Discussiones Mathematicae
Graph Theory 37 (2017) 537–545
doi:10.7151/dmgt.1939

A SUFFICIENT CONDITION FOR GRAPHS TO BE SUPER k-RESTRICTED EDGE CONNECTED[1]

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Abstract

For a subset S of edges in a connected graph G, S is a k-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. Let $\xi_k(G) = \min\{|[X, \bar{X}]| : |X| = k,\ G[X] \text{ is connected}\}$, where $\bar{X} = V(G)\backslash X$. A graph G is super k-restricted edge connected if every minimum k-restricted edge cut of G isolates a component of order exactly k. Let k be a positive integer and let G be a graph of order $\nu \geq 2k$. In this paper, we show that if $|N(u) \cap N(v)| \geq k + 1$ for all pairs u, v of nonadjacent vertices and $\xi_k(G) \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k$, then G is super k-restricted edge connected.

Keywords: graph, neighborhood, k-restricted edge connectivity, super k-restricted edge connected graph.

2010 Mathematics Subject Classification: 05C40.

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[1]This work is supported by the National Science Foundation of China (61370001).

1. Terminology and Introduction

For graph-theoretical terminology and notation not defined here we follow [1]. We consider finite, undirected and simple graphs. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order of G, denoted by $\nu = \nu(G)$, is the number of vertices in G. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$. If G' is a subgraph of G and v is a vertex of G', we define $N_{G'}(v) = N_G(v) \cap V(G')$. Unambiguously, we use $N(v)$ for $N_G(v)$. For subsets X and Y of $V(G)$, we denote by $[X, Y]$ the set of edges with one end in X and the other in Y. An edge cut of G is a subset of $E(G)$ of the from $[X, Y]$, where X is a non-empty proper subset of $V(G)$ and $Y = V(G)\backslash X$.

An interconnection network can be conveniently modeled as a graph $G = (V, E)$. A classical measurement of the fault tolerance of a network is the edge connectivity $\lambda(G)$. The edge connectivity $\lambda(G)$ of a graph G is the minimum cardinality of an edge cut of G. As a more refined index than the edge connectivity, Fàbrega and Fiol [5] proposed the more general concept of k-restricted edge connectivity. For a subset S of edges in a connected graph G, S is a k-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A minimum k-restricted edge cut is called a λ_k-cut. A connected graph G is said to be λ_k-connected if G has a k-restricted edge cut.

In view of recent studies on k-restricted edge connectivity, it seems that the larger the $\lambda_k(G)$, the more reliable the network [7–8, 10]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first and so the optimization of k-restricted edge connectivity draws a lot of attention. For details, the readers can refer to [2–4, 6, 11, 13, 15]. For any positive integer k, let $\xi_k(G) = \min\{|[X, \bar{X}]| : |X| = k,\ G[X] \text{ is connected}\}$. A λ_k-connected graph G is said to be optimally k-restricted edge connected, for short λ_k-optimal, if $\lambda_k(G) = \xi_k(G)$.

A λ_k-connected graph G is super k-restricted edge connected, for short super-λ_k, if every minimum k-restricted edge cut of G isolates a component of order exactly k. The sufficient conditions of super-λ_k have been studied by several authors, see [9, 12, 14]. Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$. By definition, if G is a super-λ_k graph, then G must be a λ_k-optimal graph. However, the converse is not true. For example, a cycle of length at least $2k + 2$ is a λ_k-optimal graph that is not super-λ_k.

Definition 1.1. Let H_1, H_2 be two complete graphs with $V(H_1) = \{x_1, x_2, x_3\}$, $V(H_2) = \{y_1, y_2, y_3, z_1, z_2\}$ and let $M = \{x_1y_1, x_1z_1, x_2y_2, x_2z_2, x_3z_2, x_3y_3\}$. Set $H_8 = (H_1 \cup H_2) + M$ and $W_8 = \{H_8, H_8 - y_1z_1\}$. The graph H_8 is shown in

Figure 1. The heavy edge between A and B indicates that each vertex in A and each vertex in B are adjacent.

Definition 1.2. Let H_1, H_2 be two complete graphs with $V(H_1) = \{x_1, x_2, x_3\}$, $V(H_2) = \{y_1, y_2, y_3, z_1, z_2, z_3\}$ and let $M = \{x_iy_i, x_iz_i : i = 1, 2, 3\}$. Set $H_9^1 = (H_1 \cup H_2) + M$ and $W_9 = \{H_9^1 - M' : M' \subseteq \{y_1z_1, y_2z_2, y_3z_3\}\}$. The graph H_9^1 is shown in Figure 2. The heavy edge between A_i and A_j $(i \neq j, i, j = 1, 2, 3)$ indicates that each vertex in A_i and each vertex in A_j are adjacent.

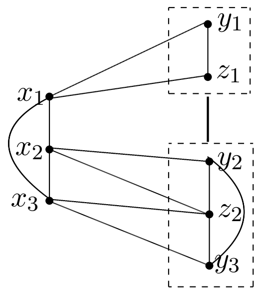

Figure 1. The graph H_8.

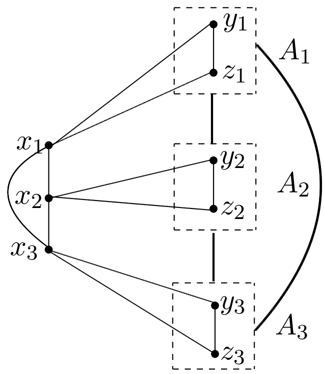

Figure 2. The graph H_9^1.

Definition 1.3. Let H_1, H_2 be two complete graphs with $V(H_1) = \{x_1, x_2, x_3, x_4\}$, $V(H_2) = \{y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4\}$, and let $M = \{x_iy_i, x_iz_i : i = 1, 2, 3, 4\}$. Set $H_{12} = (H_1 \cup H_2) + M$ and $W_{12} = \{H_{12} - M' : M' \subseteq \{y_1z_1, y_2z_2, y_3z_3, y_4z_4\}\}$. Set $E_1 = \{y_1z_1, y_2z_2, y_3z_3, y_4z_4\}$. We define $W_0 = H_{12}$ and W_i as the graph obtained from H_{12} by deleting i edges of E_1, where $i = 1, 2, 3, 4$. Set $\mathcal{W} = \{W_0, W_1, W_2, W_3, W_4\}$. The graph H_{12} is shown in Figure 3. The heavy edge between A_i and A_j $(i \neq j, i, j = 1, 2, 3, 4)$ indicates that each vertex in A_i and each vertex in A_j are adjacent.

Set $\mathcal{W}' = W_8 \cup W_9 \cup W_{12}$. In [12], Wang *et al.* gave the following sufficient condition for a graph to be super-λ_2.

Theorem 1.4 [12]**.** *Let G be a graph of order $\nu \geq 4$. If $|N(u) \cap N(v)| \geq 3$ for all pairs u, v of nonadjacent vertices and $\xi(G) \leq \left\lfloor \frac{\nu}{2} \right\rfloor + 2$, then G is super-λ_2 or in $\mathcal{W}'$.*

In this article, we extend the above result to super-λ_k with $k \geq 3$, and present a neighborhood condition for a graph to be super-λ_k.

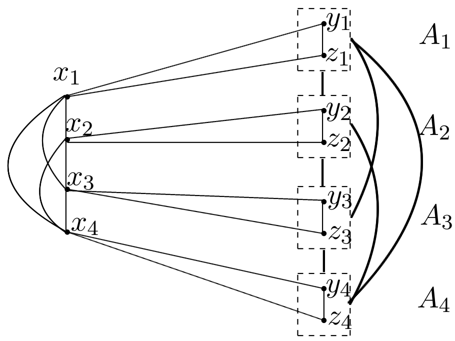

Figure 3. The graph H_{12}.

2. Main Results

Let G be a λ_k-connected graph, and let S be a λ_k-cut of G. It has been shown in [14] that there exists $X \subset V(G)$ such that $G[X]$ and $G[Y]$ are both the connected induced subgraphs of orders at least k and $S = [X, Y]$, where $Y = \bar{X} = V(G)\backslash X$. Let x be a vertex of G. We define $S(x)$ as the set of edges of S incident with x. Furthermore, we define $X_k = \{x \in X : |S(x)| \geq k\}$, $Y_k = \{y \in Y : |S(y)| \geq k\}$, $X_i = \{x \in X : |S(x)| = i\}$, $Y_i = \{y \in Y : |S(y)| = i\}$, where $i = 0, 1, 2, \ldots, k-1$.

In order to prove our main result, we first give some useful lemmas.

Lemma 2.1 [14]**.** *Let k be a positive integer. If G is a complete graph of order $\nu \geq 2k$, then G is super-λ_k.*

Lemma 2.2 [11]**.** *Let $k \geq 3$ be an integer and let $G \notin \mathcal{W}$ be a graph of order $\nu \geq 2k$. If each pair u, v of nonadjacent vertices satisfies $|N(u) \cap N(v)| \geq k$ and $\xi_k(G) \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k$, then G is λ_k-optimal.*

Theorem 2.3. *Let $k \geq 3$ be an integer and G be a graph of order $\nu \geq 2k$. If $|N(u) \cap N(v)| \geq k+1$ for all pairs u, v of nonadjacent vertices and $\xi_k(G) \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k$, then G is super-λ_k or $G \in \mathcal{W}$.*

Proof. If G contains no nonadjacent vertices, then, by Lemma 2.1, we are done. Therefore, we only consider the case that there exist nonadjacent vertices in G below. By Lemma 2.2, G is λ_k-optimal. That is, $\lambda_k(G) = \xi_k(G)$. Suppose that G is neither super-λ_k nor in $\mathcal{W}$. Then there exists a λ_k-cut $S = [X, Y]$ such that $|X| \geq k+1$ and $|Y| \geq k+1$.

Claim 1. *There exists a vertex $x \in X$ such that $|S(x)| \leq k$, and there exists a vertex $y \in Y$ such that $|S(y)| \leq k$.*

Proof. Suppose, on the contrary, that for each $x \in X$, we have $|S(x)| \geq k+1$. Let H be a connected subgraph with order k of $G[X]$. Then

$$\begin{aligned} \xi_k(G) &\leq \sum_{u \in V(H)} |S(u)| + \sum_{u \in X \setminus V(H)} |N(u) \cap V(H)| \\ &\leq \sum_{u \in V(H)} |S(u)| + k|X \setminus V(H)| < \sum_{u \in V(H)} |S(u)| + (k+1)|X \setminus V(H)| \qquad (1) \\ &\leq \sum_{u \in V(H)} |S(u)| + \sum_{v \in X \setminus V(H)} |S(v)| = |S| = \lambda_k(G), \end{aligned}$$

contradicting the fact that $\lambda_k(G) = \xi_k(G)$. □

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. We assume that $Y_0 \neq \emptyset$, say $y_0 \in Y_0$. By Claim 1, there exists a vertex $x \in X$ such that $|S(x)| \leq k$. It is easy to see that x, y_0 are nonadjacent vertices in G, and $|N(x) \cap N(y_0)| \leq k$, a contradiction to the hypothesis.

So, $Y_0 = \emptyset$. By the symmetry, we have $X_0 = \emptyset$. □

Without loss of generality, assume that $|X| \geq |Y| \geq k+1$. Then we can deduce that

$$\left\lceil \frac{\nu}{2} \right\rceil \leq |X| \leq |[X,Y]| = \lambda_k(G) = \xi_k(G) \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k \qquad (2)$$

and

$$\left\lceil \frac{\nu}{2} \right\rceil - k \leq |Y| = \nu - |X| \leq \left\lfloor \frac{\nu}{2} \right\rfloor. \qquad (3)$$

Claim 3. $|X_1| \geq 3$ *when* ν *is odd, and* $|X_1| \geq 1$ *when* ν *is even.*

Proof. Recall that $|X| \geq |Y| \geq k+1$. We have $\nu \geq 2k+3$ when ν is odd, and $\nu \geq 2k+2$ when ν is even. Combining this with the fact

$$2\left\lceil \frac{\nu}{2} \right\rceil - |X_1| \leq 2|X| - |X_1| \leq |[X,Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k,$$

we have $|X_1| \geq 3$ when ν is odd, and $|X_1| \geq 1$ when ν is even. □

Claim 4. $Y_1 = \emptyset$.

Proof. Suppose that $Y_1 \neq \emptyset$. Let $y_1 \in Y_1$ and $N(y_1) \cap X = \{x_1\}$. Then, for any $x \in X \setminus \{x_1\}$, we have

$$\begin{aligned} k+1 &\leq |N(x) \cap N(y_1)| = |N(x) \cap N(y_1) \cap X| + |N(x) \cap N(y_1) \cap Y| \\ &\leq |N(x) \cap Y| + |N(y_1) \cap X| = |N(x) \cap Y| + 1, \end{aligned}$$

which implies that $|N(x) \cap Y| \geq k$. Hence,

$$k\left(\left\lceil \frac{\nu}{2} \right\rceil - 1\right) + 1 \leq \sum_{x \in X \setminus \{x\}} |N(x) \cap Y| + 1 \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k.$$

Combining this with $k \geq 3$, we can deduce that

$$4 \leq k + 1 \leq \left\lfloor \frac{\nu}{2} \right\rfloor \leq \frac{2k-1}{k-1} = 2 + \frac{1}{k-1} < 3,$$

a contradiction. □

Claim 5. $Y_2 = \emptyset$.

Proof. By contradiction, suppose that $Y_2 \neq \emptyset$. By Claim 3, we have $|X_1| \geq 1$, say $x_1 \in X_1$ and $N(x_1) \cap Y = \{y'\}$. Then, for any $y \in Y \setminus \{y'\}$, we have

$$k + 1 \leq |N(x_1) \cap N(y)| \leq |N(x_1) \cap Y| + |N(y) \cap X| = 1 + |N(y) \cap X|,$$

and so $|N(y) \cap X| \geq k \geq 3$. It implies that $|Y_2| = 1$, and so $Y_2 = \{y'\}$. Let $N(y') \cap X = \{x_1, x_2\}$. For any $x \in X \setminus \{x_1, x_2\}$, we can deduce that

$$k + 1 \leq |N(x) \cap N(y')| \leq |N(x) \cap Y| + |N(y') \cap X| = |N(x) \cap Y| + 2,$$

which implies that $|N(x) \cap Y| \geq k - 1$. Therefore,

$$(k-1)\left(\left\lceil \frac{\nu}{2} \right\rceil - 2\right) + 2 \leq (k-1)(|X| - 2) + 2 \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k. \tag{4}$$

Consider the case that ν is odd. By (4), we have

$$(k-2)\left\lfloor \frac{\nu}{2} \right\rfloor < 2k - 3,$$

and so

$$4 \leq k + 1 \leq \left\lfloor \frac{\nu}{2} \right\rfloor < 2 + \frac{1}{k-2} \leq 3,$$

a contradiction. So, $|X| = 5, |Y| = 4$ and $k = 3$. It follows that $8 = 2|Y| < |[X, Y]| \leq \lfloor \frac{\nu}{2} \rfloor + k = 7$, a contradiction.

Consider the case that ν is even. By (4), we have

$$(k-2)\left\lfloor \frac{\nu}{2} \right\rfloor \leq 3k - 4,$$

and so

$$4 \leq k + 1 \leq \left\lfloor \frac{\nu}{2} \right\rfloor \leq 3 + \frac{2}{k-2},$$

which implies that $\nu = 8$ or $\nu = 10$. Since $|Y| \geq k+1$ and $Y_0 = Y_1 = \emptyset$ and $|Y_2| = 1$, we obtain that

$$2(k+1) \leq 2|Y| < |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k = 5 + k.$$

Hence, $k < 3$, a contradiction. □

Let m be the minimum integer such that $Y_m \neq \emptyset$. By Claims 2, 4 and 5, we obtain that $m \geq 3$. By Claim 3, we can choose a vertex $x_1 \in X_1$. Let $N(x_1) \cap Y = \{y'\}$. Then, for any $y \in Y \backslash \{y'\}$, we have

$$|N(y) \cap X| \geq k.$$

By (3), we can deduce that

$$k\left(\left\lceil \frac{\nu}{2} \right\rceil - k - 1\right) + m \leq k(|Y| - 1) + |N(y') \cap X| \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k. \tag{5}$$

By (5) and the fact $m \geq 3$, we have

$$2k + 2 \leq \nu \leq 2k + 3 + \frac{4k - 2m + 2}{k - 1} \leq 2k + 7.$$

It follows that

$$3(k+1) \leq 3|Y| \leq |[X, Y]| \leq \left\lfloor \frac{\nu}{2} \right\rfloor + k \leq 2k + 3,$$

a contradiction. ■

The graphs defined in the following example show that the bound in Theorem 2.3 is tight.

Example 2.4. Suppose $k \geq 3$ is a positive integer. Let G_1 and G_2 be two complete graphs with $V(G_1) = \{u_1, u_2, \ldots, u_{k+1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{2k^2}\}$. We define $\mathcal{F}_k = \{G' : V(G') = V(G_1) \cup V(G_2)$ and $|N(u) \cap V(G_2)| = k$ for any $u \in V(G_1)\}$. Set $\mathcal{W}^* = \{G_1 \cup G_2 \cup G_3 : G_3 \in \mathcal{F}_k\}$. Let $G \in \mathcal{W}^*$. Clearly, $V(G) = V(G_1) \cup V(G_2)$ and $|N(u) \cap V(G_2)| = k$ for any $u \in V(G_1)$. Since $2k^2 = \nu(G_2) > |[V(G_1), V(G_2)]| = (k+1)k$, there exists $v \in V(G_2)$ such that $|N(v) \cap V(G_1)| = 0$. This implies that u and v are nonadjacent for any $u \in V(G_1)$. If u is not adjacent to v, then by the definition of G, $|N(u) \cap N(v)| = k$.

Let H be a connected subgraph of G with order k such that $\xi_k(G) = |[V(H), \overline{V(H)}]|$. Assume that $|V(H) \cap V(G_1)| = s$ and $|V(H) \cap V(G_2)| = t$. If $s = k$, then $|[V(H), \overline{V(H)}]| = (k+k)k - (k-1)k = k^2 + k$. If $0 < s < k$, then $|[V(H), \overline{V(H)}]| \geq (k+1-s)s + (2k^2 - t)t > s + 2k^2 t - (k-1)t = k^2 t + k + k^2 t - kt >$

k^2+k. If $s=0$, then $t=k$, and so $|[V(H),\overline{V(H)}]| \geq (k+1-s)s+(2k^2-t)t > k^2+2k$. Hence, $\xi_k(G)=k^2+k$. Combining this with $\frac{\nu(G)}{2}+k=\frac{k+1+2k^2}{2}+k$, we have that $\xi_k(G) \leq \left\lfloor \frac{\nu(G)}{2} \right\rfloor + k$. By Lemma 2.2, G is λ_k-optimal. It implies that $\lambda_k(G)=\xi_k(G)=k^2+k$. Since $|[V(G_1),V(G_2)]|=(k+1)k$, $|[V(G_1),V(G_2)]|$ is a λ_k-cut of G. Note that $|V(G_1)|>k$ and $|V(G_2)|>k$. Hence, G is not super-λ_k.

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Received 13 August 2015
Revised 11 May 2016
Accepted 11 May 2016