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A TRIPLE OF HEAVY SUBGRAPHS ENSURING PANCYCLICITY OF 2-CONNECTED GRAPHS

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Abstract

A graph G on n vertices is said to be pancyclic if it contains cycles of all lengths k for $k \in \{3, \ldots, n\}$. A vertex $v \in V(G)$ is called super-heavy if the number of its neighbours in G is at least $(n+1)/2$. For a given graph H we say that G is H-f_1-heavy if for every induced subgraph K of G isomorphic to H and every two vertices u, $v \in V(K)$, $d_K(u,v) = 2$ implies that at least one of them is super-heavy. For a family of graphs $\mathcal{H}$ we say that G is $\mathcal{H}$-f_1-heavy, if G is H-f_1-heavy for every graph $H \in \mathcal{H}$.

Let D denote the deer, a graph consisting of a triangle with two disjoint paths P_3 adjoined to two of its vertices. In this paper we prove that every 2-connected $\{K_{1,3}, P_7, D\}$-f_1-heavy graph on $n \geq 14$ vertices is pancyclic. This result extends the previous work by Faudree, Ryjáček and Schiermeyer.

Keywords: cycle, Fan-type heavy subgraph, Hamilton cycle, pancyclicity.

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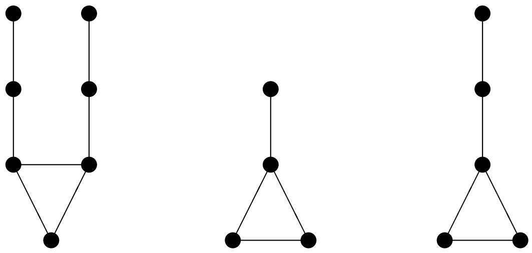
1. Introduction

We consider only finite, simple and undirected graphs. For terminology and notation not defined here see [5].

Let G be a graph on n vertices. G is said to be Hamiltonian if it contains a cycle C_n, and it is called pancyclic if it contains cycles of all possible lengths. If G does not contain an induced copy of a given graph H, we say that G is H-free. G is called H-f_i-heavy, if for every induced subgraph K of G isomorphic to H and for every two vertices x, $y \in V(K)$ satisfying $d_K(x, y) = 2$, the following



D Z_1 Z_2

Figure 1. Graphs D (deer), Z_1 and Z_2.

inequality holds: $\max\{d_G(x), d_G(y)\} \geq (n+i)/2$. For simplicity, we write f-heavy instead of f_0-heavy. For a family of graphs $\mathcal{H}$ we say that G is $\mathcal{H}$-free ($\mathcal{H}$-f_i-heavy), if G is H-free (H-f_i-heavy) for every graph $H \in \mathcal{H}$.

The complete bipartite graph $K_{1,3}$ is called a claw. The vertex of degree three in the claw is called its center vertex, and other vertices are its end vertices.

Recent decades have seen many interesting results connecting the existence of cycles in graphs with their induced subgraphs. Among them one can find the following theorem by Bedrossian (the graphs Z_1 and Z_2 are represented on Figure 1, as well as the deer).

Theorem 1 (Bedrossian [1])**.** *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R,S\}$-free implies G is pancyclic if and only if (up to the symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

One can allow these specific pairs of subgraphs to be present in a 2-connected graph, but with some requirements regarding degrees of their vertices imposed on them, and still obtain a sufficient condition for a graph to be pancyclic. Thus Bedrossian's result was later extended by numerous authors. One of these extentions involves the notion of f_i-heaviness (also called a Fan-type heaviness, due to the well-known theorem by Fan).

Theorem 2. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R,S\}$-f_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*

Z_1 (Bedrossian, Chen and Schelp [2]),

Z_2, P_4 (Ning [10]), *or*

P_5 (Wideł [12]).

One of the results regarding triples of forbidden subgraphs and pancylicity of two-connected graphs is due to Faudree *et al.*

Theorem 3 (Faudree, Ryjácek and Schiermeyer, Corollary F in [7]). *Every* 2-*connected,* $\{K_{1,3}, P_7, D\}$*-free graph on* $n \geq 14$ *vertices is pancyclic.*

Recently, Ning proved the following fact.

Theorem 4 (Ning, [9]). *Every* 2*-connected,* $\{K_{1,3}, P_7, D\}$*-*f*-heavy graph is Hamiltonian.*

Motivated by Theorems 3 and 4 and by similar results for pairs of forbidden and Fan-type heavy subgraphs, in this paper we prove the following.

Theorem 5. *Every* 2*-connected,* $\{K_{1,3}, P_7, D\}$*-*f_1*-heavy graph on* $n \geq 14$ *vertices is pancyclic.*

In Section 2 we introduce notation used further in the paper and present some of the previous results that will be of use in the proof of Theorem 5. The proof itself is postponed to Section 3.

Remark 1. It is easy to see that every graph satisfying the assumptions of Theorem 3 satisfies also the assumptions of Theorem 5. To see that Theorem 5 in fact extends Theorem 3, consider a disjoint union $K_{n/2+1}+K_{n/2-8}$ of complete graphs for even $n \geq 18$. Let $V(G) = V(K_{n/2+1} + K_{n/2-8}) \cup \{x, y, z, u, v, w, t\}$ and $E(G) = E(K_{n/2+1}+K_{n/2-8}) \cup \{xy, yz, zx, yw, wu, zt, tv\} \cup \{xx', yx', zx' \colon x' \in V(K_{n/2+1})\} \cup \{uy', vy' \colon y' \in V(K_{n/2-8})\}$. It is not difficult to see that G is not D-free, and thus not $\{K_{1,3}, P_7, D\}$-free, but it is $\{K_{1,3}, P_7, D\}$-f_1-heavy.

2. Preliminaries

The subgraph of G induced by the set of vertices $A \subset V(G)$ is denoted by $G[A]$. By $G - A$ we denote the subgraph $G[V(G) \setminus A]$. If A consists of one vertex, say $A = \{v\}$, we write $G - v$ instead of $G - \{v\}$. Let $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. If $G[A]$ is isomorphic to P_7, where $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7\}$ are the edges of this path, we say that A induces a P_7. If $A = \{v_1, v_2, v_3, v_4\}$ and $G[A]$ is isomorphic to $K_{1,3}$, we say that $\{v_1; v_2, v_3, v_4\}$ induces $K_{1,3}$ (or induces a claw), where v_1 is a center vertex and v_2, v_3 and v_4 are end vertices of a claw. Finally, if $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $G[A]$ is isomorphic to D, we say that $\{v_1, v_2, v_3; v_4, v_5; v_6, v_7\}$ induces a D, where $\{v_1, v_2, v_3\}$ induces a triangle and both $\{v_2, v_4, v_5\}$ and $\{v_3, v_6, v_7\}$ induce P_3.

For a cycle C we select one of the two possible orientations of C. We write xC^+y for the path from $x \in V(C)$ to $y \in V(C)$ following the orientation of C, and $xC^{-}y$ denotes the path from x to y opposite to the direction of C. For two positive integers k and m, where $k \leq m$, we say that G contains $[k,m]$-cycles if there are cycles C_k, $C_{k+1}, \ldots, C_m$ in G.

Let $C = v_1 v_2 \cdots v_p v_1$ be a cycle. For two positive integers k and m, satisfying $k \leq m \leq p$, by $C[v_k, v_m]$ we denote the set $\{v_k, v_{k+1}, \ldots, v_m\}$. A chord in C is an edge beetwen two vertices from $V(C)$ that do not lie next to each other on the cycle. In particular, a one-chord (two-chord) in C is an edge $v_i v_{i+2}$ ($v_i v_{i+3}$), where addition of indices is performed modulo p and $i \in \{1, \ldots, p\}$. A chord in $C[v_k, v_m]$ is a chord of C with both of its endvertices belonging to the set $\{v_k, v_{k+1}, \ldots, v_m\}$.

Let G be a graph on n vertices. Vertex $v \in V(G)$ is called heavy if $d_G(v) \geq n/2$ and super-heavy if $d_G(v) \geq (n+1)/2$. We say that two vertices u and v form a heavy-pair (super-heavy pair), if both u and v are heavy (super-heavy).

Let $A, B \subset V(G)$ be subsets of vertices of G. By $e(A,B) = |\{e = uv \in E(G) \colon u \in A, v \in B\}|$ we denote the total number of edges between A and B. If both A and B consist of one element, say $A = \{v_A\}$ and $B = \{v_B\}$, we write $e(v_A, v_B)$ instead of $e(\{v_A\}, \{v_B\})$.

Lemma 6 (Benhocine and Wojda [3])**.** *Let G be a graph on $n \geq 4$ vertices and let C be a cycle of length $n-1$ in G. If $d_G(v) \geq n/2$ for $v \in V(G) \setminus V(C)$, then G is pancyclic.*

This lemma can be extended as follows.

Lemma 7. *Let G be a graph on n vertices and let C be a cycle of length $n-i$ in G, where $i \in \{1, \ldots, n-3\}$. If $d_G(v) \geq (n+i-1)/2$ for some $v \in V(G) \setminus V(C)$, then there are $[3, n-i+1]$-cycles in G.*

Proof. Let $C = v_0 v_1 \cdots v_{n-i-1} v_0$ and let v be a vertex of degree at least $(n+i-1)/2$ such that $v \notin V(C)$. Let G' denote $G[V(C)]$. Suppose the statement is not true, i.e., that there is no cycle C_p in G for some $p \in \{3, \ldots, n-i+1\}$. Then

$$e(v, v_j) + e(v, v_{j+p-2}) \leq 1$$

for $j = 1, \ldots, n-i$, with addition of indices performed modulo $n-i$. This implies that

$$d_{G'}(v) = 1/2 \cdot \sum_{j=1}^{n-i} [e(v, v_j) + e(v, v_{j+p-2})] \leq (n-i)/2.$$

On the other hand, since there are $i-1$ possible neighbours of v outside the cycle C, we get

$$d_{G'}(v) \geq (n+i-1)/2 - i + 1 = (n-i+1)/2.$$

A contradiction. ■

Corollary 8. *Let G be a Hamiltonian graph on n vertices with a super-heavy vertex v. If there exists a cycle C of length $n-2$ in G such that $v \notin V(C)$, then G is pancyclic.*

Proof. Lemma 7 implies that there are $[3, n-1]$-cycles in G. Since G is Hamiltonian, it is pancyclic. ∎

Lemma 9 (Bondy [4])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x,y) = 1$ and $d_G(x)+d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 10 (Hakimi and Schmeichel [11])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x,y) = 1$ and $d_G(x) + d_G(y) \geq n$, then G is pancyclic unless G is bipartite or else G is missing only $(n-1)$-cycles.*

Lemma 11 (Ferrara, Jacobson and Harris [8])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x,y) = 2$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 12. *Let G be a 2-connected, $\mathcal{H}$-f_1-heavy graph on n vertices, where $\mathcal{H}$ is some family of graphs with $K_{1,3} \in \mathcal{H}$. If there exists a super-heavy vertex $u \in V(G)$ and every 2-connected $\mathcal{H}$-f-heavy graph is Hamiltonian, then either*

1. *G is pancyclic*
   or
2. *there exists $v \in V(G)$ such that $G - \{u, v\}$ consists of two components H_1 and H_2. Suppose that $|H_1| \leq |H_2|$ and $y \in H_2$ is a neighbour of u along a Hamilton cycle of G. Then*
   (a) *there are no super-heavy vertices in H_1,*
   (b) *$N_{H_2}[u] \subseteq N_G[y]$.*

Proof. If $G-u$ is 2-connected, then it is Hamiltonian (since $G-u$ is $\mathcal{H}$-f-heavy) and so G is pancyclic by Lemma 6.

Now suppose that G is not pancyclic. This implies, by the previous paragraph, that $G-u$ is not 2-connected, and so there exists a vertex $v \in V(G)$ such that $G - \{u, v\}$ consists of two components. Let $C = uy_1 \cdots y_{h_2}vx_{h_1} \cdots x_1u$ be a Hamilton cycle in G. Assume, without loss of generality, that $h_1 \leq h_2$ and consider vertex $x \in H_1 = \{x_1, \ldots, x_{h_1}\}$. Since x can be adjacent to at most u, v and every other vertex in H_1, it must be that $d_G(x) \leq 2+h_1-1 \leq 2+(n-2)/2-1 = n/2$. Hence, x cannot be super-heavy.

Now suppose there exists a vertex $y_i \in H_2 = \{y_1, \ldots, y_{h_2}\}$ adjacent to u and not adjacent to y_1, where $i \geq 2$. Then $\{u; x_1, y_1, y_i\}$ induces a claw. Since G is $K_{1,3}$-f_1-heavy and x_1 is not super-heavy, y_1 must be super-heavy. But then $d_G(u) + d_G(y_1) \geq n+1$. Since $d_C(u, y_1) = 1$, G is pancyclic by Lemma 9. ∎

Lemma 13. *Let G be a graph on n vertices. Let $u, v \in V(G)$ and let i be some nonnegative integer less than $n-1$. Let X be a set of i vertices $\{x_1, \ldots, x_i\} \subset V(G)$ such that $(N[u] \cup N[v]) \cap X = \emptyset$. Suppose there are $[n-i+1, n]$ cycles in G and $G' = G - X$ is Hamiltonian with a Hamilton cycle C. Then*

1. *if $d_C(u, v) \leq 2$ and $d_G(u) + d_G(v) \geq n - i + 1$, then G is pancyclic,*
2. *if $d_C(u, v) = 1$, $d_G(u) + d_G(v) \geq n - i$ and there is a $(|G'| - 1)$-cycle in G', then G is pancyclic.*

Proof. The first statement is true, since under these assumptions G' is pancyclic by Lemma 9 or 11. If the second case occurs, G' is pancyclic by Lemma 10. Pancyclicity of G' implies pancyclicity of G. ■

3. Proof of Theorem 5

Theorem 5. *Every 2-connected, $\{K_{1,3}, P_7, D\}$-f_1-heavy graph on $n \geq 14$ vertices is pancyclic.*

Proof. The theorem will be proved by contradiction. Suppose that a graph G on $n \geq 14$ vertices satisfies the assumptions of the theorem but is not pancyclic. Then G is not $\{K_{1,3}, P_7, D\}$-free, by Theorem 3, and so there is a super-heavy vertex in G, say u. Since G is $\{K_{1,3}, P_7, D\}$-f_1-heavy, in particular it is $\{K_{1,3}, P_7, D\}$-f-heavy, and so it is Hamiltonian by Theorem 4. Let C denote a Hamilton cycle in G. By Lemma 12, with $\mathcal{H} = \{K_{1,3}, P_7, D\}$, and Theorem 4 we can set $C = uy_1 \cdots y_{h_2} v x_{h_1} \cdots x_1 u$, where $H_1 = \{x_1, \ldots, x_{h_1}\}$ and $H_2 = \{y_1, \ldots, y_{h_2}\}$ are components of $G - \{u, v\}$ satisfying $h_1 \leq h_2$. We restate the last two pieces of information given by Lemma 12, as they will be frequently referred to in the following.

Claim 14. *There are no super-heavy vertices in H_1.*

Claim 15. $N_{H_2}[u] \subseteq N_G[y_1]$.

Claim 16. *There are no super-heavy pairs of vertices with distance one or two along a Hamilton cycle in G.*

Proof. Otherwise G is pancyclic by Lemma 9 or Lemma 11, a contradiction. □

Claim 17. *If $y_i y_{i+2} \notin E(G)$ for some vertices $y_i, y_{i+2} \in H_2$, then at least one of them is not adjacent to u.*

Proof. Otherwise $\{u; x_1, y_i, y_{i+2}\}$ induces a claw. Since G is claw-f_1-heavy and x_1 is not super-heavy by Claim 14, both y_i and y_{i+2} are super-heavy. This contradicts Claim 16. □

Claim 18. $N_{H_1}[u]$ *induces a clique in* G.

Proof. Since the statement is obvious for $h_1 = 1$ and $h_1 = 2$, assume $h_1 \geq 3$. Suppose the claim is not true, i.e., that there exist vertices x_a, $x_b \in N_{H_1}(u)$ such that $x_a x_b \notin E(G)$. Then $\{u; x_a, x_b, y_1\}$ induces a claw. Since neither x_a nor x_b is super-heavy by Claim 14, this contradicts G being claw-f_1-heavy. □

Claim 19. *Let* $v_1 v_2 \cdots v_n v_1$ *be a Hamilton cycle in* G. *If* $uv \notin E(G)$, *then* $d_G(v_i) + d_G(v_{i+1}) < n$ *for* $i \in \{1, \ldots, n\}$, *where addition of indices is performed modulo* n.

Proof. Suppose $d_G(v_i) + d_G(v_{i+1}) \geq n$ for some $i \in \{1, \ldots, n\}$. Since G is not pancyclic, Lemma 10 implies that G is either bipartite or missing a cycle of length $n-1$. Suppose the latter is true. Then $y_i y_{i+2} \notin E(G)$ and $x_j x_{j+2} \notin E(G)$ for every y_i, $y_{i+2} \in H_2$, x_j, $x_{j+2} \in H_1$. By Claim 17, u can be adjacent to at most one vertex from every pair $\{y_i, y_{i+2}\} \subset H_2$, and by Claim 18 it can be adjacent to at most one vertex from every pair $\{x_j, x_{j+2}\} \subset H_1$. Since $uv \notin E(G)$, we get

$$d_G(u) \leq \lceil h_1/2 \rceil + \lceil h_2/2 \rceil \leq (h_1 + 1)/2 + (h_2 + 1)/2 = n/2,$$

a contradiction with u being super-heavy.

Hence, there is a cycle of length $n-1$ in G. Since G is Hamiltonian, it cannot be bipartite. This contradicts Lemma 10. □

Claim 20. $N_{H_2}(u) \neq H_2$.

Proof. Otherwise there are both $[3, h_2+1]$- and $[n-h_2+1, n]$-cycles in G. If $h_2 > (n-2)/2$, this implies that G is pancyclic, a contradiction. Since $h_2 \geq (n-2)/2$, it follows that $h_2 = (n-2)/2 = h_1$ and G is missing and most (h_2+2)-cycle.

Now, if u is adjacent to some vertex $x_i \in H_1$ other than x_1, then $uy_{n-i-h_2} C^+ x_i u$ is a cycle of length $h_2 + 2$, a contradiction. Similarly, if there is an edge in H_1 that does not lie on C, say $x_i x_j \in E(G)$ with $i + 1 < j$, then $uy_{n+i-j-h_2-2} C^+ x_j x_i u$ is such a cycle. Hence, the subgraph of G induced by u and all vertices of H_1 is a path. Since $n \geq 14$, it follows that $h_2 \geq 6$ and so $\{u, x_1, x_2, x_3, x_4, x_5, x_6\}$ induces a path P_7 in G. Since u is the only super-heavy vertex of this path, by Claim 14, this contradicts G being P_7-f_1-heavy. □

By Claim 20 we can choose a vertex $y_k \in N_{H_2}(u)$ such that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$.

***Case* 1.** $h_1 = 1$.

Claim 21. $uv \in E(G)$.

Proof. Suppose the contrary. Then, by Claim 15, we have $d_G(y_1) \geq (n-1)/2$ and so $d_G(u) + d_G(y_1) \geq n$. This contradicts Claim 19. □

Recall that $N_{H_2}[u] \subseteq N_G[y_1]$ by Claim 15, implying $d_G(y_1) \geq (n+1)/2 - 2$ (since u is super-heavy and both x_1 and v are its neighbours) and $d_G(u)+d_G(y_1) \geq n-1$. We will refer to the latter implicitly in the following.

Claim 22. $N_{H_2}[u] = N_G[y_1]$.

Proof. Suppose the claim is not true. Then, by Claim 15, either there is a vertex $y \in H_2$ adjacent to y_1 and not adjacent to u or else $vy_1 \in E(G)$. In either case it follows that $d_G(y_1) \geq (n+1)/2 - 1$ and so $d_G(u) + d_G(y_1) \geq n$. Since G is Hamiltonian and uC^+vu is a cycle of length $n-1$, G is neither bipartite nor missing $(n-1)$-cycles. Lemma 10 implies that G is pancyclic, a contradiction. □

Claim 23. *There are* $[n-2, n]$*-cycles in* G.

Proof. Obviously, G is Hamiltonian and vuC^+v is an $(n-1)$-cycle. Claim 22 implies that $uy_2 \in E(G)$ and so uy_2C^+vu is a cycle of length $n-2$. □

Recall that y_k is a neighbour of u in H_2 such that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$. Choose the minimal possible k for which this property holds.

Claim 24. $h_2 \geq k+5$.

Proof. By the choice of k and the fact that $n = h_2 + 3$ we have $d_{H_2}(u) \geq k + n - h_2 - 3$, implying, by Claim 22, that $d_G(y_1) \geq k + n - h_2 - 3$. Since G is not pancyclic, it follows from Lemma 9 that $d_G(u) + d_G(y_1) < n+1$. Noting that $d_G(u) = d_{H_2}(u) + 2$ and combining these inequalities, we get

$$2(k + n - h_2 - 2) \leq d_G(u) + d_G(y_1) < n + 1,$$

implying $h_2 > k + (n-5)/2$. Since $n \geq 14$, the claim follows. □

Claim 25. $uy_{k+2} \notin E(G)$.

Proof. Suppose the statement is not true. Then $uy_{k+2} \in E(G)$, implying, by Claim 17, that $y_ky_{k+2} \in E(G)$. Consider $G' = G - y_{k+1}$, a Hamiltonian graph with a Hamilton cycle $C' = uy_1C^+y_ky_{k+2}C^+u$. Since $uy_{k+1} \notin E(G)$ it follows from Claim 22 that $y_1y_{k+1} \notin E(G)$ and so

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq n - 1 = |G'|.$$

This implies, together with the fact that vuC'^+v is an $(|G'|-1)$-cycle in G', that G' is pancyclic, by Lemma 10. But then G is pancyclic, a contradiction. □

Claim 26. y_ky_{k+2}, y_ky_{k+3}, $y_{k+1}y_{k+3} \notin E(G)$.

Proof. This is indeed true, since if any of these edges exists, say y_ay_{a+i}, Lemma 13 for u, y_1, $X = \{y_{a+1}, y_{a+i-1}\}$ and a Hamilton cycle $y_ay_{a+i}C^+y_a$ in $G - X$ implies pancyclicity of G. □

Claim 27. $uy_{k+3} \notin E(G)$.

Proof. Suppose the statement is not true. The it follows from Claim 22 that $y_1y_{k+3} \in E(G)$ and from Claim 26 that $\{u; x_1, y_k, y_{k+3}\}$ induces a claw. Since G is claw-f_1-heavy and x_1 is not super-heavy by Claim 14, y_k is super-heavy.

Consider $G' = G - \{y_{k+1}, y_{k+2}\}$ with a Hamilton cycle $y_1C^+y_kuC^-y_{k+3}y_1$. By Claims 25 and 26 and the fact that y_k is super-heavy we have

$$d_{G'}(u) + d_{G'}(y_k) = d_G(u) + d_G(y_k) - 1 \geq |G'| + 1.$$

Hence, G' is pancyclic by Lemma 9 and so there are $[3, n-2]$-cycles in G. Together with Claim 23 this gives pancyclicity of G, a contradiction. $\square$

Claim 28. y_ky_{k+4}, $y_{k+1}y_{k+4}$, $y_{k+2}y_{k+4} \notin E(G)$.

Proof. See the proof of Claim 26 (which can now be applied here due to Claim 27). $\square$

Claim 29. $uy_{k+4} \notin E(G)$.

Proof. For the proof replace y_{k+3} in the proof of Claim 27 with y_{k+4}, $G' = G - \{y_{k+1}, y_{k+2}\}$ with $G' = G - \{y_{k+1}, y_{k+2}, y_{k+3}\}$ and Claims 25 and 26 with Claims 27 and 28, respectively. $\square$

Claims 25, 26, 27, 28 and 29 imply that $\{x_1, u, y_k, y_{k+1}, y_{k+2}, y_{k+3}, y_{k+4}\}$ induces a P_7. Since G is P_7-f_1-heavy at least one vertex from each of the pairs $\{x_1, y_k\}$, $\{y_{k+1}, y_{k+3}\}$ and $\{y_{k+2}, y_{k+4}\}$ must be super-heavy. Since x_1 is not super-heavy by Claim 14, y_k is super-heavy. Claim 16 implies that neither y_{k+1} nor y_{k+2} is super heavy, and so both y_{k+3} and y_{k+4} must be super-heavy. This contradicts Claim 16 and completes the proof of this case.

***Case* 2.** $h_1 \geq 2$.

***Subcase* 2.1.** $d_{H_1}(u) = 1$. In this subcase the only neighbour of u in H_1 is x_1. As in Case 1, Claim 15 implies that $d_G(y_1) \geq (n+1)/2 - 2$ and so $d_G(u) + d_G(y_1) \geq n - 1$. Again, this fact will be implicitly referred to in the following.

Claim 30. $uv \in E(G)$.

Proof. Otherwise $uv \notin E(G)$ and so $d_G(y_1) \geq (n+1)/2 - 1$ by Claim 15. But then $d_G(u) + d_G(y_1) \geq n$, in contradiction of Claim 19. $\square$

Claim 31. *Suppose $x_ix_{i+2} \in E(G)$ for some x_i, $x_{i+2} \in H_1$. Then the only possible one-chords in C other than x_ix_{i+2} are $x_{i-1}x_{i+1}$ and $x_{i+1}x_{i+3}$.*

Proof. Suppose the claim is not true. Then there is a one-chord in C, say v_jv_{j+2}, for some $v_j \notin \{x_{i-1}, x_i, x_{i+1}\}$.

Consider $G' = G - x_{i+1}$. Obviously, $C' = uy_1C^+x_ix_{i+2}C^+u$ is a Hamilton cycle in G' with

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq |G'|.$$

Since v_jv_{j+2} is a one-chord in C', there is an $(|G'|-1)$-cycle in G' and G' is not bipartite. Hence, G' is pancyclic by Lemma 10, implying pancyclicity of G, a contradiction. □

Claim 32. *Suppose $x_ix_{i+3} \in E(G)$ for some x_i, $x_{i+3} \in H_1$. Then there are no one-chords in C.*

Proof. Otherwise there is a one-chord in C. Let $G' = G - \{x_{i+1}, x_{i+2}\}$. G' is Hamiltonian with a Hamilton cycle $uy_1C^+x_ix_{i+3}C^+u$ and

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq |G'| + 1.$$

Lemma 9 implies that G' is pancyclic and so there are $[3, n-2]$-cycles in G. Since the one-chord in C creates a cycle of length $n-1$ and G is Hamiltonian, G is pancyclic. A contradiction. □

Claim 33. *If there is a one-chord in $C[u, v]$, then there are no one-chords and no two-chords in $C[x_{h_1}, x_1]$.*

Proof. This claim is a corollary of Claim 31 and Claim 32. □

Claim 34. *Suppose there is a one-chord in $C[u, v]$. Then $h_1 \leq 3$.*

Proof. Suppose the statement is not true. Then there is a one-chord in $C[u, v]$ and $h_1 \geq 4$. Recall that $y_k \in N_{H_2}(u)$ is such a vertex that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$. Since $N_{H_1}(u) = \{x_1\}$, Claim 33 implies that $\{x_4, x_3, x_2, x_1, u, y_k, y_{k+1}\}$ induces a P_7. Since neither x_4 nor x_2 are super-heavy, by Claim 14, this contradicts G being P_7-f_1-heavy. □

Claim 35. *There are no one-chords in $C[u, v]$.*

Proof. Suppose the claim is not true. Then there is a one-chord in $C[u, v]$ and $h_1 \leq 3$, by Claim 34.

Assume $h_1 = 2$. Consider $G' = G - \{x_1, x_2\}$. By Claim 30, $C' = uy_1C^+vu$ is a Hamilton cycle in G'. Since the one-chord in $C[u, v]$ is also a one-chord in C', there is a cycle of length $|G'| - 1$ in G'. Furthermore, we have

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) - 1 + d_G(y_1) \geq |G'|,$$

and so G' is pancyclic by Lemma 10. This implies pancyclicity of G, a contradiction.

Now let $h_1 = 3$. Let $G' = G - \{x_1, x_2, x_3\}$ with a Hamilton cycle uy_1C^+vu. Since

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) - 1 + d_G(y_1) \geq |G'| + 1,$$

G' is pancyclic by Lemma 9. Hence, there are $[3, n-3]$-cycles in G. Since there is a one-chord in $C[u, v]$, G contains also $[n-1, n]$-cycles. It follows that there are no cycles of length $n-2$ in G, since we assumed G is not pancyclic. Then obviously $vx_1 \notin E(G)$. But now, in order to avoid $\{u; , x_1, v, y_1\}$ inducing a claw with neither x_1 nor y_1 being super-heavy, $vy_1 \in E(G)$. This implies, by Claim 15, that $d_G(y_1) \geq (n+1)/2 - 2 + 1$ and so $d_G(u) + d_G(y_1) \geq n$. Since there is an $(n-1)$-cycle in G, G is pancyclic by Lemma 10, a contradiction. □

Now it follows from Claim 17 and Claim 35 that u can be adjacent to at most $\lceil h_2/2 \rceil \leq (h_2 + 1)/2$ vertices in H_2. Hence, $d_G(u) \leq (h_2 + 1)/2 + 2$. If $h_1 \geq 3$, then $h_2 \leq n - 5$ and we get $d_G(u) \leq n/2$, a contradiction with u being super-heavy. Hence, $h_1 = 2$.

Claim 36. *$vy_1 \in E(G)$.*

Proof. First we show that $vx_1 \notin E(G)$. Indeed, otherwise one could consider a Hamiltonian graph $G' = G - x_2$, with a Hamilton cycle $vx_1uy_1C^+v$. Since $uv \in E(G)$ by Claim 30, vuC^+v is an $(|G'| - 1)$-cycle in G'. Finally, we have

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq |G'|,$$

and so G' is pancyclic by Lemma 10. Since vx_1C^+v is an $(n-1)$-cycle in G, G is pancyclic, a contradiction.

Hence, $vx_1 \notin E(G)$. Now suppose the claim is not true. Then $vy_1 \notin E(G)$ and so $\{u; x_1, v, y_1\}$ induces a claw. Since x_1 is not super-heavy by Claim 14, y_1 must be super heavy. But then $\{u, y_1\}$ is a super-heavy pair of vertices that lie next to each other on the cycle C, a contradiction with Claim 16. □

Since $uy_2 \notin E(G)$, by Claim 35, it follows from Claim 15 and Claim 36 that $d_G(y_1) \geq (n+1)/2$. But then $\{u, y_1\}$ is a super-heavy pair of vertices with distance one along the cycle C, a contradiction with Claim 16.

***Subcase* 2.2.** $2 \leq d_{H_1}(u) < h_1$. Note that the assumptions of this subcase imply $h_1 \geq 3$. Let $x_i \in N_{H_1}(u)$ be a vertex such that $x_{i+1} \in H_1$ and $ux_{i+1} \notin E(G)$.

Claim 37. *Suppose u is adjacent to a super-heavy vertex $y_j \in H_2$, where $j < h_2$. Then $\{y_{j+1}, \ldots, y_{h_2}\} \subset N_G[y_j]$ and $\{y_{j+1}, \ldots, y_{h_2}\} \cap N_G(y_1) = \emptyset$.*

Proof. First we show that $y_{j+1} \notin N_G(y_1)$. Indeed, suppose $y_1y_{j+1} \in E(G)$. Then $y_1y_{j+1}C^+uy_jC^-y_1$ is a Hamilton cycle in G with $d_G(u) + d_G(y_j) \geq n+1$. Lemma 9 implies G is pancyclic, a contradiction.

Assume $\{y_{j+1}, \ldots, y_{j+m}\} \subset N_G[y_j]$ and $\{y_{j+1}, \ldots, y_{j+m}\} \cap N_G(y_1) = \emptyset$ for some m such that $j + m < h_2$. We will show that this implies $y_jy_{j+m+1} \in E(G)$ and $y_1y_{j+m+1} \notin E(G)$.

Suppose y_1 is adjacent to y_{j+m+1}. Consider $G' = G - \{y_{j+1}, \ldots, y_{j+m}\}$. Obviously, $|G'| = n - m$ and $y_1y_{j+m+1}C^+uy_jC^-y_1$ is a Hamilton cycle in G'. Since none of the vertices removed from G in order to obtain G' is adjacent to y_1, it follows from Claim 15 that none of them is adjacent to u. Hence, we get

$$d_{G'}(u) + d_{G'}(y_j) = d_G(u) + d_G(y_j) - m \geq |G'| + 1,$$

and so G' is pancyclic by Lemma 9, implying that there are $[3, n-m]$-cycles in G. Note that the cycle $y_jy_{j+m}C^+y_j$ of length $n - m + 1$ can be extended to the $(n - m + 2)$-cycle $y_jy_{j+m-1}y_{j+m}C^+y_j$. Appending vertices $y_{j+m-2}, \ldots, y_{j+1}$ to this cycle, one-by-one, in the similar manner, gives $[n-m+3, n]$-cycles. It follows that G is pancyclic, a contradiction.

Hence, $y_1y_{j+m+1} \notin E(G)$, implying, by Claim 15, $uy_{j+m+1} \notin E(G)$. Now, if $y_jy_{j+m+1} \notin E(G)$, $\{y_1, u, y_j; x_i, x_{i+1}; y_{j+m}, y_{j+m+1}\}$ induces a D. Since G is D-f_1-heavy and x_i is not super-heavy, by Claim 14, y_1 must be super-heavy. But then $\{u, y_1\}$ is a super-heavy pair of vertices, a contradiction with Claim 16. So it must be $y_jy_{j+m+1} \in E(G)$. By mathematical induction the claim is true. □

Claim 38. *$N_{H_2}[u]$ induces a clique and u is adjacent to at most one super-heavy vertex in H_2.*

Proof. Note that it follows from Claim 37 and Claim 15 that if u is adjacent to some super-heavy vertex $y_j \in H_2$, then $\{y_{j+1}, \ldots, y_{h_2}\} \cap N_G(u) = \emptyset$. Suppose there are two super-heavy neighbours of u in H_2, say y_j and y_m, where $j < m$. Then obviously $y_m \in \{y_{j+1}, \ldots, y_{h_2}\}$, a contradiction.

Now suppose the first part of the claim is not true. Then there exist two neighbours of u, say y_a and y_b, such that $y_ay_b \notin E(G)$. But then $\{u; x_1, y_a, y_b\}$ induces a claw. Since x_1 is not super-heavy by Claim 14 and at most one vertex from the pair $\{y_a, y_b\}$ can be super-heavy, this contradicts G being $K_{1,3}$-f_1-heavy. □

Claim 39. *There are $[3, 5]$-cycles in G.*

Proof. Since $n \geq 14$ and u is super-heavy, $d_G(u) \geq 8$. Hence, u has at least four neighbours either in H_1 or in H_2. Both $N_{H_1}[u]$ and $N_{H_2}[u]$ induce cliques, by Claim 18 and Claim 38, respectively, implying that there is an induced clique on at least five vertices in G. The claim follows. □

Claim 40. *Let $A = \{x_{a+1}, \ldots, x_{a+p}\}$ be a maximal set of consecutive non-neighbours of u in H_1 (i.e., $x_a \in N_{H_1}(u)$ and either $x_{a+p+1} \in N_{H_1}(u)$ or $x_{a+p+1} = v$). Then $x_a x_{a+j} \in E(G)$ for $j = 1, \ldots, p$.*

Proof. Since the statement is obvious for $p = 1$, assume $p \geq 2$. Suppose this is not true. Then $x_a x_{a+j} \in E(G)$ and $x_a x_{a+j+1} \notin E(G)$ for some $1 < j < p - 1$. We divide the proof of this claim into three subclaims.

Claim 40.1. *Let $B = \{y_{b+1}, \ldots, y_{b+q}\}$ be a maximal set of consecutive non-neighbours of u in H_2. Then $y_b y_{b+l} \in E(G)$ for $l = 1, \ldots, q$.*

Proof. Again, assume $q \geq 2$, since the statement is obviously true for $q = 1$, and suppose it is not true. Then there are vertices $y_{b+l}, y_{b+l+1} \in B$ such that $y_b y_{b+l} \in E(G)$ and $y_b y_{b+l+1} \notin E(G)$. But now $\{x_{a+j+1}, x_{a+j}, x_a, u, y_b, y_{b+l}, y_{b+l+1}\}$ induces P_7. Since neither x_{a+j+1} nor x_a is super-heavy, this contradicts G being P_7-f_1-heavy. □

Claim 40.2. $d_{H_1}(u, x_{h_1}) = 3$.

Proof. Suppose the statement is not true. First assume $d_{H_1}(u, x_{h_1}) \geq 4$. Then there is an induced path P_5 in H_1 connecting u with x_{h_1}, say $uxx'x''x_{h_1}$. Recall that $y_k \in N_{H_2}(u)$ is a vertex such that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$. It follows that $\{x_{h_1}, x'', x', x, u, y_k, y_{k+1}\}$ induces a P_7, a contradiction with G being P_7-f_1-heavy (by Claim 14).

Now assume $d_{H_1}(u, x_{h_1}) \leq 2$. First we note that whether or not u is adjacent to x_{h_1}, there is a vertex $x \in H_1$ such that uxx_{h_1} is a path P_3 (not necessarily the induced one). It is obviously true when $ux_{h_1} \notin E(G)$; if the opposite is true, it follows from Claim 18 and the fact that $d_{H_1}(u) \geq 2$.

Furthermore, the same is true for y_{h_2}: whether or not this vertex is adjacent to u, there is $y \in H_2$ such that uyy_{h_2} is a path P_3. If $uy_{h_2} \in E(G)$, it follows from Claim 38 for $y = y_1$. Otherwise it is a corollary from Claim 40.1.

Hence, $uyy_{h_2}vx_{h_1}xu$ is a cycle of length 6. Since neighbours of u in H_2 induce a clique, by Claim 38, they can be appended to this cycle one-by-one between u and y, creating at least $[6, d_{H_2}+4]$-cycles. Consider the longest cycle of those just obtained. By Claim 18, the neighbours of u from H_1 can be added to this cycle in a similar manner. Finally the vertices from the gaps between the neighbours of u in $C[y_1, y_{h_2}]$ can be appended to this cycle (again, one-by-one), due to Claim 40.1. In this way we obtain $[6, h_2 + d_{H_1}(u) + 2]$-cycles.

Note that $uyy_{h_2}C^+u$ is a cycle of length $n - h_2 + 2$. To this cycle we also can append all vertices from H_2, in the way described above, thus obtaining $[n - h_2 + 2, n]$-cycles. Since G is not pancyclic and it contains $[3, 5]$- (by Claim 38) and $[6, h_2 + d_{H_1}(u) + 2]$-cycles, it must be

$$h_2 + d_{H_1}(u) + 2 < n - h_2 + 2 = h_1 + 4 \leq h_2 + 4,$$

implying $d_{H_1}(u) < 2$. This contradicts the assumptions of this subcase.

Hence, it must be that $d_{H_1}(u, x_{h_1}) = 3$. □

Claim 40.3. *There are cycles of length* 6 *and* 7 *in* G*, where the cycle on seven vertices is* $uyy_{h_2}vx_{h_1}x'xu$ *for some* $y \in H_2$ *and* $x', x \in H_1$*.*

Proof. Obiously, since $d_{H_1}(u, x_{h_1}) = 3$, there are vertices $x, x' \in H_1$ such that $uxx'x_{h_1}$ is a path P_4. Now, if $uy_{h_2} \in E(G)$, then, by Claim 38, there is a path $uy_1y_{h_2}$ and we can set $y = y_1$. Otherwise let y be the last (i.e., with the highest index) neighbour of u in H_2. It is adjacent to y_{h_2} by Claim 40.1, and so $uyy_{h_2}vx_{h_1}x'xu$ is a cycle on seven vertices. Denote this cycle by C'.

Now suppose the first part of the statement is not true, that is that there are no cycles of length six in G. Then there are no one-chords in C', in particular $xx_{h_1} \notin E(G)$ and $uy_{h_2} \notin E(G)$.

Remark 41. $vx \notin E(G)$.

Otherwise $\{v; x_{h_1}, y_{h_2}, x\}$ induces a claw with neither x nor x_{h_1} being super-heavy, a contradiction with G being claw-f_1-heavy.

Remark 42. $N_G(u) \cap N_G(v) = \emptyset$.

If there exists a common neighbour of u and v, say w, then by the previous remark we have $w \neq x$ and so $uxx'x_{h_1}vwu$ is a cycle C_6, a contradiction.

Remark 43. $N_{H_2}(u) \subset N_G(y_{h_2})$.

Suppose there is a vertex $y'' \in H_2$ adjacent to u but not adjacent to y_{h_2}. Then it follows from the previous observations that $\{y'', u, x, x', x_{h_1}, v, y_{h_2}\}$ induces a P_7. Since neither x nor x_{h_1} is super-heavy, by Claim 14, this contradicts G being P_7-f_1-heavy.

Remark 44. $d_{H_2}(u) \leq 3$.

Indeed, if the opposite was true, then u and four of its neighbours from H_2 would induce a clique, by Claim 38. By the previous remark y_{h_2} is adjacent to every neighbour of u, and so we obtain a cycle C_6, a contradiction.

Remark 45. $N_{H_1}(u) \subset N_G(x')$.

Otherwise there is a vertex $x'' \in H_1$ adjacent to u and not adjacent to x'. Furthermore, $xx'' \in E(G)$, by Claim 18, and $x_{h_1}x'' \notin E(G)$, by Claim 40.2. Hence, $\{x'', u, y, y_{h_2}, v, x_{h_1}, x'\}$ induces a P_7. Since neither x' nor x_{h_1} is super-heavy and G is P_7-f_1-heavy, it follows that $\{v, y_{h_2}\}$ is a super-heavy pair of vertices. This contradicts Claim 16.

Remark 46. $d_{H_1}(u) \leq 3$.

If the opposite was true, then u and four of its neighbours from H_1 induce a clique, by Claim 18. By the previous remark x' is adjacent to every vertex of $N_{H_1}[u]$, and so we obtain a cycle C_6, a contradiction.

It follows from Remarks 44 and 46 that $d_G(u) \leq 7$. Since $n \geq 14$, this contradicts u being super-heavy. □

By Claims 39 and 40.3 there are $[3,7]$-cycles in G. Consider now the cycle $C' = uyy_{h_2}vx_{h_1}x'xu$. We can extend C' by appending to it, one-by-one, vertices from $N_{H_2}(u)$ (by Claim 38), then the remaining vertices from H_2 (by Claim 40.1) and finally all neighbours of u from H_1 (by Claim 18). In this way we obtain $[7, h_2 + d_{H_1}(u) + 4]$-cycles.

Note that $uyy_{h_2}C^+u$ is a cycle of length $h_1 + 4$. This cycle also can be extended with vertices from $N_{H_2}(u)$ and then the remaining vertices from H_2. This procedure gives $[h_1 + 4, n]$-cycles.

Since G is not pancyclic, it must be $h_2 + d_{H_1}(u) + 4 < h_1 + 4$. But by the choice of h_1 we have also $h_1 \leq h_2$. These inequalities imply that $d_{H_1}(u) < 0$, an obvious contradiction. □

Claim 47. *Let $A = \{y_{a+1}, \ldots, y_{a+p}\}$ be a set of consecutive non-neighbours of u in H_2 such that $uy_a \in E(G)$ and $y_a y_{a+p+1} \in E(G)$ (where we assume $y_{h_2+1} = v$). Let $P = v_1v_2 \ldots v_m$ be a path with $m \geq 3$, $v_1 = y_a$, $v_m = y_{a+p+1}$ and $v_i \in A$ for $i = 2, \ldots, m-1$. Finally, let C' be a cycle of length q in G such that $u, v \in V(C')$, $C'[v,u] = \{v, x_{h_1}, x_{h_1-1}, \ldots, x_1, u\}$, $A \cap V(C') = \emptyset$ and $y_a y_{a+p+1}$ is an edge of C'.*

Then one can obtain $[q+1, q+m-2]$-cycles by appending some of the vertices from the path P to the cycle C' and omitting at most one vertex from $V(C')$.

Proof. If y_a is super-heavy, it is adjacent to every vertex from A, by Claim 37, and so the statement follows. Now assume that y_a is not super-heavy.

First we show that there is a vertex in $V(C')$ the omitting of which along C' results in a cycle of length $q-1$. Clearly, if $ux_2 \in E(G)$, then x_1 is such a vertex (namely, the cycle of length $q-1$ is $x_2uC'^+x_2$). If $ux_2 \notin E(G)$, then $x_1x_3 \in E(G)$ (it follows from Claim 18 if $ux_3 \in E(G)$, or from Claim 40 if $ux_3 \notin E(G)$) and the vertex that can be omitted is x_2.

The proof of the claim goes by induction with respect to m. For $m = 3$ we need to point out only a cycle of length $q+1$. Obviously, $uC'^+y_av_2y_{a+p+1}C'^+u$ is such a cycle. For the case when $m = 4$ we want to find cycles of lengths $q+1$ and $q+2$. The previous is $uC'^+y_av_2v_3y_{a+p+1}C'^+\hat{x}C'^+u$ (where $\hat{x}$ stands for omitting either x_1 or x_2) and the latter is $uC'^+y_av_2v_3y_{a+p+1}C'^+u$.

Now assume the statement is true for some fixed $m \geq 4$ and consider $P = v_1 \cdots v_{m+1}$. In order to avoid $\{x_{i+1}, x_i, u, y_a, v_2, v_3, v_4\}$ inducing a P_7 with neither x_i nor y_a being super-heavy, there must be one of the edges y_av_3, y_av_4 or v_2v_4. If $y_av_3 \in E(G)$ (or $v_2v_4 \in E(G)$), $P' = y_av_3P^+y_{a+p+1}$ (or $P' = y_av_2v_4P^+y_{a+p+1}$)

is a path on m vertices that allows us to obtain $[q+1, q+m-2]$-cycles. In order to obtain a cycle of length $q+m-1$, we simply append all vertices from P to C' (i.e., this cycle is $uC'^{+}y_a v_2 \cdots v_m y_{a+p+1} C'^{+}u$).

If there is an edge $y_a v_4$, it creates a path $P' = y_a v_4 P^{+} y_{a+p+1}$ on $m-1$ vertices, and so there are $[q+1, q+m-3]$-cycles. To obtain a cycle of length $q+m-1$, simply append all vertices from P to C'. Finally, omitting x_1 or x_2 in this last cycle creates a $(q+m-2)$-cycle. □

So far we know the structure of u neighbourhoods in H_1 and H_2 and the parts of the cycle C that lie between u's neighbours. To describe the remaining part of C, let y_j denote the last (i.e., the one with the highest index) neighbour of u in H_2.

Claim 48. *$y_j \neq y_{h_2}$ and $y_j y_{h_2} \notin E(G)$.*

Proof. Suppose the statement is not true. Then, by Claim 40 and the fact that $d_{H_1}(u) \geq 2$, there is a cycle $uy_{h_2}vx_{h_1}xu$ (if $y_j = y_{h_2}$) of length five or a cycle $uy_j y_{h_2} v x_{h_1} xu$ (if $y_j y_{h_2} \in E(G)$) of length six. Since neighbours of u in H_1 induce a clique, by Claim 18, they can be appended to this cycle, one-by-one. Then the same can be done with the remaining vertices from H_1, by Claim 40, and subsequently with neighbours of u from H_2, as they also induce a clique, by Claim 38.

In this manner we obtain at least $[6, h_1 + d_{H_2}(u) + 2]$-cycles, the longest of which contains all vertices from G but the non-neighbours of u in H_2. These remaining vertices can be divided into disjoint maximal sets of connsecutive non-neighbours of u along C. Applying Claim 47 to C' with the first of these sets as A (where the path P from Claim 47 consists of all vertices from A), gives a cycle C'' with $V(C'') = V(C') \cup A$, and every cycle shorter than C''. Applying Claim 47 to C'' and the remaining sets of non-neighbours of u, one-by-one, we finally arrive at the Hamilton cycle C. Since this procedure guarantees creating cycles of all lengths from $h_1 + d_{H_2}(u) + 2$ up to n, there are $[6, n]$-cycles in G. Since there are also $[3, 5]$-cycles, by Claim 39, G is pancyclic, a contradiction. □

Note that if y_j was super-heavy, it would be adjacent to y_{h_2} by Claim 37. Hence it follows from Claim 48 that y_j is not super-heavy.

Claim 49. *Let y_m be the last neighbour (i.e., with the highest index) of y_j in $C[y_j, y_{h_2}]$. Then $y_m y \in E(G)$ for $y \in \{y_{m+1}, \ldots, y_{h_2}\}$.*

Proof. Note that $m \leq h_2 - 1$ by Claim 48. Since the statement is obvious for $m = h_2 - 1$, assume $m \leq h_2 - 2$. Suppose the claim is not true. Then there is some vertex $y_b \in \{y_{m+1}, \ldots, y_{h_2-1}\}$ such that $y_b y_m \in E(G)$ and $y_m y_{b+1} \notin E(G)$. But then $\{x_{i+1}, x_i, u, y_j, y_m, y_b, y_{b+1}\}$ induces a P_7 with neither x_i nor y_j being super-heavy. A contradiction. □

Now it follows from Claims 40, 48 and 49 and the fact that $d_{H_1}(u) \geq 2$ that there is a cycle $C' = uy_jy_my_{h_2}vx_{h_1}xu$, where x is the neighbour of u in H_1 with the highest index if $ux_{h_1} \notin E(G)$ and $x = x_1$ otherwise. To this cycle C_7 we can append neighbours of u, one-by-one, by Claim 18 and Claim 38 and then non-neighbours of u from H_1, by Claim 40. Vertices from the set $\{y_{m+1}, \ldots, y_{h_2-1}\}$ can then be added to the cycle due to Claim 49. Finally, Claim 47 allows us to extend the longest of just created cycles using the non-neighbours of u in H_2 (just like in the proof of Claim 48) up to the Hamiltonian cycle C. Hence, there are $[7, n]$-cycles in G. Recall that there are also $[3, 5]$-cycles, by Claim 39.

Suppose there are no cycles of length six in G. Then there are no one-chords in C', in particular $vx, vy_m, ux_{h_1} \notin E(G)$. Recall that by the choice of j and m we also have uy_m, $uy_{h_2} \notin E(G)$.

Remark 50. $uv \notin E(G)$.

Assume the contrary. Note that since u is super-heavy and $n \geq 14$, we have $d_G(u) \geq 8$. It follows that u has at least three neighbours either in H_1 or in H_2. If there are three vertices in H_1 adjacent to u, say x, x' and x'', then, by Claim 18, $uvx_{h_1}xx'x''u$ is a cycle C_6, a contradiction. Hence, u has at least three neighbours in H_2. But then $uy_1y_jy_my_{h_2}vu$ is a cycle of length six, by Claim 38.

Remark 51. $vy_j \notin E(G)$.

Otherwise, since $x \neq x_1$ under assumptions of this subcase and $xx_1 \in E(G)$ by Claim 18, $vy_jux_1xx_{h_1}v$ is a cycle C_6.

Remark 52. $N_G(u) \cap N_G(v) = \emptyset$.

If there exists a common neighbour of u and v, say w, then from the previous remark it follows that $w \neq y_j$, and from the choice of j we have $w \neq y_{h_2}$. Obviously we also have $w \neq y_m$, since y_m is adjacent neither to u nor to v. But then $uy_jy_my_{h_2}vwu$ is a cycle C_6, a contradiction.

Remark 53. $N_{H_1}(u) \subset N_G(x_{h_1})$.

Otherwise there is a vertex $x' \in H_1$ adjacent to u and not adjacent to x_{h_1}. Furthermore, $xx' \in E(G)$, by Claim 18, and $vx' \notin E(G)$ by the previous remark. Hence, $\{x', u, x; y_j, y_m; x_{h_1}, v\}$ induces a deer. Since neither x' nor x_{h_1} is super-heavy, this contradicts G being D-f_1-heavy.

Remark 54. $d_{H_1}(u) \leq 3$.

If the opposite was true, then u and four of its neighbours from H_1 induce a clique, by Claim 18. By the previous remark x_{h_1} is adjacent to every vertex from of $N_{H_1}[u]$, and so we obtain a cycle C_6, a contradiction.

Remark 55. $N_{H_2}(u) \subset N_G(y_m)$.

Suppose there is a vertex $y \in H_2$ adjacent to u but not adjacent to y_m. Note that $yy_{h_2} \notin E(G)$, since otherwise $yuxx_{h_1}vy_{h_2}y$ would be a cycle C_6. Then it follows from the previous observations that $\{y, u, x, x_{h_1}, v, y_{h_2}, y_m\}$ induces a P_7. Since neither x nor x_{h_1} is super-heavy, by Claim 14, and G is P_7-f_1-heavy, it follows that $\{v, y_{h_2}\}$ is a super-heavy pair of vertices. This contradicts Claim 16.

Remark 56. $d_{H_2}(u) \leq 3$.

Indeed, if the opposite was true, then u and four of its neighbours from H_2 would induce a clique, by Claim 38. By the previous remark y_m is adjacent to every neighbour of u, and so we obtain a cycle C_6, a contradiction.

It follows from Remarks 50, 54 and 56 that $d_G(u) \leq 6$. Since $n \geq 14$, this contradicts u being super-heavy. Hence, there is a cycle C_6 in G and so G is pancyclic. This contradiction completes the proof of this subcase.

***Subcase* 2.3.** $h_1 \geq 2$, $d_{H_1}(u) = h_1$.

Claim 57. *None of the neighbours of u in H_2 is super-heavy.*

Proof. Assume the contrary. Then u is adjacent to some super-heavy vertex $y_j \in H_2$. Note that $j \geq 3$, by Claim 16, $y_{j-1}y_{j+1} \notin E(G)$, by Lemma 6, and $y_1y_j \in E(G)$, by Claim 15. Furthermore, it must be $y_1y_{j+1} \notin E(G)$, since otherwise $C' = y_1y_{j+1}C^+uy_jC^-y_1$ would be a Hamilton cycle in G with $d_{C'}(u, y_j) = 1$ and $d_G(u) + d_G(y_j) \geq n + 1$, and thus G would be pancyclic by Lemma 9.

Claim 16 implies that neither y_{j-1} nor y_{j+1} is super-heavy. Since G is claw-f_1-heavy, it follows that $\{y_j; u, y_{j-1}, y_{j+1}\}$ cannot induce a claw. Hence, u is adjacent to y_{j-1} or y_{j+1}.

Suppose $uy_{j+1} \in E(G)$. Since $y_1y_{j+1} \notin E(G)$, Claim 15 implies that $y_{j+1} \notin H_2$ and so $j = h_2$ and $y_{j+1} = v$. Consider $G' = G - H_1$. G' is obviously Hamiltonian with a Hamilton cycle $C' = y_jvuC^+y_j$. Since

$$d_{G'}(u) + d_{G'}(y_j) \geq (n+1)/2 - h_1 + (n+1)/2 \geq |G'| + 1,$$

G' is pancyclic by Lemma 11. Appending vertices from H_1 to C', one-by-one, creates cycles of all lengths greater than $|G'|$ and so G is pancyclic, a contradiction.

Hence, $uy_{j+1} \notin E(G)$ and $uy_{j-1} \in E(G)$.

Suppose now that $uv \notin E(G)$. Consider $G' = G - \{x_1, \ldots, x_{h_1-1}\}$, a Hamiltonian graph with a Hamilton cycle $C' = y_1y_jC^+x_{h_1}uy_{j-1}C^-y_1$. First we show that G' is pancyclic. Indeed, if $uy_2 \notin E(G)$, then $y_2 \in N_G(y_1) \setminus N_G(u)$ and Claim 15 together with the fact that $uv \notin E(G)$ imply $d_G(y_1) \geq (n+1)/2 - h_1 + 1$. Hence, $d_{G'}(y_1) + d_{G'}(y_j) \geq |G'| + 1$, and pancyclicity of G' follows from Lemma 9.

If $uy_2 \in E(G)$, then a similar argument leads to the inequality $d_{G'}(y_1)+d_{G'}(y_j) \geq |G'|$. This inequality together with the cycle $uy_2C'^{+}u$ of length $|G'|-1$ implies that G' is pancyclic by Lemma 10. It follows that there are $[3,|G'|]$-cycles in G. Since the vertices from H_1 can be appended to the cycle C' one-by-one, thus creating $[|G'|,n]$-cycles, G is pancyclic, a contradiction.

Now assume $uv \in E(G)$ and consider $G' = G - H_1$ with a Hamilton cycle $C' = y_1y_jC^+vuy_{j-1}C^-y_1$. Again, depending on whether or not u is adjacent to y_2, we have $d_G(y_1) \geq (n+1)/2 - h_1 - 1$ (if it is) or $d_G(y_1) \geq (n+1)/2 - h_1$. In the previous case $uy_2C'^{+}u$ is a $(|G'|-1)$-cycle in G' and the inequality $d_{G'}(y_1)+d_{G'}(y_j) \geq |G'|$ holds, implying that G' is pancyclic by Lemma 10. In the latter case we have $d_{G'}(y_1)+d_{G'}(y_j) \geq |G'|+1$ and so G' is pancyclic by Lemma 9. Again, pancyclicity of G' implies pancyclicity of G, since the vertices from H_1 can be appendend to C' one-by-one. Thus G is pancyclic, a contradiction. $\square$

Claim 58. *$N_{H_2}[u]$ induces a clique in G.*

Proof. For the proof replace h_1 with h_2, y_1 with x_1 and x_a, $x_b \in H_1$ with y_a, $y_b \in H_2$ (which are not super-heavy due to Claim 57) in the proof of Claim 18. $\square$

Claim 59. *There are $[3,5]$-cycles in G.*

Proof. Since u is super-heavy and $n \geq 14$, we have $d_G(u) \geq 8$. Obviously, u has at least four neighbours in H_1 or H_2. Both $N_{H_1}[u]$ and $N_{H_2}[u]$ are complete subgraphs of G, by Claims 18 and 58, respectively, and so the claim follows. $\square$

Claim 60. *Let $A = \{y_{a+1},\ldots,y_{a+p}\}$ be a set of consecutive non-neighbours of u in H_2 such that $uy_a \in E(G)$ and $y_ay_{a+p+1} \in E(G)$ (where we assume $y_{h_2+1} = v$). Let $C' = uC^+y_ay_{a+p+1}C^+u$ be a cycle of length $q = n-p$. Finally, let $P = v_1v_2\cdots v_m$ be a path with $m \geq 3$, $v_1 = y_a$, $v_m = y_{a+p+1}$ and $v_i \in A$ for $i = 2,\ldots,m-1$.*

Then one can obtain $[q+1,q+m-2]$-cycles by appending some of the vertices from the path P to the cycle C' and omitting at most two neighbours of u belonging to $V(C')$.

Proof. The proof is by induction on m. For the case when $m = 3$ we only need to point out a cycle of length $q+1$. It is easy to see that $y_av_2y_{a+p+1}C'^{+}y_a$ is such a cycle.

Assume $m = 4$. By the assumptions of this subcase u is adjacent to x_2 and so $y_av_2v_3y_{a+p+1}C'^{+}x_2uC'^{+}y_a$ is a cycle of length $q+1$. Append x_2 to this cycle in order to obtain a cycle on $q+2$ vertices.

Now let $m = 5$. Obviously, the cycle $C'' = y_av_2v_3v_4y_{a+p+1}C'^{+}y_a$ has length $q+3$. Using the edge ux_2 to omit vertex x_1 we obtain a cycle of length $q+2$. If $h_1 \geq 3$, then the chord ux_3 in the cycle C'' creates a cycle of length $q+1$. Otherwise $h_1 = 2$. Now, if u is adjacent to v, then the edge uv is a two-chord in C'', and so there is a $(q+1)$-cycle in G. If $uv \notin E(G)$ and $uy_2 \notin E(G)$, it follows from Claim 15 that $d_G(y_1) \geq (n+1)/2 - 1$ and so $d_G(u) + d_G(y_1) \geq n$, a contradiction with Claim 19. Finally, if $uv \notin E(G)$ and $uy_2 \in E(G)$, then $uy_2C'^{+}y_av_2v_3v_4y_{a+p+1}C'^{+}x_2u$ is a cycle of length $q+1$.

Assume the claim is true for some $m \geq 5$ and consider a path P of length $m+1$ that satisfies the assumptions. If $\{x_1, u, y_a, v_2, v_3, v_4, v_5\}$ induces a P_7, this contradicts G being P_7-f_1-heavy, since neither x_1 nor y_a is super-heavy (by Claims 14 and 57). Hence, there is an edge in $G[\{y_a, v_2, v_3, v_4, v_5\}]$ that does not belong to the path P. This edge creates a shorter path, of length at least $m-2$, that satisfies the assumptions of the claim. It follows that we can obtain $[q+1, q+m-4]$-cycles in a desired manner. Obviously, $C'' = y_aP^{+}y_{a+p+1}C'^{+}y_a$ is a cycle of length $q+m-1$. To obtain cycles of lengths $q+m-3$ and $q+m-2$ use chords of C'' as described in the case of $m=5$. □

From now on let y_j denote the neighbour of u in H_2 with the highest index.

Claim 61. *$j \leq h_2 - 3$ and y_j is adjacent neither to y_{h_2} nor y_{h_2-1}.*

Proof. Suppose the first part of the claim is not true. Then $j \in \{h_2-2, h_2-1, h_2\}$ and one of the cycles $uy_{h_2-2}y_{h_2-1}y_{h_2}vx_{h_1}u$, $uy_{h_2-1}y_{h_2}vx_{h_1}u$, $uy_{h_2}vx_{h_1}u$ exists. Let C' denote that cycle. Neighbours of u both in H_1 and in H_2 induce cliques (by Claims 18 and 58, respectively), and so they can be appended to C', one-by-one. Let C'' be the cycle C' with all neighbours of u appended to it. The remaining vertices are non-neighbours of u in H_2. Let $\{y^1, \ldots, y^{d_{H_2}(u)}\}$ be the neighbours of u sorted by their indices in ascending order. Applying Claim 60 to the cycle C'' and the set $C[y^1, y^2]$ we obtain cycles longer than C'' up to the cycle $C''' = y^1C^{+}y^2C''^{+}y^1$. Now we can apply Claim 60 to the cycle C''' and the set $C[y^2, y^3]$. Repeating this procedure up to the set $C[y^{d_{H_2}(u)-1}, y^{d_{H_2}(u)}]$, we finally arrive at the cycle C. It follows that there are $[|C'|, n]$-cycles in G. Since $|C'| \leq 6$, together with Claim 59 this implies that G is pancyclic, a contradiction.

If y_j was adjacent to either y_{h_2-1} or y_{h_2}, the similar argument as presented above applied to the cycle $uy_jy_{h_2-1}y_{h_2}vx_{h_1}u$ or $uy_jy_{h_2}vx_{h_1}u$ leads to the pancyclicity of G, contradicting our assumptions. Note that Claim 60 works also for the sets $A = \{y_{j+1}, \ldots, y_{h_2-2}\}$ and $A = \{y_{j+1}, \ldots, y_{h_2-1}\}$. □

Consider now the neighbour of y_j in H_2 with the highest index. Let y_m denote this vertex. It follows from Claim 61 that $m \leq h_2 - 2$ and so it makes sense to consider also the neighbour of y_m with the highest index, say $y_{m'} \in H_2$. Note that the choice of j, m and m' implies that $\{x_{h_1}, u, y_j, y_m, y_{m'}\}$ induces a P_5.

Claim 62. *$y_{m'}y \in E(G)$ for every $y \in C[y_{m'+1}, y_{h_2}]$.*

Proof. Assume the contrary and let $G' = G[C[y_{m'}, y_{h_2}]]$. It follows that there exist vertices y', $y'' \in C[y_{m'}, y_{h_2}]$ such that $d_{G'}(y_{m'}, y) \geq 2$ and $\{y_{m'}, y', y''\}$ induces

P_3. By the choice of y', y'', j, m and m' it follows that $\{x_{h_1}, u, y_j, y_m, y_{m'}, y', y''\}$ induces P_7. Since neither x_{h_1} nor y_j is super-heavy, by Claims 14 and 57, this contradicts G being P_7-f_1-heavy. □

Claim 63. *Assume the cycle $C' = y_m y_{m'} y_{h_2} C^+ y_m$ has length q. Let $P = v_1 \cdots v_l$ be a path with $l \geq 3$, $v_1 = y_m$, $v_l = y_{m'}$ and $v_i \in C[y_m, y_{m'}]$ for $i = 2, \ldots, l-1$.*

Then one can obtain $[q+1, q+l-2]$-cycles by appending some of the vertices from P to C' and omitting at most x_1.

Proof. Since the claim is obviously true for $l = 3$, consider $l = 4$. Then $y_m v_2 v_3 y_{m'} C^+ y_m$ is a cycle of length $q+2$ and $y_m v_2 v_3 y_{m'} C^+ x_2 u C^+ y_m$ is a cycle of length $q+1$.

Assume the statement is true for some fixed $l_0 \geq 4$ and for every $l \leq l_0$. Consider now a path $P = v_1 \cdots v_{l_0+1}$ satisfying the assumptions of the claim. Since G is P_7-f_1-heavy and neither x_1 nor y_j is super-heavy (by Claims 14 and 57), $\{x_1, u, y_j, y_m, v_2, v_3, v_4\}$ cannot induce a P_7. Note that by the choice of j and m both u and y_j have no neighbours in the set $C[y_{m+1}, y_{m'}]$. It follows that there exists an edge in $G[\{y_m, v_2, v_3, v_4\}]$ that does not belong to the path P. This edge, say $v'v''$, creates a path $P' = y_m P^+ v' v'' P^+ y_{m'}$ of length at most l_0 and at least $l_0 - 1$. The validity of the claim for $l \leq l_0$ implies that there are $[q+1, q+l_0-3]$-cycles in G, created in the manner desired. Obviously, the cycle $y_m P^+ y_{m'} C^+ y_m$ has length $q + l_0 - 1$ and the cycle $y_m P^+ y_{m'} C^+ x_2 u C^+ y_m$ has length $q + l_0 - 2$. By mathematical induction the claim is true. □

Claim 64. *There are $[7, n]$-cycles in G.*

Proof. Claim 62 implies that $y_{m'} y_{h_2} \in E(G)$. Hence, $C' = u y_j y_m y_{m'} y_{h_2} v x_{h_1} u$ is a cycle C_7. Let $\{y^1, \ldots, y^{d_{H_2}(u)}\}$ denote the neighbours of u in H_2 sorted by their indices in ascending order.

Just as in the proof of Claim 61 we can extend the cycle C' by appending to it all neighbours of u (since $N_{H_1}[u]$ and $N_{H_2}[u]$ induce cliques in G) and then all non-neighbours of u that belong to one of the sets $C[y^l, y^{l+1}]$ for $l \in \{1, \ldots, d_{H_2}(u) - 1\}$ or to the set $C[y_j, y_m]$ (by Claim 60), as well as those belonging to the set $C[y_{m+1}, y_{m'-1}]$ (by Claim 62). To the longest of just created cycles, that is the cycle $y_{h_2} C^+ y_{m'} y_{h_2}$, we can then add all vertices from the set $C[y_{m'+1}, y_{h_2}]$, also one-by-one, by Claim 63, thus arriving finally at the cycle C. □

It follows from Claims 59 and 64 that G is missing only cycles of length six. Suppose this is indeed true and recall the cycle $C' = u y_j y_m y_{m'} y_{h_2} v x_{h_1} u$ of length seven. It follows that uv, $y_{m'} v \notin E(G)$.

Remark 65. C' is an induced cycle.

Proof. To prove this fact we need to show that $v y_m$, $v y_j \notin E(G)$ (by the choice of j, m, m' and the fact that v is adjacent neither to u nor to $y_{m'}$).

If $vy_m \in E(G)$, then $vy_my_jux_1x_{h_1}v$ is a cycle C_6 (since $d_{H_1}(u) \geq 2$ and $N_{H_1}[u]$ induces a clique).

Since $n \geq 14$, $uv \notin E(G)$ and u is super-heavy, it follows that u has at least four neighbours in H_1 or H_2. If $vy_j \in E(G)$, these neighbours can be used to obtain a cycle C_6 from the cycle $uy_jvx_{h_1}x_1u$. □

Remark 66. $N_{H_1}(u) \subset N_{H_1}(v)$.

Proof. Indeed, if some vertex $x \in N_{H_1}(u)$ is not adjacent to v, then it follows from the previous remark that $\{x, u, y_j, y_m, y_{m'}, y_{h_2}, v\}$ induces a P_7. Since neither x nor y_j is super-heavy, this contradicts G being P_7-f_1-heavy. □

Remark 67. $d_{H_1}(u) \leq 3$.

Proof. Assume the contrary. Since $N_{H_1}(u) = H_1$ and the neighbours of u in H_1 induce a clique, by Claim 18, and they are adjacent to v by the previous remark, it follows that four of them together with u and v form a cycle C_6. A contradiction. □

Since $n \geq 14$, u is super-heavy and $uv \notin E(G)$, the last remark implies that $d_{H_2}(u) \geq 5$. But $N_{H_2}[u]$ induces a clique, by Claim 58, and so there is a cycle C_6 in G. This final contradiction completes the proof. ■

4. Propositions of Further Research

Similarly to Theorems 3 and 4 we have the following results.

Theorem 68 (Faudree *et al.*, [7]). *Every 2-connected, $\{K_{1,3}, P_6\}$-free graph on $n \geq 10$ vertices is pancyclic.*

Theorem 69 (Chen *et al.*, [6]). *Every 2-connected, $\{K_{1,3}, P_6\}$-f-heavy graph is Hamiltonian.*

It seems natural to propose the following conjecture.

Conjecture 70. *Every 2-connected, $\{K_{1,3}, P_6\}$-f_1-heavy graph on $n \geq 10$ vertices is pancyclic.*

Note that in the proof of Theorem 5 we used the assumption of G being D-f_1-heavy only twice (in Claim 37 and in Remark 53). It seems that it would suffice to slightly modify our proof in order to prove the above Conjecture.

In the light of results for pairs and triples of forbidden and heavy subgraphs we propose another, more general, conjecture.

Conjecture 71. *Let $\mathcal{H}$ be a family of graphs. If every 2-connected, $\mathcal{H}$-free graph on $n \geq n_0$ vertices is pancyclic and every 2-connected, $\mathcal{H}$-f-heavy graph is Hamiltonian, then every 2-connected, $\mathcal{H}$-f_1-heavy graph on $n \geq n_0$ vertices is pancyclic.*

As the proofs of the results obtained so far made extensive use of the specific forbidden (or heavy) graphs, the proof in the general case seems to be much more difficult.

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