslant 2 + 2 = 4$. Hence, by (4)

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 13 + 4 + 6 = 23 < 25.$$

Otherwise, $P_2 \nsubseteq H[W]$ and consequently one can show that $|H[W]| \leqslant |W_1| - 2 = 3$. Therefore, by (4) and (14),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 13 + 7 + 3 = 23 < 25.$$

For $F_1^2 = \emptyset$ we have $F_1^0 \neq \emptyset$. Hence, since for each $h \in F_1^0$ the pair $h \cap W_1$ is nonseparable, $|H_1| = |F_1^0| \leqslant 2$ and by Fact 1, $|H[W]| \leqslant 4$. Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant \binom{7-1}{2} + 2 + 4 = 21 < 25.$$

We move to $|W_1| = 6$. Then $|W_0| = 1$, $|U \cup W_0| = 6$ and by (6), $|H[W]| \leqslant \binom{6-1}{2} = 10$. Let us again start with the case $F_1^2 \neq \emptyset$. By (16) we get $|H[U \cup W_0]| \leqslant 8$. If $P_2 \subseteq H[W]$, then since H is P-free, $|H_1| = |F_1^0| + |F_1^2| \leqslant 2 + 4 = 6$. Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 8 + 6 + 10 = 24 < 25.$$

Otherwise, $P_2 \nsubseteq H[W]$ and therefore one can show that $|H[W]| \leqslant |W_1| - 2 = 4$. By (14), $|H_1| \leqslant 9$ and consequently by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 8 + 9 + 4 = 21 < 25.$$

For $F_1^2 = \emptyset$ we have $F_1^0 \neq \emptyset$, thus by Fact 1, $|H[W]| \leqslant 8$ and by (11), $|H_1| = |F_1^0| \leqslant 6$. Therefore, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 10 + 6 + 8 = 24 < 25.$$

Finally, $|W_1| = 7$, $W_0 = \emptyset$ and by (6), $|H[W]| \leqslant \binom{7-1}{2} = 15$. First assume that $H[W]$ is a subset of the star S_7 with the center $y \in W_1$. Further assume also that $F^2 \setminus F^2(y) \neq \emptyset$, say, there exists $v \in W_1$, $v \neq y$, such that $F^2(v) \neq \emptyset$. Let $h \in H[W_1]$ be the edge containing y and v (because $H[W] \subseteq S_7$). Since H is P-free, for any $h' \in H[W_1]$, $h' \neq h$, it holds that $|h' \cap h| \geqslant 2$ and $y \in h' \cap h$. Thus $|H[W_1]| \leqslant 1 + 2(|W_1| - 3) = 9$. By (15), we have $|H| \leqslant 2|W_1| - 1 + 9 = 22 < 25$. Otherwise, $|F^2| = |F^2(y)| \leqslant 2$. If additionally $F_1^0 = \emptyset$, then by (3), (7) and (12),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 6 + 2 + 15 = 23 < 25.$$

Otherwise $F_1^0 \neq \emptyset$, thus $|H_1| = |F_1^0| + |F_1^2| \leqslant 3+2 = 5$ and by Fact 1, $|H[W]| \leqslant 7$. Therefore by (3) and (12),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 6 + 5 + 7 = 18 < 25.$$

The last case we have to consider is $H[W] \nsubseteq S_7$. If $M \subseteq H[W]$, then by Lemma 14, $|H[W]| \leqslant \mathrm{ex}(7; \{P, C\}|M) = 10$. Otherwise, by Lemma 16, $|H[W]| \leqslant \mathrm{ex}^{(2)}(7; \{M, C\}) = 10$. Hence by (3) and (15),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 13 + 10 = 23 < 25. \qquad \blacksquare$$

Fact 7. For $n \geqslant 12$, if $|W_1| \geqslant 5$ and $H_1 \neq \emptyset$, then

$$|H| < \binom{n-6}{2} + 10. \tag{17}$$

Proof. The proof is by induction on n with the initial step $n = 12$ done in Fact 6. Let $n \geqslant 13$. For $W_0 = \emptyset$ the inequality (17) results from Fact 3. Otherwise, there exist a vertex $v \in W_0$. Notice that since $|W_1| \geqslant 5$, we have $|W_0| \leqslant n - 10$ and consequently, by (10), $|H(v)| \leqslant 4 + \max\{2, |W_0| - 1\} \leqslant 4 + n - 11 = n - 7$. Finally, by the induction assumption we get $|H - v| < \binom{n-7}{2} + 10$. Therefore,

$$|H| = |H(v)| + |H - v| < n - 7 + \binom{n-7}{2} + 10 = \binom{n-6}{2} + 10. \qquad \blacksquare$$

Acknowledgments

I would like to thank both referees for reading the manuscript and for their remarks. I am especially grateful to Referee 4 for his/her valuable comments and suggestions leading to a great improvement in the presentation of the paper.

References

[1] R. Csákány and J. Kahn, *A homological approach to two problems on finite sets*, J. Algebraic Combin. **9** (1999) 141–149.
doi:10.1023/A:1018630111976

[2] P. Erdős, C. Ko and R. Rado, *Intersection theorems for systems of finite sets*, Q. J. Math. **12** (1961) 313–320.
doi:10.1093/qmath/12.1.313

[3] P. Frankl and Z. Füredi, *Exact solution of some Turán-type problems*, J. Combin. Theory Ser. A **45** (1987) 226–262.
doi:10.1016/0097-3165(87)90016-1

[4] P. Frankl and Z. Füredi, *Non-trivial intersecting families*, J. Combin. Theory Ser. A **41** (1986) 150–153.
doi:10.1016/0097-3165(86)90121-4

[5] Z. Füredi, T. Jiang and R. Seiver, *Exact solution of the hypergraph Turán problem for k-uniform linear paths*, Combinatorica **34** (2014) 299–322. doi:10.1007/s00493-014-2838-4

[6] A. Gyárfás and G. Raeisi, *The Ramsey number of loose triangles and quadrangles in hypergraphs*, Electron. J. Combin. **19** (2012) # R30.

[7] J. Han and Y. Kohayakawa, *The maximum size of a non-trivial intersecting uniform family that is not a subfamily of the Hilton-Milner family*, Proc. Amer. Math. Soc. **145** (2017) 73–87. doi:101090/proc/13221

[8] A.J.W. Hilton, and E.C. Milner, *Some intersection theorems for systems of finite sets*, Q. J. Math. **18** (1967) 369–384. doi:10.1093/qmath/18.1.369

[9] E. Jackowska, *The* 3*-color Ramsey number for a* 3*-uniform loose path of length* 3, Australas. J. Combin. **63** (2015) 314–320.

[10] E. Jackowska, J. Polcyn and A. Ruciński, *Turán numbers for linear* 3*-uniform paths of length* 3, Electron. J. Combin. **23(2)** (2016) #P2.30.

[11] E. Jackowska, J. Polcyn and A. Ruciński, *Multicolor Ramsey numbers and restricted Turán numbers for the loose* 3*-uniform path of length three*, submitted.

[12] A. Kostochka, D. Mubayi and J. Verstraëte, *Turán problems and shadows* I: *paths and cycles*, J. Combin. Theory Ser. A **129** (2015) 57–79 doi:10.1016/j.jcta.2014.09.005

[13] G.R. Omidi and M. Shahsiah, *Ramsey numbers of* 3*-uniform loose paths and loose cycles*, J. Combin. Theory Ser. A **121** (2014) 64–73. doi:10.1016/j.jcta.2013.09.003

[14] J. Polcyn and A. Ruciński, *A hierarchy of maximal intersecting triple systems*, Opuscula Math. **37(4)** (2017), to appear. arXiv:1608.06114

[15] J. Polcyn and A. Ruciński, *Refined Turán and Ramsey numbers for the loose* 3*-uniform path of length three*, Discrete Math. **340** (2017) 107–118. doi:10.1016/j.disc.2016.08.006

Received 28-12-2015
Revised 9-05-2016
Accepted 9-05-2016