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ON THE LAPLACIAN COEFFICIENTS OF TRICYCLIC GRAPHS WITH PRESCRIBED MATCHING NUMBER

Jing Luo, Zhongxun Zhu*

Department of Mathematics and Statistics
South Central University for Nationalities
Wuhan 430074, P.R. China

and

Runze Wan

College of Computer, Hubei University of Education
Wuhan 430205, P.R. China

e-mail: zzxun73@mail.scuec.edu.cn

Abstract

Let $\phi(L(G)) = \det(xI - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)x^{n-k}$ be the Laplacian characteristic polynomial of G. In this paper, we characterize the minimal graphs with the minimum Laplacian coefficients in $\mathscr{G}_{n,n+2}(i)$ (the set of all tricyclic graphs with fixed order n and matching number i). Furthermore, the graphs with the minimal Laplacian-like energy, which is the sum of square roots of all roots on $\phi(L(G))$, is also determined in $\mathscr{G}_{n,n+2}(i)$.

Keywords: Laplacian characteristic polynomial, Laplacian-like energy, tricyclic graph.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges. Denote by $\mathscr{G}_{n,m}$ the set of all simple connected graphs of order n and size m. If $m = n - 1 + c$, then G is called a *c-cyclic graph*. If $c = 0, 1, 2$ and 3, then G is a tree, unicyclic graph, bicyclic graph and tricyclic graph, respectively. Let P_n, C_n and

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* Corresponding author.

S_n be the path, the cycle and the star on n vertices, respectively. Furthermore, let $\mathscr{G}_{n,m}(i)$ be the set of all simple connected graphs with order n, size m and matching number i.

Let $L(G) = D(G) - A(G)$ be the *Laplacian matrix* of G, where $A(G)$ is its $(0,1)$-adjacency matrix and $D(G)$ its degree diagonal matrix. While the Laplacian polynomial of G is the characteristic polynomial of $L(G)$, $\phi(L(G)) = \det(xI - L(G))$. Let $c_k(G)$ $(0 \le k \le n)$ be the absolute values of the coefficients of $\phi(L(G))$, so that

$$\phi(L(G)) = \det(xI - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)x^{n-k}.$$

For $G, H \in \mathscr{G}_{n,m}$, we write $G \preceq H$ if the Laplacian coefficients $c_k(G) \le c_k(H)$ for $k = 0, 1, 2, \ldots, n$, and we write $G \prec H$ if $G \preceq H$ and $c_{k_0}(G) < c_{k_0}(H)$ for some $0 \le k_0 \le n$.

Recently, the study of the structure and properties on the Laplacian coefficients have attracted much attention. As for n-vertex trees, Mohar [6] proved that P_n has the maximal Laplacian coefficients and S_n has the minimal Laplacian coefficients, respectively. As for n-vertex unicyclic graphs, Stevanović and Ilić [8] showed that C_n has the maximal Laplacian coefficients and S'_n has the minimal Laplacian coefficients, where S'_n is the graph obtained from S_n by joining two of its pendant vertices with an edge. As for n-vertex bicyclic graphs, He and Shan [3] obtained that the Laplacian coefficients are the smallest when the graph is obtained from C_4 by adding one edge connecting two non-adjacent vertices and adding $n-4$ pendent vertices attached to the vertex of degree 3. As for n-vertex tricyclic graphs, Pai *et al.* [7] determined that the coefficients are the smallest when the graph is obtained from the complete graph K_4 by adding $n-4$ pendent vertices attached to the vertex of degree 3. Furthermore, in $\mathscr{G}_{n,m}(i)$, Ilić [4] characterized the minimal trees with the minimum Laplacian coefficients for $m = n-1$; Tan [9, 10] obtained the graphs with the minimum Laplacian coefficients for $m = n, n+1$, respectively. Motivated by all these works, in the present paper we are devoted to find the graphs with the minimum Laplacian coefficients for $m = n+2$.

In order to state our results, we introduce some notation and terminology. For other undefined notation we refer to Bollobás [1]. Let $N_G(v) = \{u|uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$. Denote by $d_G(v) = |N_G(v)|$ the degree of the vertex v of G. If $E_0 \subset E(G)$, we denote by $G - E_0$ the subgraph of G obtained by deleting the edges in E_0. If E_1 is the subset of the edge set of the complement of G, $G + E_1$ denotes the graph obtained from G by adding the edges in E_1. Similarly, if $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If $E = \{xy\}$ and $W = \{v\}$, we write $G - xy$ and $G - v$ instead of $G - \{xy\}$ and $G - \{v\}$, respectively.

2. Preliminaries

In this section, we introduce some graphic transformations and lemmas, which will be used to prove our main results.

For any graph G and $v \in V(G)$, let $L_v(G)$ denote the principal submatrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v.

Lemma 2.1 [2]**.** *Let $G = G_1u : vG_2$ be the graph obtained from two disjoint graphs G_1 and G_2 by joining a vertex u of the graph G_1 to a vertex v of the graph G_2 by an edge. Then*

$$\phi(L(G)) = \phi(L(G_1))\phi(L(G_2)) - \phi(L(G_1))\phi(L_v(G_2)) - \phi(L_u(G_1))\phi(L(G_2)).$$

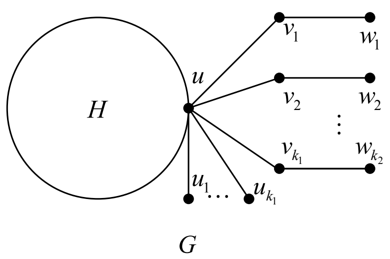

Figure 1. The graph in Lemma 2.2.

Lemma 2.2. *Let H be a graph and u a vertex of it. Let G be a graph of order n, which is obtained from H by attaching k_1 pendent edges and k_2 pendent paths of length 2 at u (as shown in Figure 1). Then*

$$\begin{aligned}\phi(L(G)) \;=\; & \left(x^2-3x+1\right)^{k_2}\left[(x-1)^{k_1}\phi(L(H)) - k_1x(x-1)^{k_1-1}\phi(L_u(H))\right] \\ & - k_2\left(x^2-3x+1\right)^{k_2-1}\left(x^2-2x\right)(x-1)^{k_1}\phi(L_u(H)).\end{aligned}$$

Proof. We label the rows and columns of $L(G)$ as the vertices $v_1, w_1, \ldots, v_{k_2}, w_{k_2}$, $u_1, \ldots, v_{k_1}, u, V(H-u)$. Let $G'_i = G - \bigcup_{k=1}^{i}\{v_k, w_k\}$; by Lemma 2.1, we have

$$\begin{aligned}\phi(L(G'_1)) \;=\; & \phi(L(G'_2))\phi(L(K_2)) - \phi(L(G'_2))\phi(L_{v_2}(K_2)) - \phi(L_u(G'_2))\phi(L(K_2)) \\ =\; & \phi(L(G'_2))(x^2-3x+1) - \phi(L_u(G'_2))(x^2-2x), \\ & \vdots \\ \phi(L(G'_{k_2-1})) \;=\; & \phi(L(G'_{k_2}))\phi(L(K_2)) - C(L(G'_{k_2}))\phi(L_{v_{k_2}}(K_2)) \\ & -\phi(L_u(G'_{k_2}))\phi(L(K_2)) \\ =\; & \phi(L(G'_{k_2}))(x^2-3x+1) - \phi(L_u(G'_{k_2}))(x^2-2x),\end{aligned}$$

$$\begin{aligned}
\phi(L(G)) &= \phi(L(G'_1))(x^2-3x+1)-\phi(L_u(G'_1))(x^2-2x)\\
&= (x^2-3x+1)[(x^2-3x+1)\phi(L(G'_2))-\phi(L_u(G'_2))(x^2-2x)]\\
&\quad -\phi(L_u(G'_1))(x^2-2x)\\
&= (x^2-3x+1)^2\phi(L(G'_2))-(x^2-3x+1)(x^2-2x)\phi(L_u(G'_2))\\
&\quad -\phi(L_u(G'_1))(x^2-2x)\\
&= \cdots\\
&= (x^2-3x+1)^{k_2}\phi(L(G'_{k_2}))-(x^2-3x+1)^{k_2-1}(x^2-2x)\phi(L_u(G'_{k_2}))\\
&\quad -\cdots-(x^2-3x+1)(x^2-2x)\phi(L_u(G'_2))-\phi(L_u(G'_1))(x^2-2x).
\end{aligned}$$

Note that

$$\begin{aligned}
\phi(L_u(G'_1)) &= \phi(L_u(G'_{k_2}))[(x-2)(x-1)-1]^{k_2-1}\\
&= \phi(L_u(G'_{k_2}))(x^2-3x+1)^{k_2-1},\\
\phi(L_u(G'_2)) &= \phi(L_u(G'_{k_2}))(x^2-3x+1)^{k_2-2},\\
&\vdots\\
\phi(L_u(G'_{k_2-1})) &= \phi(L_u(G'_{k_2}))(x^2-3x+1),
\end{aligned}$$

so we have

$$\begin{aligned}
\phi(L(G)) &= (x^2-3x+1)^{k_2}\phi(L(G'_{k_2}))\\
&\quad - k_2(x^2-3x+1)^{k_2-1}(x^2-2x)\phi(L_u(G'_{k_2})).
\end{aligned}$$

Furthermore, we have $|V(H)|=n-k_1-2k_2$ and

$$\begin{aligned}
\phi(L_u(G'_{k_2})) &= (x-1)^{k_1}\phi(L_u(H)),\\
\phi(L(G'_{k_2})) &= (x-1)^{k_1+2k_2}\phi(L(H))-(k_1+2k_2)x(x-1)^{k_1+2k_2-1}\phi(L_u(H)),\\
\phi(L_u(G'_{k_2})) &= (x-1)^{k_1}\phi(L_u(H)).
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(L(G)) &= (x^2-3x+1)^{k_2}[(x-1)^{k_1}\phi(L(H))-k_1x(x-1)^{k_1-1}\phi(L_u(H))]\\
&\quad - k_2(x^2-3x+1)^{k_2-1}(x^2-2x)(x-1)^{k_1}\phi(L_u(H)).
\end{aligned}$$

■

Definition 1 [9]**.** Let G be a simple connected graph with n vertices, and uv be a non-pendent edge which is not contained in any cycle of length 3. Let G_{uv} be the graph obtained from G in the following way: (1) Delete the edge uv; (2) Identify u and v, and denote the new vertex by w; (3) Add a pendent edge ww' to w. We say that G_{uv} is a I-edge-growing transform of G at uv.

Lemma 2.3 [10]**.** *Let G and G_{uv} be the two graphs defined in Definition* 1. *Let E_{uv}^u denote the set of edges incident to u except the edge uv. Then $|M(G_{uv})| = |M(G)|$ when $M(G) \cap E_{uv}^u = \emptyset$ or $M(G) \cap E_{uv}^v = \emptyset$.*

Lemma 2.4 [9]**.** *Let G and G_{uv} be the two graphs presented in Definition* 1. *Then $G_{uv} \prec G$, i.e., $c_k(G_{uv}) \leq c_k(G)$, $k = 0, 1, \ldots, n$, with equality if and only if either $k \in \{0, 1, n-1, n\}$ when uv is a cut edge, or $k \in \{0, 1, n\}$ otherwise.*

Definition 2. Let G be a simple connected graph with n vertices, and uv be an edge of G which is not contained in any cycle of length 3, $d_G(u) \geq 3, d_G(v) \geq 3$ and uu' is a pendent edge. Let G'_{uv} be the graph obtained from G in the following way: (1) Delete the edge uv and vertex u'; (2) Identify u and v, and denote the new vertex by w; (3) Add a pendent path $ww'u'$ to w. We say that G'_{uv} is a II-edge-growing transform of G at uv.

Remark 1 [9]**.** Let G and G'_{uv} be the two graphs presented in Definition 2. Then $|M(G)| \leq |M(G'_{uv})| \leq |M(G)| + 1$.

Lemma 2.5. *Let G and G'_{uv} be the two graphs presented in Definition* 2. *Then $G_{uv} \prec G$, i.e., $c_k(G'_{uv}) \leq c_k(G)$, $k = 0, 1, \ldots, n$, with equality if and only if either $k \in \{0, 1, n-1, n\}$ when uv is a cut edge, or $k \in \{0, 1, n\}$ otherwise.*

Proof. The proof is similar to that of Theorem 2.5 in [9]. Thus we omit it. ■

Remark 2. Lemma 2.5 is a generalization of Theorem 2.5 from [9] and Theorem 2.1 from [10].

Definition 3 [10]**.** Let H, G_1, G_2 be three connected graphs and let v_1, v_2 be two vertices of H. Let G be the graph of order n obtained from H, G_1, G_2 by identifying v_i and a vertex $\widetilde{v_i}$ of G_i (still denote this new vertex by v_i) $(i = 1, 2)$ and adding a pendant edge v_2v to v_2. Let $z_1, z_2, \ldots, z_t$ be all adjacent vertices of $\widetilde{v_i} = v_2$ in G_2 and let G' be the graph obtained from G by deleting edges $v_2z_1, v_2z_2, \ldots, v_2z_t$ and adding edges v_1z_1, $v_1z_1, v_1z_2, \ldots, v_1z_t$. We say that G' is an α_2-transform of G from v_2 to v_1.

Lemma 2.6 [10]**.** *Let G and G' be the two graphs presented in Definition* 3 *such that $N_H(v_2) - \{v_1\} \subseteq N_H(v_1) - \{v_2\}, o(G_2) \geq 2$ and either $o(G_1) \geq 3$ or $o(G_1) = 2$ and $N_H(v_2) - \{v_1\} \subset N_H(v_1) - \{v_2\}$. Then $c_k(G) \geq c_k(G'), k = 0, 1, \ldots, n$, with equality if and only if $k \in \{0, 1, n-1, n\}$.*

Definition 4 [10]**.** Let H, G_1, G_2 be three connected graphs and let v_1, v_2 be two vertices of H. Let G be the graph of order n obtained from H, G_1, G_2 by identifying v_i and a vertex $\widetilde{v_i}$ of G_i (still denote this new vertex by v_i) $(i = 1, 2)$. Let $z_1, z_2, \ldots, z_t$ be all adjacent vertices of $\widetilde{v_i} = v_2$ in G_2 and let G' be the graph obtained from G by deleting edges $v_2z_1, v_2z_2, \ldots, v_2z_t$ and adding edges v_1z_1, $v_1z_1, v_1z_2, \ldots, v_1z_t$. We say that G' is an α_3-transform of G from v_2 to v_1.

Lemma 2.7 [10]**.** *Let G and G' be the two graphs presented in Definition 4 such that $N_H(v_2) - \{v_1\} \subseteq N_H(v_1) - \{v_2\}$ and both G_1 and G_2 have at least two vertices. Then $c_k(G) \geq c_k(G'), k = 0, 1, \ldots, n$, with equality if and only if $k \in \{0, 1, n-1, n\}$.*

Lemma 2.8 [10]**.** *Let $f(\lambda)$ and $g(\lambda)$ be two real polynomials arranged according to decreasing exponents. If their coefficients are alternately positive and negative, then the coefficients of $f(\lambda)g(\lambda)$ are also alternately positive and negative.*

3. MAIN RESULTS

Let G be a tricyclic graph. The base of G, denoted by $\widehat{G}$, is the minimal tricyclic subgraph of G. Obviously, $\widehat{G}$ is the unique tricyclic subgraph of G containing no pendant vertex, and G can be obtained from $\widehat{G}$ by planting trees to some vertices of $\widehat{G}$. By [5], we know that tricyclic graphs have the following four types of bases (as shown in Figures 2–4): G_j^3 $(j = 1, \ldots, 7)$, G_j^4 $(j = 1, \ldots, 4)$, G_j^6 $(j = 1, \ldots, 3)$ and G_1^7. Let

$$\mathscr{G}_{n,n+2}^3 = \{G | \widehat{G} \cong G_j^3, j \in \{1, \ldots, 7\}\}; \qquad \mathscr{G}_{n,n+2}^4 = \{G | \widehat{G} \cong G_j^4, j \in \{1, \ldots, 4\}\};$$
$$\mathscr{G}_{n,n+2}^6 = \{G | \widehat{G} \cong G_j^6, j \in \{1, \ldots, 3\}\}; \qquad \mathscr{G}_{n,n+2}^7 = \{G | \widehat{G} \cong G_1^7\}.$$

Then $\mathscr{G}_{n,n+2} = \mathscr{G}_{n,n+2}^3 \cup \mathscr{G}_{n,n+2}^4 \cup \mathscr{G}_{n,n+2}^6 \cup \mathscr{G}_{n,n+2}^7$.

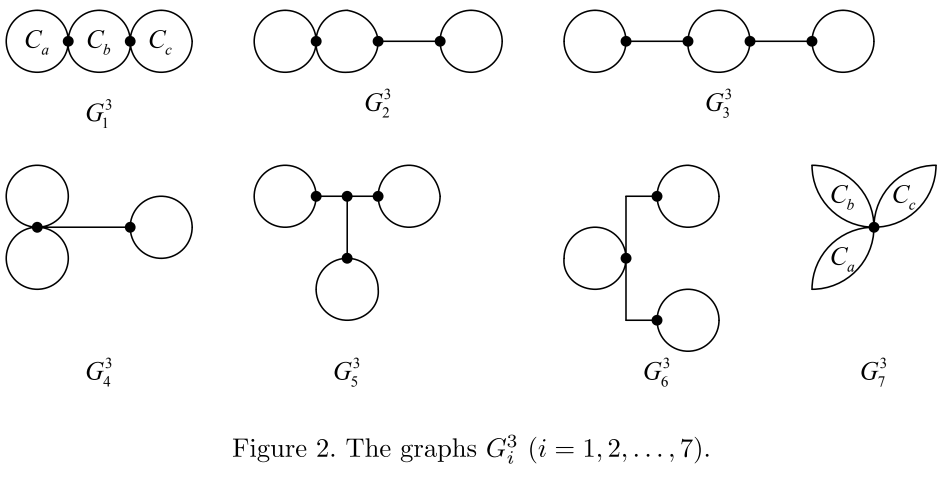

Figure 2. The graphs G_i^3 $(i = 1, 2, \ldots, 7)$.

Lemma 3.1. *Let G^* be the minimal element in $\mathscr{G}_{n,n+2}(i)$ under the partial order $\preceq$. Then*

(i) *each vertex of G^* not on $\widehat{G^*}$ has degree at most 2;*
(ii) *each pendent path of G^* has length at most 2;*

(iii) *there is no cut-edge in* $\widehat{G^*}$;

(iv) *the length of an internal path is at most* 2 *in* $\widehat{G^*}$.

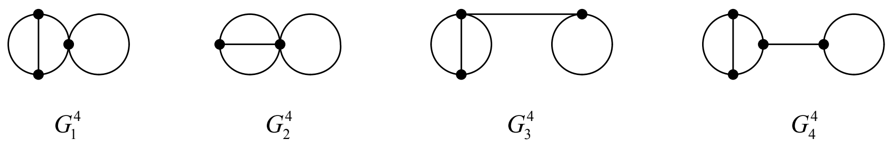

Figure 3. The graphs G_i^4 $(i = 1, 2, \ldots, 4)$.

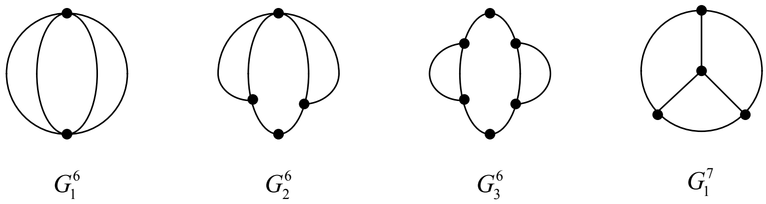

Figure 4. The graphs G_i^6 $(i = 1, 2, 3)$ and G_1^7.

Proof. Let $M(G^*)$ be a maximum matching of G^* containing the most pendent edges. Similarly to the proof in [9], we can prove (i) and (ii). Now we only prove (iii) and (iv).

(iii) Suppose, for contradiction, that there is a cut-edge uv in $\widehat{G^*}$. Obviously, it is also a cut-edge of G^*.

Case 1. If $uv \in M(G^*)$, by I-edge-growing transform of G^* at uv, we can get a connected tricyclic graph G^*_{uv} which is also in $\mathscr{G}_{n,n+2}(i)$, where $M(G^*_{uv}) = M(G^*) - uv + ww'$. By Lemma 2.4, we have $G^*_{uv} \prec G^*$; it is a contradiction.

Case 2. If $uv \notin M(G^*)$ and $E^u_{uv} \cap M(G^*) = \emptyset$ or $E^v_{uv} \cap M(G^*) = \emptyset$, by I-edge-growing transform of G^* at uv, by Lemma 2.3, G^*_{uv} is also in $\mathscr{G}_{n,n+2}(i)$. Further by Lemma 2.4, we have $G^*_{uv} \prec G^*$; it is also a contradiction.

Case 3. Suppose $uv \notin M(G^*)$ and $E^u_{uv} \cap M(G^*) \neq \emptyset$ and $E^v_{uv} \cap M(G^*) \neq \emptyset$.

Case 3.1. If the edge e_0 in $E^u_{uv} \cap M(G^*)$ or $E^v_{uv} \cap M(G^*)$ is not in $E(\widehat{G^*})$, by (i), (ii) and the choice of $M(G^*)$, e_0 must be a pendent edge. By II-edge-growing transform of G^* at uv, we can get a connected tricyclic graph $G^{*'}_{uv}$; similarly to the proof of Theorem 3.3 in [9], we also can obtain a graph $W \prec G^*$, a contradiction, too.

Case 3.2. Suppose the edge e_0 in $E^u_{uv} \cap M(G^*)$ or $E^v_{uv} \cap M(G^*)$ is in $E(\widehat{G^*})$. By the choice of $M(G^*)$, there is no pendent edge at u or v in G^*. If e_0 is also a cut-edge in $\widehat{G^*}$, by I-edge-growing transform of G^* at e_0, following Case 1, we can obtain a contradiction. Further by Lemma 2.4, e_0 must be on a triangle $\widetilde{C_3}$ in $\widehat{G^*}$; without loss of generality, let $\widetilde{C_3} = uyz$, where $e_0 = uy$.

(1) If there is no pendent edge at z, let $M = M(G^*) - e_0 + yz$. By I-edge-growing transform of G^* at uv, we have $G^*_{uv} \prec G^*$, a contradiction.

(2) If there is a pendent edge at z, let $\breve{G}$ be the graph obtained by deleting edge e_0 and adding edge zv. By Lemma 2.6, we have $\breve{G} \prec G^*$, a contradiction.

(iv) By (iii), we know that every edge in an internal path of $\widehat{G^*}$ must be in a cycle. Further by Lemmas 2.4 and 2.5, we can obtain the desirable result. ■

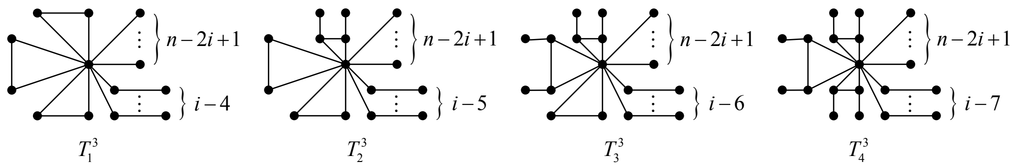

Figure 5. The graphs T_i^3 $(i = 1, 2, 3, 4)$.

Lemma 3.2. *Let T_i^3 $(i = 1, 2, 3, 4)$ be the graphs as shown in Figure 5. Then $T_1^3 \prec T_2^3 \prec T_3^3 \prec T_4^3$.*

Proof. Let H be the graph obtained from T_1^3 by deleting all the vertices in the pendent edges and pendent paths. By Lemma 2.2, we have

(1)

$$\begin{aligned}
&\phi(L(T_1^3))\\
&= (x^2-3x+1)^{i-4}[(x-1)^{n-2i+1}\phi(L(H)) - (n-2i)x(x-1)^{n-2i}\phi(L_u(H))]\\
&\quad - (i-4)(x^2-3x+1)^{i-5}(x^2-2x)(x-1)^{n-2i+1}\phi(L_u(H))\\
&= x(x^2-3x+1)^{i-5}(x-1)^{n-2i}[(x-1)(x^2-3x+1)\\
&\quad (189-594x+711x^2-412x^3+123x^4-18x^5+x^6)\\
&\quad - (n-2i+1)(x^2-3x+1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6)\\
&\quad - (i-4)(x-2)(x-1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6)]\\
&= x(x^2-3x+1)^{i-5}(x-1)^{n-2i}g(x),
\end{aligned}$$

where

$$\begin{aligned}
&g(x)\\
&= (x-1)(x^2-3x+1)(189-594x+711x^2-412x^3+123x^4-18x^5+x^6)\\
&\quad - (n-2i+1)(x^2-3x+1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6)\\
&\quad - (i-4)(x-2)(x-1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6).
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&\phi(L(T_2^3))\\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i}[(x-1)(x^2-3x+1)\\
&\quad (243x-1404x^2+3195x^3-3714x^4+2414x^5-908x^6+195x^7-22x^8+x^9)\\
&\quad -(n-2i+1)x(x^2-3x+1)(27-198x+573x^2-860x^3+734x^4\\
&\quad -366x^5+105x^6-16x^7+x^8)-(i-5)(x^2-2x)(x-1)(27-198x+573x^2\\
&\quad -860x^3+734x^4-366x^5+105x^6-16x^7+x^8)],\\
&\phi(L(T_3^3))\\
&= (x^2-3x+1)^{i-7}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(297x-2574x^2+9147x^3\\
&\quad -17480x^4+19797x^5-13866x^6+6117x^7-1692x^8+283x^9-26x^{10}+x^{11})\\
&\quad -(n-2i+1)x(x^2-3x+1)(27-288x+1275x^2-3064x^3+4403x^4\\
&\quad -3940x^5+2225x^6-788x^7+169x^8-20x^9+x^{10})\\
&\quad -(i-6)(x^2-2x)(x-1)(27-288x+1275x^2-3064x^3\\
&\quad +4403x^4-3940x^5+2225x^6-788x^7+169x^8-20x^9+x^{10})],\\
&\phi(L(T_4^3))\\
&= (x^2-3x+1)^{i-8}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(351x-4104x^2+20367x^3\\
&\quad -56390x^4+96504x^5-107124x^6+79003x^7-39114x^8+12976x^9\\
&\quad -2828x^{10}+387x^{11}-30x^{12}+x^{13})-(n-2i+1)x(x^2-3x+1)\\
&\quad (27-378x+2277x^2-7748x^3+16464x^4-22854x^5+21133x^6-13092x^7\\
&\quad +5412x^8-1466x^9+249x^{10}-24x^{11}+x^{12})-(i-7)(x^2-2x)(x-1)\\
&\quad (27-378x+2277x^2-7748x^3+16464x^4-22854x^5+21133x^6-13092x^7\\
&\quad +5412x^8-1466x^9+249x^{10}-24x^{11}+x^{12})].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\phi(L(T_2^3))-\phi(L(T_1^3))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i}[(n-i-1)x^8-(14n-16-14i)x^7\\
&\quad +(81n-80i-111)x^6-(250n-239i-432)x^5+(444n-397i-1016)x^4\\
&\quad -(458n-1448-360i)x^3+(265n-162i-1191)x^2\\
&\quad -(78n-504-27i)x+9n].
\end{aligned}
$$

By Lemma 2.8, $A=\phi(L(T_2^3))-\phi(L(T_1^3))$ is a polynomial of order $n-2$ whose coefficients are alternately positive and negative. Let $A=\sum_{j=0}^{n}(-1)^j b_j x^{n-j}$, where $b_0=b_1=b_{n-1}=b_n=0$ and $b_j>0$ for $2\le j\le n-2$. Then

$$\phi(L(T_2^3))=\phi(L(T_1^3))+A=\sum_{j=0}^{n}(-1)^j(c_j(T_1^3)+b_j)x^{n-j}.$$

Hence $c_j(T_2^3) = c_j(T_1^3) + b_j$ for $0 \le j \le n$. It follows that $c_j(T_2^3) = c_j(T_1^3)$ if $j = 0, 1, n-1, n$ and $c_j(T_2^3) > c_j(T_1^3)$ if $2 \le j \le n$. Thus we have $T_1^3 \prec T_2^3$. Note that

$$\begin{aligned}
&\phi(L(T_3^3)) - \phi(L(T_2^3))\\
&= x^2(x^2-3x+1)^{i-7}(x-1)^{n-2i}[(n-i-2)x^{10} - (18n-18i-38)x^9\\
&\quad + (137n-136i-311)x^8 - (576n-561i-1439)x^7\\
&\quad + (1467n-1376i-4147)x^6 - (2340n-2052i-7720)x^5\\
&\quad + (2347n-1835i-9310)x^4 - (1458n-942i-7102)x^3\\
&\quad + (539n-252i-3249)x^2 - (108n-27i-801)x + (9n-81)],\\
&\phi(L(T_4^3)) - \phi(L(T_3^3))\\
&= x^2(x^2-3x+1)^{i-8}(x-1)^{n-2i}[(n-3-i)x^{12} - (22n-22i-68)x^{11}\\
&\quad + (209n-208i-671)x^{10} - (1126n-1107i-3794)x^9\\
&\quad + (3802n-3651i-13620)x^8 - (8406n-7752i-32520)x^7\\
&\quad + (12385n-10696i-52659)x^6 - (12202n-9517i-57998)x^5\\
&\quad + (7994n-5353i-43016)x^4 - (3418n-1824i-20960)x^3\\
&\quad + (913n-342i-6387)x^2 - (138n-27i-1098)x + (9n-81)].
\end{aligned}$$

Similarly, we have $T_2^3 \prec T_3^3 \prec T_4^3$. So we have $T_1^3 \prec T_2^3 \prec T_3^3 \prec T_4^3$. ∎

Theorem 3.3. *For $G \in \mathscr{G}_{n,n+2}^3(i)$, $c_k(G) \ge c_k(T_1^3), k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. Let G^* be the minimal element in $\mathscr{G}_{n,n+2}^3(i)$ under the partial order $\preceq$. Now we only need to prove $G^* \cong T_1^3$.

Let $M(G^*)$ be a maximum matching of G^* containing the most pendent edges. By Lemma 3.1, we have $\widehat{G^*} \cong G_1^3$ or $\widehat{G^*} \cong G_7^3$ and $a = b = c = 3$.

Case 1. If $\widehat{G^*} \cong G_1^3$, let $H = C_b = xyz$, G_1 be the component of $G^* - \{xy, xz, yz\}$ containing y and G_2 be the component of $G^* - \{xy, xz, yz\}$ containing x. If there exist pendent edges at x, by the choice of $M(G^*)$, we know that there is a pendent edge xx' belonging to $M(G^*)$; let $M'(G^*) = M(G^*) - xx' + xz$. By an α_3-transform of G^* from x to y, we can obtain a graph $\widetilde{G}$. Obviously, $N_H(x) - \{y\} \subseteq N_H(y) - \{x\}$, by Lemma 2.7, we have $\widetilde{G} \prec G^*$, it is contradict to the choice of G^*.

Case 2. If $\widehat{G^*} \cong G_7^3$, then $G^* \cong T_i^3$ for some $i \in \{1, 2, 3, 4\}$ (as shown in Figure 5). Further by Lemma 3.2, we have $G^* \cong T_1^3$. ∎

Lemma 3.4. *Let T_i^4 $(i = 1, 2, \ldots, 8)$ be the graphs as shown in Figure 6. Then $T_2^4 \prec T_i^4$ for $i = 1, 3, \ldots, 8$.*

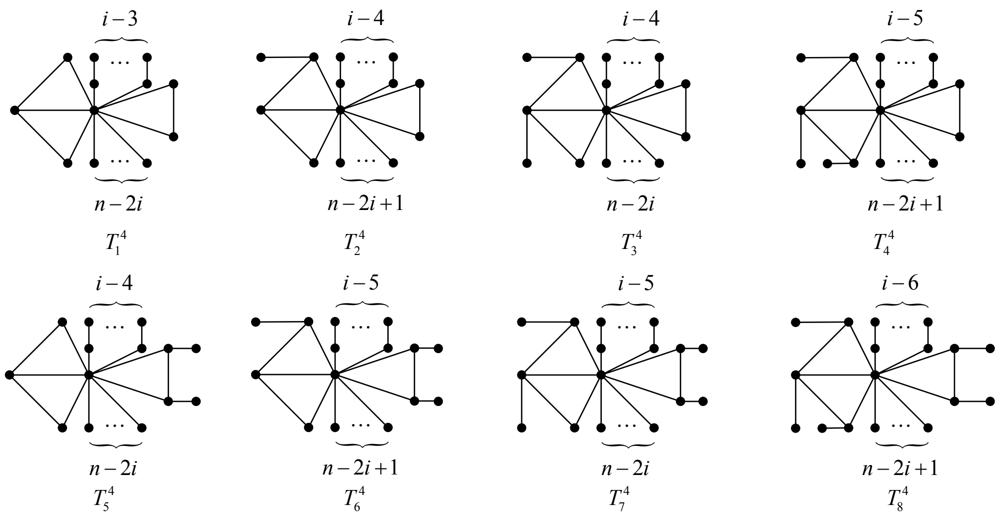

Figure 6. The graphs T_i^4 $(i = 1, 2, \dots, 8)$.

Proof. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\phi(L(T_2^4)) = {} & x(x^2-3x+1)^{i-5}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(168-584x \\
& + 728x^2 - 424x^3 + 125x^4 - 18x^5 + x^6) \\
& - (n-2i+1)(x^2-3x+1)(24-113x+194x^2-158x^3+65x^4 \qquad (2) \\
& - 13x^5 + x^6) - (i-4)(x-2)(x-1)(24-113x+194x^2-158x^3 \\
& + 65x^4 - 13x^5 + x^6)] \\
= {} & x(x^2-3x+1)^{i-5}(x-1)^{n-2i}h(x)
\end{aligned}$$

where

$$\begin{aligned}
h(x) = {} & (x-1)(x^2-3x+1)(168-584x+728x^2-424x^3+125x^4-18x^5+x^6) \\
& - (n-2i+1)(x^2-3x+1)(24-113x+194x^2-158x^3 \\
& + 65x^4 - 13x^5 + x^6) - (i-4)(x-2)(x-1)(24-113x+194x^2 \\
& - 158x^3 + 65x^4 - 13x^5 + x^6).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
& \phi(L(T_1^4)) \\
= {} & (x^2-3x+1)^{i-4}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1) \\
& (-144x+324x^2-260x^3+95x^4-16x^5+x^6) \\
& - (n-2i)x(x^2-3x+1)(x^5-11x^4+45x^3-85x^2+74x-24) \\
& -(i-3)(x^2-2x)(x-1)(x^5-11x^4+45x^3-85x^2+74x-24)],
\end{aligned}$$

$$
\begin{aligned}
&\phi(L(T_3^4)) \\
&= (x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(-192x+889x^2 \\
&\quad -1574x^3+1366x^4-632x^5+158x^6-20x^7+x^8)-(n-2i)x(x^2-3x+1) \\
&\quad (-24+149x-353x^2+414x^3-260x^4+88x^5-15x^6+x^7) \\
&\quad -(i-4)(x^2-2x)(x-1)(-24+149x-353x^2+414x^3-260x^4 \\
&\quad +88x^5-15x^6+x^7)], \\
&\phi(L(T_4^4)) \\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(216x-1284x^2+3026x^3 \\
&\quad -3634x^4+2411x^5-914x^6+196x^7-22x^8+x^9)-(n-2i+1)x \\
&\quad (x^2-3x+1)(24-188x+582x^2-924x^3+817x^4-411x^5 \\
&\quad +116x^6-17x^7+x^8)-(i-5)(x^2-2x)(x-1)(24-188x+582x^2-924x^3 \\
&\quad +817x^4-411x^5+116x^6-17x^7+x^8)], \\
&\phi(L(T_5^4)) \\
&= (x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(-192x+920x^2-1646x^3 \\
&\quad +1413x^4-644x^5+159x^6-20x^7+x^8)-(n-2i)x(x^2-3x+1) \\
&\quad (-24+154x-369x^2+431x^3-267x^4+89x^5-15x^6+x^7)-(i-4) \\
&\quad (x^2-2x)(x-1)(-24+154x-369x^2+431x^3-267x^4+89x^5-15x^6+x^7)], \\
&\phi(L(T_6^4)) \\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(216x-1338x^2+3184x^3 \\
&\quad -3792x^4+2481x^5-928x^6+197x^7-22x^8+x^9) \\
&\quad -(n-2i+1)x(x^2-3x+1)(24-193x+608x^2-968x^3+847x^4-420x^5 \\
&\quad +117x^6-17x^7+x^8)-(i-5)(x^2-2x)(x-1)(24-193x+608x^2 \\
&\quad -968x^3+847x^4-420x^5+117x^6-17x^7+x^8)], \\
&\phi(L(T_7^4)) \\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(-240x+1795x^2 \\
&\quad -5354x^3+8332x^4-7436x^5+3959x^6-1268x^7+238x^8-24x^9+x^{10}) \\
&\quad -(n-2i)x(x^2-3x+1)(-24+229x-887x^2+1810x^3-2124x^4+1479x^5 \\
&\quad -614x^6+148x^7-19x^8+x^9) \\
&\quad -(i-5)(x^2-2x)(x-1)(-24+229x-887x^2+1810x^3-2124x^4 \\
&\quad +1479x^5-614x^6+148x^7-19x^8+x^9)], \\
&\phi(L(T_8^4)) \\
&= (x^2-3x+1)^{i-7}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(88x-900x^2+3762x^3 \\
&\quad -8370x^4+10891x^5-8646x^6+4270x^7-1308x^8+240x^9 \\
&\quad -24x^{10}+x^{11})-(n-2i+1)x(x^2-3x+1)
\end{aligned}
$$

$$(8 - 100x + 522x^2 - 1480x^3 + 2491x^4 - 2571x^5 + 1640x^6 - 643x^7 + 150x^8 - 19x^9 + x^{10}) - (i-6)(x^2 - 2x)(x-1)(8 - 100x + 522x^2 - 1480x^3 + 2491x^4 - 2571x^5 + 1640x^6 - 643x^7 + 150x^8 - 19x^9 + x^{10})].$$

Then

$$\begin{aligned}
&\phi(L(T_1^4)) - \phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[x^7 - (n+14-i)x^6\\
&\quad + (11n+80-11i)x^5 - (47n+235-46i)x^4 + (98n+365-90i)x^3\\
&\quad - (103n+272-81i)x^2 + (51n+66-27i)x - (9n-9)],\\
&\phi(L(T_3^4)) - \phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(n-i-2)x^7 - (13n-13i-26)x^6\\
&\quad + (68n-67i-140)x^5 - (183n-173i-406)x^4 + (269n-232i-686)x^3\\
&\quad - (212n-150i-672)x^2 + (82n-36i-348)x - (12n-72)],\\
&\phi(L(T_4^4)) - \phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i}[(n-i-3)x^8 - (14n-14i-46)x^7\\
&\quad + (79n-78i-295)x^6 - (230n-219i-1026)x^5 + (368n-2094-323i)x^4\\
&\quad - (322n-2528-238i)x^3 + (149n-1727-78i)x^2 - (34n-600-9i)x\\
&\quad + (3n-81)],\\
&\phi(L(T_5^4)) - \phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(n-i-1)x^7 - (14n-14i-14)x^6\\
&\quad + (78n-77i-81)x^5 - (222n-257-211i)x^4 + (343n-491-299i)x^3\\
&\quad - (282n-203i-561)x^2 + (113n-340-51k)x - (17n-81)],\\
&\phi(L(T_6^4)) - \phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i}[(n-i-2)x^8 - (15n-32-15i)x^7\\
&\quad + (91n-90i-213)x^6 - ((288n-276i-768)x^5 + (511n-457i-1627)x^4\\
&\quad - (510n-396i-2042)x^3 + (276n-161i-1449)x^2 - (75n-24i-520)x\\
&\quad + (8n-72)],\\
&\phi(L(T_7^4)) - \phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i-1}[(2n-2i-5)x^9 - (33n-33i-85)x^8\\
&\quad + (226n-224i-608)x^7 - (836n-809i-2394)x^6\\
&\quad + (1812n-1678i-5692)x^5 - (2394n-2014i-8424)x^4\\
&\quad + (1883n-1345i-7709)x^3 - (855n-455i-4183)x^2\\
&\quad + (205n-60i-1216)x - (20n-144)],
\end{aligned}$$

$$
\begin{aligned}
&\phi(L(T_8^4)) - \phi(L(T_2^4))\\
&= x(x^2-3x+1)^{i-7}(x-1)^{n-2i}[2x^{12} - 48x^{11} + (4n-4i+492)x^{10}\\
&\quad - (66n+2846-66i)x^9 + (462n+10302-458i)x^8\\
&\quad - (1789n+24395-1735i)x^7 + (4195n-3899i+38323)x^6\\
&\quad - (6150n-5303i+39656)x^5 + (5659n-4301i+26317)x^4\\
&\quad - (3228n+10626-1999i)x^3 + (1101n-487i+2351)x^2\\
&\quad - (205n+217-48i)x + 16n].
\end{aligned}
$$

Similarly to the procedure of Lemma 3.2, we have $T_2^4 \prec T_i^4$ for $i = 1, 3, \ldots, 8$. ∎

Theorem 3.5. *For $G \in \mathscr{G}_{n,n+2}^4(i)$, $c_k(G) \geq c_k(T_2^4)$, $k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. Let G^* be the minimal element in $\mathscr{G}_{n,n+2}^4(i)$ under the partial order $\preceq$. Repeated by Lemmas 2.7 and 3.1, we have $G^* \cong T_i^4$ for some $i \in \{1, 2, \ldots, 8\}$. Further by Lemma 3.4, we have our desirable results. ∎

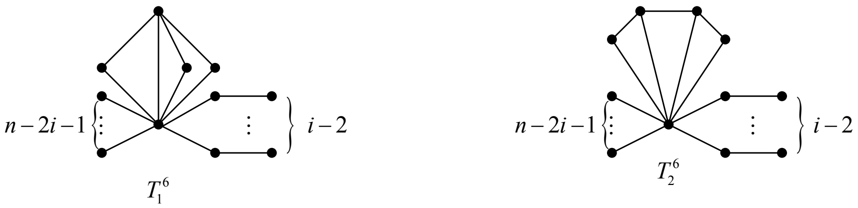

Figure 7. The graphs T_i^6 $(i = 1, 2)$.

Lemma 3.6. *Let T_i^6 $(i = 1, 2)$ be the graphs as shown in Figure 7. Then $T_2^6 \prec T_1^6$.*

Proof. By direct calculation, we have

$$
\begin{aligned}
\phi(L(T_1^6)) = {}& (x^2-3x+1)^{i-3}(x-1)^{n-2i-2}[(x-1)(x^2-3x+1)(x^5-14x^4\\
&+ 69x^3 - 140x^2 + 100x)\\
&- (n-2i-1)x(x^2-3x+1)(x^4-10x^3+33x^2-44x+20)\\
&- (i-2)(x^2-2x)(x-1)(x^4-10x^3+33x^2-44x+20)]
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(L(T_2^6)) = {}& (x^2-3x+1)^{i-3}(x-1)^{n-2i-2}[(x-1)(x^2-3x+1)(x^5-14x^4\\
&+ 70x^3 - 146x^2 + 105x)\\
&- (n-2i-1)x(x^2-3x+1)(-10x^3+x^4+34x^2-46x+21)\\
&- (i-2)(x^2-2x)(x-1)(x^4-10x^3+34x^2-46x+21)].
\end{aligned} \tag{3}
$$

Then

$$\begin{aligned}&\phi(L(T_1^6)) - \phi(L(T_2^6))\\ &= x^2(x^2-3x+1)^{i-3}(x-1)^{n-2i-2}[x^8 - 19x^7 + 148x^6 - 613x^5 + 1465x^4\\ &\quad - (i+2050)x^3 + (5i+1622)x^2 - (7i+652)x + (3i+98)].\end{aligned}$$

Hence $T_2^6 \prec T_1^6$. ■

Theorem 3.7. *For $G \in \mathscr{G}^6_{n,n+2}(i)$, $c_k(G) \geq c_k(T_1^6)$, $k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. Let G^* be the minimal element in $\mathscr{G}^6_{n,n+2}(\beta)$ under the partial order $\preceq$. Repeated by Lemmas 2.7 and 3.1, we have $G^* \cong T_i^6$ for some $i \in \{1, 2\}$. Further by Lemma 3.6, we have our desirable results. ■

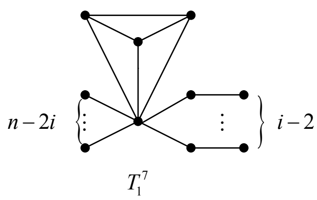

Figure 8. The graph T_1^7.

Theorem 3.8. *For $G \in \mathscr{G}^7_{n,n+2}(i)$, $c_k(G) \geq c_k(T_1^7)$, $k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. By Lemma 3.1, it is easy to obtain our desirable results. ■

Theorem 3.9. *T_1^3, T_2^4, T_1^7 are the only three minimal elements in the partial set $(\mathscr{G}_{n,n+2}(i), \preceq)$.*

Proof. For any graph $G \in \mathscr{G}_{n,n+2}(i)$, by Theorems 3.3, 3.5, 3.7 and 3.8, we have

$$c_k(G) \geq \min\{c_k(T_1^3), c_k(T_2^4), c_k(T_2^6), c_k(T_1^7)\}$$

for $k = 0, 1, \ldots, n$. By direct calculation, we have

$$\begin{aligned}\phi(L(T_1^7)) &= x(x^2-3x+1)^{i-3}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(x^3-12x^2\\ &\quad + 48x - 64) - (n-2i)(x^2-3x+1)(x^3-9x^2+24x-16)\\ &\quad - (i-2)(x-2)(x-1)(x^3-9x^2+24x-16)]\\ &= x(x^2-3x+1)^{i-3}(x-1)^{n-2i-1}r(x),\end{aligned} \tag{4}$$

where

$$\begin{aligned} r(x) = {} & (x-1)(x^2-3x+1)(x^3-12x^2+48x-64) \\ & -(n-2i)(x^2-3x+1)(x^3-9x^2+24x-16) \\ & -(i-2)(x-2)(x-1)(x^3-9x^2+24x-16). \end{aligned}$$

By equations (3) and (4), we have

$$\begin{aligned} & \phi(L(T_2^6)) - \phi(L(T_1^7)) \\ = {} & x(x^2-3x+1)^{i-3}(x-1)^{n-2i-1}[5n-(16n-15i+35)x \\ & +(8n-8i+32)x^2-(n-i+10)x^3+x^4], \end{aligned}$$

hence $T_1^7 \prec T_2^6$.

Further by equations (1)–(4), we have

$$\begin{aligned} & \phi(L(T_2^4)) - \phi(L(T_1^3)) \\ = {} & x(x^2-3x+1)^{i-5}(x-1)^{n-2i}[(12+3n-3i)-(4n-4i-453)x \\ & -(35n-35i+1928)x^2+(96n-96i-1871)x^3-(97n-97i+352)x^4 \\ & +(47n-47i-68)-(11n-11i-13)x^6+(n-i-1)x^7], \\ & \phi(L(T_2^4)) - \phi(L(T_1^7)) \\ = {} & x(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(432-40n)+(257n-120i-3047)x \\ & -(654n-451i-9277)x^2+(905n-746i+15877)x^3 \\ & -(745n-680i-16666)x^4+(367n-354i-11128)x^5 \\ & -(105n-104i-4803)x^6+(16n-16i-1336)x^7-(n-i-232)x^8 \\ & -23x^9+x^{10}], \\ & \phi(L(T_1^7)) - \phi(L(T_1^3)) \\ = {} & x(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[-11n+(48n+63i-488)x \\ & -(6n+552i-2867)x^2-(243n-1605i+5540)x^3 \\ & +(446n-2201i+49200)x^4-(344n-1622i+2453)x^5 \\ & +(134n-679i+902)x^6-(26n-161i+240)x^7 \\ & +(2n-20i+34)x^8-(2-i)x^9]. \end{aligned}$$

Obviously, T_1^3, T_2^4, T_1^7 are incomparable, thus we obtain our desirable results. ∎

4. The Laplacian-Like Energy of Tricyclic Graphs with Prescribed Matching Number

Let G be a graph. The Laplacian matrix $L(G)$ has non-negative eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. The *Laplacian-like energy* of graph G,

$LEL(G)$ for short, is defined as follows:

$$LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k(G)}.$$

Stevanović [11] proved a connection between Laplacian-like energy and Laplacian coefficients of a graph G.

Theorem 4.1 [11]**.** *Let G and H be two n-vertex graphs. If $c_k(G) \leq c_k(H)$ for $k = 1, 2, \ldots, n-1$, then $LEL(G) \leq LEL(H)$. Furthermore, if a strict inequality $c_k(G) < c_k(H)$ holds for some $1 \leq k \leq n-1$, then $LEL(G) < LEL(H)$.*

By Theorems 3.9 and 4.1, we have the following result.

Theorem 4.2. *For $G \in \mathscr{G}_{n,n+2}(i)$, we have $LEL(G) \geq \min\{LEL(T_1^3), LEL(T_2^4), LEL(T_1^7)\}$. The equality holds if and only if $G \cong T_1^3$, $G \cong T_2^4$ or $G \cong T_1^7$.*

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