

A SUFFICIENT CONDITION FOR GRAPHS TO BE SUPER k -RESTRICTED EDGE CONNECTED¹

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Abstract

For a subset S of edges in a connected graph G , S is a k -restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k -restricted edge connectivity of G , denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k -restricted edge cut. Let $\xi_k(G) = \min\{|[X, \bar{X}]| : |X| = k, G[X] \text{ is connected}\}$, where $\bar{X} = V(G) \setminus X$. A graph G is super k -restricted edge connected if every minimum k -restricted edge cut of G isolates a component of order exactly k . Let k be a positive integer and let G be a graph of order $\nu \geq 2k$. In this paper, we show that if $|N(u) \cap N(v)| \geq k + 1$ for all pairs u, v of nonadjacent vertices and $\xi_k(G) \leq \lfloor \frac{\nu}{2} \rfloor + k$, then G is super k -restricted edge connected.

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1. TERMINOLOGY AND INTRODUCTION

For graph-theoretical terminology and notation not defined here we follow [1]. We consider finite, undirected and simple graphs. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order of G , denoted by $\nu = \nu(G)$, is the number of vertices in G . The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$. If G' is a subgraph of G and v is a vertex of G' , we define $N_{G'}(v) = N_G(v) \cap V(G')$. Unambiguously, we use $N(v)$ for $N_G(v)$. For subsets X and Y of $V(G)$, we denote by $[X, Y]$ the set of edges with one end in X and the other in Y . An edge cut of G is a subset of $E(G)$ of the form $[X, Y]$, where X is a non-empty proper subset of $V(G)$ and $Y = V(G) \setminus X$.

An interconnection network can be conveniently modeled as a graph $G = (V, E)$. A classical measurement of the fault tolerance of a network is the edge connectivity $\lambda(G)$. The edge connectivity $\lambda(G)$ of a graph G is the minimum cardinality of an edge cut of G . As a more refined index than the edge connectivity, Fàbrega and Fiol [5] proposed the more general concept of k -restricted edge connectivity. For a subset S of edges in a connected graph G , S is a k -restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k -restricted edge connectivity of G , denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k -restricted edge cut. A minimum k -restricted edge cut is called a λ_k -cut. A connected graph G is said to be λ_k -connected if G has a k -restricted edge cut.

In view of recent studies on k -restricted edge connectivity, it seems that the larger the $\lambda_k(G)$, the more reliable the network [7–8, 10]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first and so the optimization of k -restricted edge connectivity draws a lot of attention. For details, the readers can refer to [2–4, 6, 11, 13, 15]. For any positive integer k , let $\xi_k(G) = \min\{|[X, \bar{X}]| : |X| = k, G[X] \text{ is connected}\}$. A λ_k -connected graph G is said to be optimally k -restricted edge connected, for short λ_k -optimal, if $\lambda_k(G) = \xi_k(G)$.

A λ_k -connected graph G is super k -restricted edge connected, for short super- λ_k , if every minimum k -restricted edge cut of G isolates a component of order exactly k . The sufficient conditions of super- λ_k have been studied by several authors, see [9, 12, 14]. Let G be a λ_k -connected graph with $\lambda_k(G) \leq \xi_k(G)$. By definition, if G is a super- λ_k graph, then G must be a λ_k -optimal graph. However, the converse is not true. For example, a cycle of length at least $2k + 2$ is a λ_k -optimal graph that is not super- λ_k .

Definition 1.1. Let H_1, H_2 be two complete graphs with $V(H_1) = \{x_1, x_2, x_3\}$, $V(H_2) = \{y_1, y_2, y_3, z_1, z_2\}$ and let $M = \{x_1y_1, x_1z_1, x_2y_2, x_2z_2, x_3z_2, x_3y_3\}$. Set $H_8 = (H_1 \cup H_2) + M$ and $W_8 = \{H_8, H_8 - y_1z_1\}$. The graph H_8 is shown in

Figure 1. The heavy edge between A and B indicates that each vertex in A and each vertex in B are adjacent.

Definition 1.2. Let H_1, H_2 be two complete graphs with $V(H_1) = \{x_1, x_2, x_3\}$, $V(H_2) = \{y_1, y_2, y_3, z_1, z_2, z_3\}$ and let $M = \{x_i y_i, x_i z_i : i = 1, 2, 3\}$. Set $H_9^1 = (H_1 \cup H_2) + M$ and $W_9 = \{H_9^1 - M' : M' \subseteq \{y_1 z_1, y_2 z_2, y_3 z_3\}\}$. The graph H_9^1 is shown in Figure 2. The heavy edge between A_i and A_j ($i \neq j, i, j = 1, 2, 3$) indicates that each vertex in A_i and each vertex in A_j are adjacent.

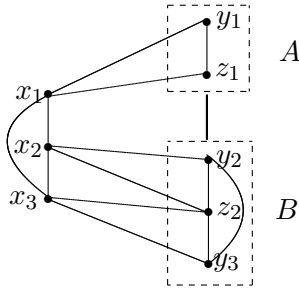


Figure 1. The graph H_8 .

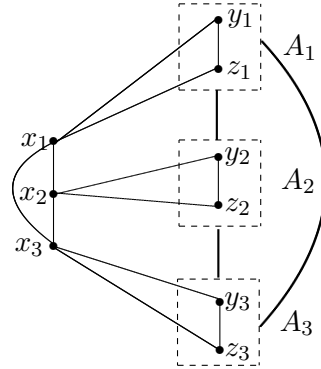


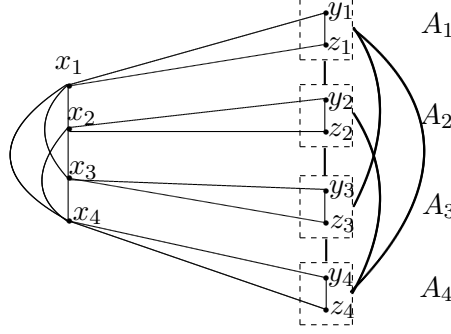
Figure 2. The graph H_9^1 .

Definition 1.3. Let H_1, H_2 be two complete graphs with $V(H_1) = \{x_1, x_2, x_3, x_4\}$, $V(H_2) = \{y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4\}$, and let $M = \{x_i y_i, x_i z_i : i = 1, 2, 3, 4\}$. Set $H_{12} = (H_1 \cup H_2) + M$ and $W_{12} = \{H_{12} - M' : M' \subseteq \{y_1 z_1, y_2 z_2, y_3 z_3, y_4 z_4\}\}$. Set $E_1 = \{y_1 z_1, y_2 z_2, y_3 z_3, y_4 z_4\}$. We define $W_0 = H_{12}$ and W_i as the graph obtained from H_{12} by deleting i edges of E_1 , where $i = 1, 2, 3, 4$. Set $\mathcal{W} = \{W_0, W_1, W_2, W_3, W_4\}$. The graph H_{12} is shown in Figure 3. The heavy edge between A_i and A_j ($i \neq j, i, j = 1, 2, 3, 4$) indicates that each vertex in A_i and each vertex in A_j are adjacent.

Set $\mathcal{W}' = W_8 \cup W_9 \cup W_{12}$. In [12], Wang *et al.* gave the following sufficient condition for a graph to be super- λ_2 .

Theorem 1.4 [12]. *Let G be a graph of order $\nu \geq 4$. If $|N(u) \cap N(v)| \geq 3$ for all pairs u, v of nonadjacent vertices and $\xi(G) \leq \lfloor \frac{\nu}{2} \rfloor + 2$, then G is super- λ_2 or in \mathcal{W}' .*

In this article, we extend the above result to super- λ_k with $k \geq 3$, and present a neighborhood condition for a graph to be super- λ_k .

Figure 3. The graph H_{12} .

2. MAIN RESULTS

Let G be a λ_k -connected graph, and let S be a λ_k -cut of G . It has been shown in [14] that there exists $X \subset V(G)$ such that $G[X]$ and $G[Y]$ are both the connected induced subgraphs of orders at least k and $S = [X, Y]$, where $Y = \bar{X} = V(G) \setminus X$. Let x be a vertex of G . We define $S(x)$ as the set of edges of S incident with x . Furthermore, we define $X_k = \{x \in X : |S(x)| \geq k\}$, $Y_k = \{y \in Y : |S(y)| \geq k\}$, $X_i = \{x \in X : |S(x)| = i\}$, $Y_i = \{y \in Y : |S(y)| = i\}$, where $i = 0, 1, 2, \dots, k-1$.

In order to prove our main result, we first give some useful lemmas.

Lemma 2.1 [14]. *Let k be a positive integer. If G is a complete graph of order $\nu \geq 2k$, then G is super- λ_k .*

Lemma 2.2 [11]. *Let $k \geq 3$ be an integer and let $G \notin \mathcal{W}$ be a graph of order $\nu \geq 2k$. If each pair u, v of nonadjacent vertices satisfies $|N(u) \cap N(v)| \geq k$ and $\xi_k(G) \leq \lfloor \frac{\nu}{2} \rfloor + k$, then G is λ_k -optimal.*

Theorem 2.3. *Let $k \geq 3$ be an integer and G be a graph of order $\nu \geq 2k$. If $|N(u) \cap N(v)| \geq k+1$ for all pairs u, v of nonadjacent vertices and $\xi_k(G) \leq \lfloor \frac{\nu}{2} \rfloor + k$, then G is super- λ_k or $G \in \mathcal{W}$.*

Proof. If G contains no nonadjacent vertices, then, by Lemma 2.1, we are done. Therefore, we only consider the case that there exist nonadjacent vertices in G below. By Lemma 2.2, G is λ_k -optimal. That is, $\lambda_k(G) = \xi_k(G)$. Suppose that G is neither super- λ_k nor in \mathcal{W} . Then there exists a λ_k -cut $S = [X, Y]$ such that $|X| \geq k+1$ and $|Y| \geq k+1$.

Claim 1. *There exists a vertex $x \in X$ such that $|S(x)| \leq k$, and there exists a vertex $y \in Y$ such that $|S(y)| \leq k$.*

Proof. Suppose, on the contrary, that for each $x \in X$, we have $|S(x)| \geq k + 1$. Let H be a connected subgraph with order k of $G[X]$. Then

$$\begin{aligned}
 \xi_k(G) &\leq \sum_{u \in V(H)} |S(u)| + \sum_{u \in X \setminus V(H)} |N(u) \cap V(H)| \\
 (1) \quad &\leq \sum_{u \in V(H)} |S(u)| + k|X \setminus V(H)| < \sum_{u \in V(H)} |S(u)| + (k+1)|X \setminus V(H)| \\
 &\leq \sum_{u \in V(H)} |S(u)| + \sum_{v \in X \setminus V(H)} |S(v)| = |S| = \lambda_k(G),
 \end{aligned}$$

contradicting the fact that $\lambda_k(G) = \xi_k(G)$. \square

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. We assume that $Y_0 \neq \emptyset$, say $y_0 \in Y_0$. By Claim 1, there exists a vertex $x \in X$ such that $|S(x)| \leq k$. It is easy to see that x, y_0 are nonadjacent vertices in G , and $|N(x) \cap N(y_0)| \leq k$, a contradiction to the hypothesis.

So, $Y_0 = \emptyset$. By the symmetry, we have $X_0 = \emptyset$. \square

Without loss of generality, assume that $|X| \geq |Y| \geq k + 1$. Then we can deduce that

$$(2) \quad \left\lceil \frac{\nu}{2} \right\rceil \leq |X| \leq |[X, Y]| = \lambda_k(G) = \xi_k(G) \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k$$

and

$$(3) \quad \left\lceil \frac{\nu}{2} \right\rceil - k \leq |Y| = \nu - |X| \leq \left\lfloor \frac{\nu}{2} \right\rfloor.$$

Claim 3. $|X_1| \geq 3$ when ν is odd, and $|X_1| \geq 1$ when ν is even.

Proof. Recall that $|X| \geq |Y| \geq k + 1$. We have $\nu \geq 2k + 3$ when ν is odd, and $\nu \geq 2k + 2$ when ν is even. Combining this with the fact

$$2 \left\lceil \frac{\nu}{2} \right\rceil - |X_1| \leq 2|X| - |X_1| \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k,$$

we have $|X_1| \geq 3$ when ν is odd, and $|X_1| \geq 1$ when ν is even. \square

Claim 4. $Y_1 = \emptyset$.

Proof. Suppose that $Y_1 \neq \emptyset$. Let $y_1 \in Y_1$ and $N(y_1) \cap X = \{x_1\}$. Then, for any $x \in X \setminus \{x_1\}$, we have

$$\begin{aligned}
 k + 1 &\leq |N(x) \cap N(y_1)| = |N(x) \cap N(y_1) \cap X| + |N(x) \cap N(y_1) \cap Y| \\
 &\leq |N(x) \cap Y| + |N(y_1) \cap X| = |N(x) \cap Y| + 1,
 \end{aligned}$$

which implies that $|N(x) \cap Y| \geq k$. Hence,

$$k \left(\left\lceil \frac{\nu}{2} \right\rceil - 1 \right) + 1 \leq \sum_{x \in X \setminus \{x\}} |N(x) \cap Y| + 1 \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k.$$

Combining this with $k \geq 3$, we can deduce that

$$4 \leq k + 1 \leq \left\lfloor \frac{\nu}{2} \right\rfloor \leq \frac{2k-1}{k-1} = 2 + \frac{1}{k-1} < 3,$$

a contradiction. □

Claim 5. $Y_2 = \emptyset$.

Proof. By contradiction, suppose that $Y_2 \neq \emptyset$. By Claim 3, we have $|X_1| \geq 1$, say $x_1 \in X_1$ and $N(x_1) \cap Y = \{y'\}$. Then, for any $y \in Y \setminus \{y'\}$, we have

$$k + 1 \leq |N(x_1) \cap N(y)| \leq |N(x_1) \cap Y| + |N(y) \cap X| = 1 + |N(y) \cap X|,$$

and so $|N(y) \cap X| \geq k \geq 3$. It implies that $|Y_2| = 1$, and so $Y_2 = \{y'\}$. Let $N(y') \cap X = \{x_1, x_2\}$. For any $x \in X \setminus \{x_1, x_2\}$, we can deduce that

$$k + 1 \leq |N(x) \cap N(y')| \leq |N(x) \cap Y| + |N(y') \cap X| = |N(x) \cap Y| + 2,$$

which implies that $|N(x) \cap Y| \geq k - 1$. Therefore,

$$(4) \quad (k-1) \left(\left\lceil \frac{\nu}{2} \right\rceil - 2 \right) + 2 \leq (k-1)(|X| - 2) + 2 \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k.$$

Consider the case that ν is odd. By (4), we have

$$(k-2) \left\lfloor \frac{\nu}{2} \right\rfloor < 2k-3,$$

and so

$$4 \leq k + 1 \leq \left\lfloor \frac{\nu}{2} \right\rfloor < 2 + \frac{1}{k-2} \leq 3,$$

a contradiction. So, $|X| = 5, |Y| = 4$ and $k = 3$. It follows that $8 = 2|Y| < |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k = 7$, a contradiction.

Consider the case that ν is even. By (4), we have

$$(k-2) \left\lfloor \frac{\nu}{2} \right\rfloor \leq 3k-4,$$

and so

$$4 \leq k + 1 \leq \left\lfloor \frac{\nu}{2} \right\rfloor \leq 3 + \frac{2}{k-2},$$

which implies that $\nu = 8$ or $\nu = 10$. Since $|Y| \geq k + 1$ and $Y_0 = Y_1 = \emptyset$ and $|Y_2| = 1$, we obtain that

$$2(k + 1) \leq 2|Y| < |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k = 5 + k.$$

Hence, $k < 3$, a contradiction. \square

Let m be the minimum integer such that $Y_m \neq \emptyset$. By Claims 2, 4 and 5, we obtain that $m \geq 3$. By Claim 3, we can choose a vertex $x_1 \in X_1$. Let $N(x_1) \cap Y = \{y'\}$. Then, for any $y \in Y \setminus \{y'\}$, we have

$$|N(y) \cap X| \geq k.$$

By (3), we can deduce that

$$(5) \quad k \left(\left\lceil \frac{\nu}{2} \right\rceil - k - 1 \right) + m \leq k(|Y| - 1) + |N(y') \cap X| \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k.$$

By (5) and the fact $m \geq 3$, we have

$$2k + 2 \leq \nu \leq 2k + 3 + \frac{4k - 2m + 2}{k - 1} \leq 2k + 7.$$

It follows that

$$3(k + 1) \leq 3|Y| \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k \leq 2k + 3,$$

a contradiction. \blacksquare

The graphs defined in the following example show that the bound in Theorem 2.3 is tight.

Example 2.4. Suppose $k \geq 3$ is a positive integer. Let G_1 and G_2 be two complete graphs with $V(G_1) = \{u_1, u_2, \dots, u_{k+1}\}$ and $V(G_2) = \{v_1, v_2, \dots, v_{2k^2}\}$. We define $\mathcal{F}_k = \{G' : V(G') = V(G_1) \cup V(G_2) \text{ and } |N(u) \cap V(G_2)| = k \text{ for any } u \in V(G_1)\}$. Set $\mathcal{W}^* = \{G_1 \cup G_2 \cup G_3 : G_3 \in \mathcal{F}_k\}$. Let $G \in \mathcal{W}^*$. Clearly, $V(G) = V(G_1) \cup V(G_2)$ and $|N(u) \cap V(G_2)| = k$ for any $u \in V(G_1)$. Since $2k^2 = \nu(G_2) > |[V(G_1), V(G_2)]| = (k+1)k$, there exists $v \in V(G_2)$ such that $|N(v) \cap V(G_1)| = 0$. This implies that u and v are nonadjacent for any $u \in V(G_1)$. If u is not adjacent to v , then by the definition of G , $|N(u) \cap N(v)| = k$.

Let H be a connected subgraph of G with order k such that $\xi_k(G) = |[V(H), \overline{V(H)}]|$. Assume that $|V(H) \cap V(G_1)| = s$ and $|V(H) \cap V(G_2)| = t$. If $s = k$, then $|[V(H), \overline{V(H)}]| = (k+k)k - (k-1)k = k^2 + k$. If $0 < s < k$, then $|[V(H), \overline{V(H)}]| \geq (k+1-s)s + (2k^2-t)t > s + 2k^2t - (k-1)t = k^2t + k + k^2t - kt >$

$k^2 + k$. If $s = 0$, then $t = k$, and so $||[V(H), \overline{V(H)}]| \geq (k+1-s)s + (2k^2-t)t > k^2 + 2k$. Hence, $\xi_k(G) = k^2 + k$. Combining this with $\frac{\nu(G)}{2} + k = \frac{k+1+2k^2}{2} + k$, we have that $\xi_k(G) \leq \left\lfloor \frac{\nu(G)}{2} \right\rfloor + k$. By Lemma 2.2, G is λ_k -optimal. It implies that $\lambda_k(G) = \xi_k(G) = k^2 + k$. Since $||[V(G_1), V(G_2)]| = (k+1)k$, $||[V(G_1), V(G_2)]|$ is a λ_k -cut of G . Note that $|V(G_1)| > k$ and $|V(G_2)| > k$. Hence, G is not super- λ_k .

REFERENCES

- [1] J.A. Bondy and U.S.R. Murty, *Graph Theory* (New York, Springer, 2008).
- [2] C. Balbuena, P. García-Vázquez and X. Marcote, *Sufficient conditions for λ' -optimality in graphs with girth g* , J. Graph Theory **52** (2006) 73–86.
doi:10.1002/jgt.20150
- [3] N.-W. Chang, C.-Y. Tsai and S.-Y. Hsieh, *On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes*, IEEE Trans. Comput. **63** (2014) 1594–1600.
doi:10.1109/TC.2013.10
- [4] A.-H. Esfahanian and S.L. Hakimi, *On computing a conditional edge-connectivity of a graph*, Inform. Process. Lett. **27** (1988) 195–199.
doi:10.1016/0020-0190(88)90025-7
- [5] J. Fàbrega and M.A. Fiol, *Extraconnectivity of graphs with large girth*, Discrete Math. **127** (1994) 163–170.
doi:10.1016/0012-365X(92)00475-7
- [6] Q. Liu, X. Huang and Z. Zhang, *Optimally restricted edge connected elementary Harary graphs*, Theoret. Comput. Sci. **497** (2013) 131–138.
doi:10.1016/j.tcs.2011.12.015
- [7] J. Meng, *Optimally super-edge-connected transitive graphs*, Discrete Math. **260** (2003) 239–248.
doi:10.1016/S0012-365X(02)00675-1
- [8] J. Meng and Y. Ji, *On a kind of restricted edge connectivity of graphs*, Discrete Appl. Math. **117** (2002) 183–193.
doi:10.1016/S0166-218X(00)00337-1
- [9] L. Shang and H. Zhang, *Super restricted edge-connectivity of graphs with diameter 2*, Discrete Appl. Math. **161** (2013) 445–451.
doi:10.1016/j.dam.2012.08.030
- [10] M. Wang and Q. Li, *Conditional edge connectivity properties, reliability comparisons and transitivity of graphs*, Discrete Math. **258** (2002) 205–214.
doi:10.1016/S0012-365X(02)00299-6
- [11] S. Wang, L. Zhang and S. Lin, *A neighborhood condition for graphs to be maximally k -restricted edge connected*, Inform. Process. Lett. **112** (2012) 95–97.
doi:10.1016/j.ipl.2011.10.012

- [12] S. Wang, J. Li, L. Wu and S. Lin, *Neighborhood conditions for graphs to be super restricted edge connected*, Networks **56** (2010) 11–19.
doi:10.1002/net.20343
- [13] S. Wang and L. Zhang, *Sufficient conditions for k -restricted edge connected graphs*, Theoret. Comput. Sci. **557** (2014) 66–75.
doi:10.1016/j.tcs.2014.08.018
- [14] S. Wang, S. Lin and C. Li, *Sufficient conditions for super k -restricted edge connectivity in graphs of diameter 2*, Discrete Math. **309** (2009) 908–919.
doi:10.1016/j.disc.2008.01.037
- [15] M. Zhang, J. Meng, W. Yang and Y. Tian, *Reliability analysis of bijective connection networks in terms of the extra edge-connectivity*, Inform. Sci. **279** (2014) 374–382.
doi:10.1016/j.ins.2014.03.125

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