Discussiones Mathematicae
Graph Theory 37 (2017) 505–522
doi:10.7151/dmgt.1937

ON THE LAPLACIAN COEFFICIENTS OF TRICYCLIC GRAPHS WITH PRESCRIBED MATCHING NUMBER

Jing Luo, Zhongxun Zhu*

Department of Mathematics and Statistics
South Central University for Nationalities
Wuhan 430074, P.R. China

and

Runze Wan

College of Computer, Hubei University of Education
Wuhan 430205, P.R. China

e-mail: zzxun73@mail.scuec.edu.cn

Abstract

Let $\phi(L(G)) = \det(xI - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)x^{n-k}$ be the Laplacian characteristic polynomial of G. In this paper, we characterize the minimal graphs with the minimum Laplacian coefficients in $\mathscr{G}_{n,n+2}(i)$ (the set of all tricyclic graphs with fixed order n and matching number i). Furthermore, the graphs with the minimal Laplacian-like energy, which is the sum of square roots of all roots on $\phi(L(G))$, is also determined in $\mathscr{G}_{n,n+2}(i)$.

Keywords: Laplacian characteristic polynomial, Laplacian-like energy, tricyclic graph.

2010 Mathematics Subject Classification: 05C12, 05C50.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges. Denote by $\mathscr{G}_{n,m}$ the set of all simple connected graphs of order n and size m. If $m = n - 1 + c$, then G is called a *c-cyclic graph*. If $c = 0, 1, 2$ and 3, then G is a tree, unicyclic graph, bicyclic graph and tricyclic graph, respectively. Let P_n, C_n and

This project is supported by the Foundation of State Ethnic Affairs (14ZNZ023), Natural Science Foundation of Hubei Province (2015CFB405) and Hubei Provincial Department of Education Scientific Research Programs for Youth Project (Q20153003).

* Corresponding author.

S_n be the path, the cycle and the star on n vertices, respectively. Furthermore, let $\mathscr{G}_{n,m}(i)$ be the set of all simple connected graphs with order n, size m and matching number i.

Let $L(G) = D(G) - A(G)$ be the *Laplacian matrix* of G, where $A(G)$ is its $(0,1)$-adjacency matrix and $D(G)$ its degree diagonal matrix. While the Laplacian polynomial of G is the characteristic polynomial of $L(G)$, $\phi(L(G)) = \det(xI - L(G))$. Let $c_k(G)$ $(0 \leq k \leq n)$ be the absolute values of the coefficients of $\phi(L(G))$, so that

$$\phi(L(G)) = \det(xI - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)x^{n-k}.$$

For $G, H \in \mathscr{G}_{n,m}$, we write $G \preceq H$ if the Laplacian coefficients $c_k(G) \leq c_k(H)$ for $k = 0, 1, 2, \ldots, n$, and we write $G \prec H$ if $G \preceq H$ and $c_{k_0}(G) < c_{k_0}(H)$ for some $0 \leq k_0 \leq n$.

Recently, the study of the structure and properties on the Laplacian coefficients have attracted much attention. As for n-vertex trees, Mohar [6] proved that P_n has the maximal Laplacian coefficients and S_n has the minimal Laplacian coefficients, respectively. As for n-vertex unicyclic graphs, Stevanović and Ilić [8] showed that C_n has the maximal Laplacian coefficients and S_n' has the minimal Laplacian coefficients, where S_n' is the graph obtained from S_n by joining two of its pendant vertices with an edge. As for n-vertex bicyclic graphs, He and Shan [3] obtained that the Laplacian coefficients are the smallest when the graph is obtained from C_4 by adding one edge connecting two non-adjacent vertices and adding $n-4$ pendent vertices attached to the vertex of degree 3. As for n-vertex tricyclic graphs, Pai *et al.* [7] determined that the coefficients are the smallest when the graph is obtained from the complete graph K_4 by adding $n-4$ pendent vertices attached to the vertex of degree 3. Furthermore, in $\mathscr{G}_{n,m}(i)$, Ilić [4] characterized the minimal trees with the minimum Laplacian coefficients for $m = n - 1$; Tan [9, 10] obtained the graphs with the minimum Laplacian coefficients for $m = n, n + 1$, respectively. Motivated by all these works, in the present paper we are devoted to find the graphs with the minimum Laplacian coefficients for $m = n + 2$.

In order to state our results, we introduce some notation and terminology. For other undefined notation we refer to Bollobás [1]. Let $N_G(v) = \{u|uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$. Denote by $d_G(v) = |N_G(v)|$ the degree of the vertex v of G. If $E_0 \subset E(G)$, we denote by $G - E_0$ the subgraph of G obtained by deleting the edges in E_0. If E_1 is the subset of the edge set of the complement of G, $G + E_1$ denotes the graph obtained from G by adding the edges in E_1. Similarly, if $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If $E = \{xy\}$ and $W = \{v\}$, we write $G - xy$ and $G - v$ instead of $G - \{xy\}$ and $G - \{v\}$, respectively.

2. PRELIMINARIES

In this section, we introduce some graphic transformations and lemmas, which will be used to prove our main results.

For any graph G and $v \in V(G)$, let $L_v(G)$ denote the principal submatrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v.

Lemma 2.1 [2]. *Let $G = G_1u : vG_2$ be the graph obtained from two disjoint graphs G_1 and G_2 by joining a vertex u of the graph G_1 to a vertex v of the graph G_2 by an edge. Then*

$$\phi(L(G)) = \phi(L(G_1))\phi(L(G_2)) - \phi(L(G_1))\phi(L_v(G_2)) - \phi(L_u(G_1))\phi(L(G_2)).$$

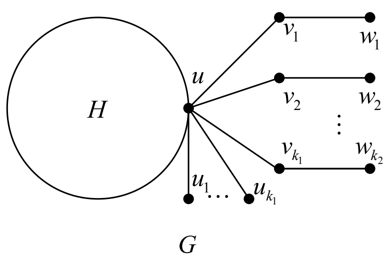

Figure 1. The graph in Lemma 2.2.

Lemma 2.2. *Let H be a graph and u a vertex of it. Let G be a graph of order n, which is obtained from H by attaching k_1 pendent edges and k_2 pendent paths of length 2 at u (as shown in Figure 1). Then*

$$\begin{aligned}\phi(L(G)) \;=\; & \left(x^2-3x+1\right)^{k_2}\left[(x-1)^{k_1}\phi(L(H)) - k_1x(x-1)^{k_1-1}\phi(L_u(H))\right] \\ & - k_2\left(x^2-3x+1\right)^{k_2-1}\left(x^2-2x\right)(x-1)^{k_1}\phi(L_u(H)).\end{aligned}$$

Proof. We label the rows and columns of $L(G)$ as the vertices $v_1, w_1, \ldots, v_{k_2}, w_{k_2}$, $u_1, \ldots, v_{k_1}, u, V(H-u)$. Let $G'_i = G - \bigcup_{k=1}^{i}\{v_k, w_k\}$; by Lemma 2.1, we have

$$\begin{aligned}\phi(L(G'_1)) \;=\; & \phi(L(G'_2))\phi(L(K_2)) - \phi(L(G'_2))\phi(L_{v_2}(K_2)) - \phi(L_u(G'_2))\phi(L(K_2)) \\ =\; & \phi(L(G'_2))(x^2-3x+1) - \phi(L_u(G'_2))(x^2-2x), \\ & \vdots \\ \phi(L(G'_{k_2-1})) \;=\; & \phi(L(G'_{k_2}))\phi(L(K_2)) - C(L(G'_{k_2}))\phi(L_{v_{k_2}}(K_2)) \\ & -\phi(L_u(G'_{k_2}))\phi(L(K_2)) \\ =\; & \phi(L(G'_{k_2}))(x^2-3x+1) - \phi(L_u(G'_{k_2}))(x^2-2x),\end{aligned}$$

$$\begin{aligned}\phi(L(G)) &= \phi(L(G_1'))(x^2-3x+1)-\phi(L_u(G_1'))(x^2-2x)\\ &= (x^2-3x+1)[(x^2-3x+1)\phi(L(G_2'))-\phi(L_u(G_2'))(x^2-2x)]\\ &\quad -\phi(L_u(G_1'))(x^2-2x)\\ &= (x^2-3x+1)^2\phi(L(G_2'))-(x^2-3x+1)(x^2-2x)\phi(L_u(G_2'))\\ &\quad -\phi(L_u(G_1'))(x^2-2x)\\ &= \cdots\\ &= (x^2-3x+1)^{k_2}\phi(L(G_{k_2}'))-(x^2-3x+1)^{k_2-1}(x^2-2x)\phi(L_u(G_{k_2}'))\\ &\quad -\cdots-(x^2-3x+1)(x^2-2x)\phi(L_u(G_2'))-\phi(L_u(G_1'))(x^2-2x).\end{aligned}$$

Note that

$$\begin{aligned}\phi(L_u(G_1')) &= \phi(L_u(G_{k_2}'))[(x-2)(x-1)-1]^{k_2-1}\\ &= \phi(L_u(G_{k_2}'))(x^2-3x+1)^{k_2-1},\\ \phi(L_u(G_2')) &= \phi(L_u(G_{k_2}'))(x^2-3x+1)^{k_2-2},\\ &\vdots\\ \phi(L_u(G_{k_2-1}')) &= \phi(L_u(G_{k_2}'))(x^2-3x+1),\end{aligned}$$

so we have

$$\begin{aligned}\phi(L(G)) &= (x^2-3x+1)^{k_2}\phi(L(G_{k_2}'))\\ &\quad -k_2(x^2-3x+1)^{k_2-1}(x^2-2x)\phi(L_u(G_{k_2}')).\end{aligned}$$

Furthermore, we have $|V(H)|=n-k_1-2k_2$ and

$$\begin{aligned}\phi(L_u(G_{k_2}')) &= (x-1)^{k_1}\phi(L_u(H)),\\ \phi(L(G_{k_2}')) &= (x-1)^{k_1+2k_2}\phi(L(H))-(k_1+2k_2)x(x-1)^{k_1+2k_2-1}\phi(L_u(H)),\\ \phi(L_u(G_{k_2}')) &= (x-1)^{k_1}\phi(L_u(H)).\end{aligned}$$

Hence

$$\begin{aligned}\phi(L(G)) &= (x^2-3x+1)^{k_2}[(x-1)^{k_1}\phi(L(H))-k_1x(x-1)^{k_1-1}\phi(L_u(H))]\\ &\quad -k_2(x^2-3x+1)^{k_2-1}(x^2-2x)(x-1)^{k_1}\phi(L_u(H)).\end{aligned}$$

■

Definition 1 [9]**.** Let G be a simple connected graph with n vertices, and uv be a non-pendent edge which is not contained in any cycle of length 3. Let G_{uv} be the graph obtained from G in the following way: (1) Delete the edge uv; (2) Identify u and v, and denote the new vertex by w; (3) Add a pendent edge ww' to w. We say that G_{uv} is a I-edge-growing transform of G at uv.

Lemma 2.3 [10]. *Let G and G_{uv} be the two graphs defined in Definition 1. Let E^u_{uv} denote the set of edges incident to u except the edge uv. Then $|M(G_{uv})| = |M(G)|$ when $M(G) \cap E^u_{uv} = \emptyset$ or $M(G) \cap E^v_{uv} = \emptyset$.*

Lemma 2.4 [9]. *Let G and G_{uv} be the two graphs presented in Definition 1. Then $G_{uv} \prec G$, i.e., $c_k(G_{uv}) \leq c_k(G)$, $k = 0, 1, \ldots, n$, with equality if and only if either $k \in \{0, 1, n-1, n\}$ when uv is a cut edge, or $k \in \{0, 1, n\}$ otherwise.*

Definition 2. Let G be a simple connected graph with n vertices, and uv be an edge of G which is not contained in any cycle of length 3, $d_G(u) \geq 3, d_G(v) \geq 3$ and uu' is a pendent edge. Let G'_{uv} be the graph obtained from G in the following way: (1) Delete the edge uv and vertex u'; (2) Identify u and v, and denote the new vertex by w; (3) Add a pendent path $ww'u'$ to w. We say that G'_{uv} is a II-edge-growing transform of G at uv.

Remark 1 [9]. Let G and G'_{uv} be the two graphs presented in Definition 2. Then $|M(G)| \leq |M(G'_{uv})| \leq |M(G)| + 1$.

Lemma 2.5. *Let G and G'_{uv} be the two graphs presented in Definition 2. Then $G_{uv} \prec G$, i.e., $c_k(G'_{uv}) \leq c_k(G)$, $k = 0, 1, \ldots, n$, with equality if and only if either $k \in \{0, 1, n-1, n\}$ when uv is a cut edge, or $k \in \{0, 1, n\}$ otherwise.*

Proof. The proof is similar to that of Theorem 2.5 in [9]. Thus we omit it. ■

Remark 2. Lemma 2.5 is a generalization of Theorem 2.5 from [9] and Theorem 2.1 from [10].

Definition 3 [10]. Let H, G_1, G_2 be three connected graphs and let v_1, v_2 be two vertices of H. Let G be the graph of order n obtained from H, G_1, G_2 by identifying v_i and a vertex $\widetilde{v_i}$ of G_i (still denote this new vertex by v_i) $(i = 1, 2)$ and adding a pendant edge v_2v to v_2. Let $z_1, z_2, \ldots, z_t$ be all adjacent vertices of $\widetilde{v_i} = v_2$ in G_2 and let G' be the graph obtained from G by deleting edges $v_2z_1, v_2z_2, \ldots, v_2z_t$ and adding edges v_1z_1, $v_1z_1, v_1z_2, \ldots, v_1z_t$. We say that G' is an α_2-transform of G from v_2 to v_1.

Lemma 2.6 [10]. *Let G and G' be the two graphs presented in Definition 3 such that $N_H(v_2) - \{v_1\} \subseteq N_H(v_1) - \{v_2\}, o(G_2) \geq 2$ and either $o(G_1) \geq 3$ or $o(G_1) = 2$ and $N_H(v_2) - \{v_1\} \subset N_H(v_1) - \{v_2\}$. Then $c_k(G) \geq c_k(G'), k = 0, 1, \ldots, n$, with equality if and only if $k \in \{0, 1, n-1, n\}$.*

Definition 4 [10]. Let H, G_1, G_2 be three connected graphs and let v_1, v_2 be two vertices of H. Let G be the graph of order n obtained from H, G_1, G_2 by identifying v_i and a vertex $\widetilde{v_i}$ of G_i (still denote this new vertex by v_i) $(i = 1, 2)$. Let $z_1, z_2, \ldots, z_t$ be all adjacent vertices of $\widetilde{v_i} = v_2$ in G_2 and let G' be the graph obtained from G by deleting edges $v_2z_1, v_2z_2, \ldots, v_2z_t$ and adding edges v_1z_1, $v_1z_1, v_1z_2, \ldots, v_1z_t$. We say that G' is an α_3-transform of G from v_2 to v_1.

Lemma 2.7 [10]**.** *Let G and G' be the two graphs presented in Definition 4 such that $N_H(v_2) - \{v_1\} \subseteq N_H(v_1) - \{v_2\}$ and both G_1 and G_2 have at least two vertices. Then $c_k(G) \geq c_k(G'), k = 0, 1, \ldots, n$, with equality if and only if $k \in \{0, 1, n-1, n\}$.*

Lemma 2.8 [10]**.** *Let $f(\lambda)$ and $g(\lambda)$ be two real polynomials arranged according to decreasing exponents. If their coefficients are alternately positive and negative, then the coefficients of $f(\lambda)g(\lambda)$ are also alternately positive and negative.*

3. Main Results

Let G be a tricyclic graph. The base of G, denoted by $\widehat{G}$, is the minimal tricyclic subgraph of G. Obviously, $\widehat{G}$ is the unique tricyclic subgraph of G containing no pendant vertex, and G can be obtained from $\widehat{G}$ by planting trees to some vertices of $\widehat{G}$. By [5], we know that tricyclic graphs have the following four types of bases (as shown in Figures 2–4): G_j^3 $(j = 1, \ldots, 7)$, G_j^4 $(j = 1, \ldots, 4)$, G_j^6 $(j = 1, \ldots, 3)$ and G_1^7. Let

$$\mathscr{G}_{n,n+2}^3 = \{G | \widehat{G} \cong G_j^3, j \in \{1, \ldots, 7\}\}; \quad \mathscr{G}_{n,n+2}^4 = \{G | \widehat{G} \cong G_j^4, j \in \{1, \ldots, 4\}\};$$

$$\mathscr{G}_{n,n+2}^6 = \{G | \widehat{G} \cong G_j^6, j \in \{1, \ldots, 3\}\}; \quad \mathscr{G}_{n,n+2}^7 = \{G | \widehat{G} \cong G_1^7\}.$$

Then $\mathscr{G}_{n,n+2} = \mathscr{G}_{n,n+2}^3 \cup \mathscr{G}_{n,n+2}^4 \cup \mathscr{G}_{n,n+2}^6 \cup \mathscr{G}_{n,n+2}^7$.

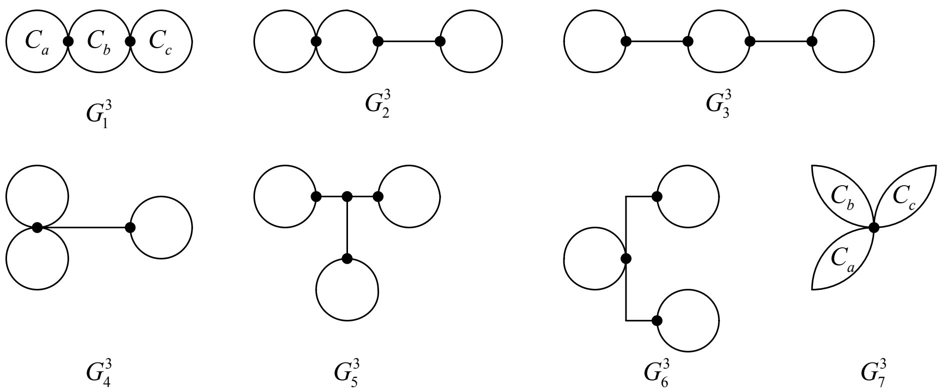

Figure 2. The graphs G_i^3 $(i = 1, 2, \ldots, 7)$.

Lemma 3.1. *Let G^* be the minimal element in $\mathscr{G}_{n,n+2}(i)$ under the partial order $\preceq$. Then*

(i) *each vertex of G^* not on $\widehat{G^*}$ has degree at most 2;*
(ii) *each pendent path of G^* has length at most 2;*

(iii) *there is no cut-edge in* $\widehat{G^*}$;

(iv) *the length of an internal path is at most* 2 *in* $\widehat{G^*}$.

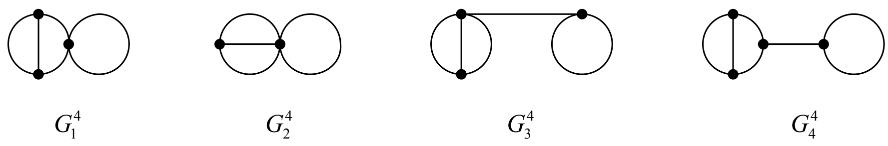

Figure 3. The graphs G_i^4 $(i = 1, 2, \dots, 4)$.

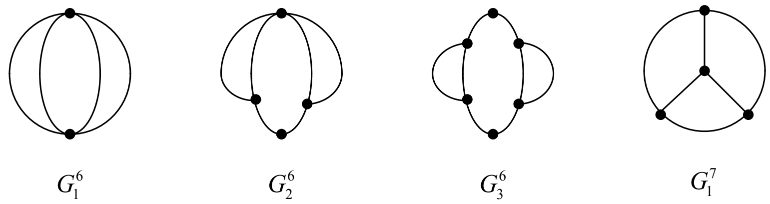

Figure 4. The graphs G_i^6 $(i = 1, 2, 3)$ and G_1^7.

Proof. Let $M(G^*)$ be a maximum matching of G^* containing the most pendent edges. Similarly to the proof in [9], we can prove (i) and (ii). Now we only prove (iii) and (iv).

(iii) Suppose, for contradiction, that there is a cut-edge uv in $\widehat{G^*}$. Obviously, it is also a cut-edge of G^*.

Case 1. If $uv \in M(G^*)$, by I-edge-growing transform of G^* at uv, we can get a connected tricyclic graph G^*_{uv} which is also in $\mathscr{G}_{n,n+2}(i)$, where $M(G^*_{uv}) = M(G^*) - uv + ww'$. By Lemma 2.4, we have $G^*_{uv} \prec G^*$; it is a contradiction.

Case 2. If $uv \notin M(G^*)$ and $E^u_{uv} \cap M(G^*) = \emptyset$ or $E^v_{uv} \cap M(G^*) = \emptyset$, by I-edge-growing transform of G^* at uv, by Lemma 2.3, G^*_{uv} is also in $\mathscr{G}_{n,n+2}(i)$. Further by Lemma 2.4, we have $G^*_{uv} \prec G^*$; it is also a contradiction.

Case 3. Suppose $uv \notin M(G^*)$ and $E^u_{uv} \cap M(G^*) \neq \emptyset$ and $E^v_{uv} \cap M(G^*) \neq \emptyset$.

Case 3.1. If the edge e_0 in $E^u_{uv} \cap M(G^*)$ or $E^v_{uv} \cap M(G^*)$ is not in $E(\widehat{G^*})$, by (i), (ii) and the choice of $M(G^*)$, e_0 must be a pendent edge. By II-edge-growing transform of G^* at uv, we can get a connected tricyclic graph $G^{*'}_{uv}$; similarly to the proof of Theorem 3.3 in [9], we also can obtain a graph $W \prec G^*$, a contradiction, too.

Case 3.2. Suppose the edge e_0 in $E^u_{uv} \cap M(G^*)$ or $E^v_{uv} \cap M(G^*)$ is in $E(\widehat{G^*})$. By the choice of $M(G^*)$, there is no pendent edge at u or v in G^*. If e_0 is also a cut-edge in $\widehat{G^*}$, by I-edge-growing transform of G^* at e_0, following Case 1, we can obtain a contradiction. Further by Lemma 2.4, e_0 must be on a triangle $\widetilde{C_3}$ in $\widehat{G^*}$; without loss of generality, let $\widetilde{C_3} = uyz$, where $e_0 = uy$.

(1) If there is no pendent edge at z, let $M = M(G^*) - e_0 + yz$. By I-edge-growing transform of G^* at uv, we have $G^*_{uv} \prec G^*$, a contradiction.

(2) If there is a pendent edge at z, let $\breve{G}$ be the graph obtained by deleting edge e_0 and adding edge zv. By Lemma 2.6, we have $\breve{G} \prec G^*$, a contradiction.

(iv) By (iii), we know that every edge in an internal path of $\widehat{G^*}$ must be in a cycle. Further by Lemmas 2.4 and 2.5, we can obtain the desirable result. ■

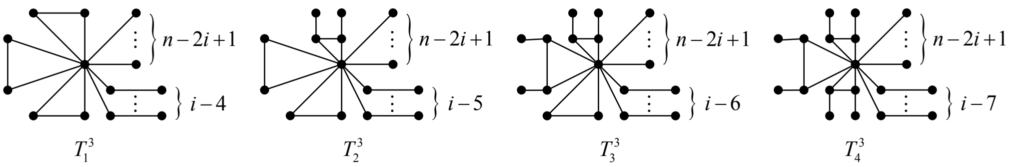

Figure 5. The graphs T_i^3 $(i = 1, 2, 3, 4)$.

Lemma 3.2. *Let* T_i^3 $(i = 1, 2, 3, 4)$ *be the graphs as shown in Figure* 5. *Then* $T_1^3 \prec T_2^3 \prec T_3^3 \prec T_4^3$.

Proof. Let H be the graph obtained from T_1^3 by deleting all the vertices in the pendent edges and pendent paths. By Lemma 2.2, we have

(1)

$$\begin{aligned}
&\phi(L(T_1^3))\\
&= (x^2-3x+1)^{i-4}[(x-1)^{n-2i+1}\phi(L(H)) - (n-2i)x(x-1)^{n-2i}\phi(L_u(H))]\\
&\quad - (i-4)(x^2-3x+1)^{i-5}(x^2-2x)(x-1)^{n-2i+1}\phi(L_u(H))\\
&= x(x^2-3x+1)^{i-5}(x-1)^{n-2i}[(x-1)(x^2-3x+1)\\
&\quad (189-594x+711x^2-412x^3+123x^4-18x^5+x^6)\\
&\quad - (n-2i+1)(x^2-3x+1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6)\\
&\quad - (i-4)(x-2)(x-1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6)]\\
&= x(x^2-3x+1)^{i-5}(x-1)^{n-2i}g(x),
\end{aligned}$$

where

$$\begin{aligned}
&g(x)\\
&= (x-1)(x^2-3x+1)(189-594x+711x^2-412x^3+123x^4-18x^5+x^6)\\
&\quad - (n-2i+1)(x^2-3x+1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6)\\
&\quad - (i-4)(x-2)(x-1)(27-108x+171x^2-136x^3+57x^4-12x^5+x^6).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\phi(L(T_2^3))\\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i}[(x-1)(x^2-3x+1)\\
&\quad (243x-1404x^2+3195x^3-3714x^4+2414x^5-908x^6+195x^7-22x^8+x^9)\\
&\quad -(n-2i+1)x(x^2-3x+1)(27-198x+573x^2-860x^3+734x^4\\
&\quad -366x^5+105x^6-16x^7+x^8)-(i-5)(x^2-2x)(x-1)(27-198x+573x^2\\
&\quad -860x^3+734x^4-366x^5+105x^6-16x^7+x^8)],\\
&\phi(L(T_3^3))\\
&= (x^2-3x+1)^{i-7}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(297x-2574x^2+9147x^3\\
&\quad -17480x^4+19797x^5-13866x^6+6117x^7-1692x^8+283x^9-26x^{10}+x^{11})\\
&\quad -(n-2i+1)x(x^2-3x+1)(27-288x+1275x^2-3064x^3+4403x^4\\
&\quad -3940x^5+2225x^6-788x^7+169x^8-20x^9+x^{10})\\
&\quad -(i-6)(x^2-2x)(x-1)(27-288x+1275x^2-3064x^3\\
&\quad +4403x^4-3940x^5+2225x^6-788x^7+169x^8-20x^9+x^{10})],\\
&\phi(L(T_4^3))\\
&= (x^2-3x+1)^{i-8}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(351x-4104x^2+20367x^3\\
&\quad -56390x^4+96504x^5-107124x^6+79003x^7-39114x^8+12976x^9\\
&\quad -2828x^{10}+387x^{11}-30x^{12}+x^{13})-(n-2i+1)x(x^2-3x+1)\\
&\quad (27-378x+2277x^2-7748x^3+16464x^4-22854x^5+21133x^6-13092x^7\\
&\quad +5412x^8-1466x^9+249x^{10}-24x^{11}+x^{12})-(i-7)(x^2-2x)(x-1)\\
&\quad (27-378x+2277x^2-7748x^3+16464x^4-22854x^5+21133x^6-13092x^7\\
&\quad +5412x^8-1466x^9+249x^{10}-24x^{11}+x^{12})].
\end{aligned}$$

Then

$$\begin{aligned}
&\phi(L(T_2^3))-\phi(L(T_1^3))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i}[(n-i-1)x^8-(14n-16-14i)x^7\\
&\quad +(81n-80i-111)x^6-(250n-239i-432)x^5+(444n-397i-1016)x^4\\
&\quad -(458n-1448-360i)x^3+(265n-162i-1191)x^2\\
&\quad -(78n-504-27i)x+9n].
\end{aligned}$$

By Lemma 2.8, $A=\phi(L(T_2^3))-\phi(L(T_1^3))$ is a polynomial of order $n-2$ whose coefficients are alternately positive and negative. Let $A=\sum_{j=0}^{n}(-1)^j b_j x^{n-j}$, where $b_0=b_1=b_{n-1}=b_n=0$ and $b_j>0$ for $2\le j\le n-2$. Then

$$\phi(L(T_2^3))=\phi(L(T_1^3))+A=\sum_{j=0}^{n}(-1)^j(c_j(T_1^3)+b_j)x^{n-j}.$$

Hence $c_j(T_2^3) = c_j(T_1^3) + b_j$ for $0 \le j \le n$. It follows that $c_j(T_2^3) = c_j(T_1^3)$ if $j = 0, 1, n-1, n$ and $c_j(T_2^3) > c_j(T_1^3)$ if $2 \le j \le n$. Thus we have $T_1^3 \prec T_2^3$. Note that

$$\begin{aligned}
&\phi(L(T_3^3)) - \phi(L(T_2^3)) \\
&= x^2(x^2-3x+1)^{i-7}(x-1)^{n-2i}[(n-i-2)x^{10} - (18n-18i-38)x^9 \\
&\quad + (137n-136i-311)x^8 - (576n-561i-1439)x^7 \\
&\quad + (1467n-1376i-4147)x^6 - (2340n-2052i-7720)x^5 \\
&\quad + (2347n-1835i-9310)x^4 - (1458n-942i-7102)x^3 \\
&\quad + (539n-252i-3249)x^2 - (108n-27i-801)x + (9n-81)], \\
&\phi(L(T_4^3)) - \phi(L(T_3^3)) \\
&= x^2(x^2-3x+1)^{i-8}(x-1)^{n-2i}[(n-3-i)x^{12} - (22n-22i-68)x^{11} \\
&\quad + (209n-208i-671)x^{10} - (1126n-1107i-3794)x^9 \\
&\quad + (3802n-3651i-13620)x^8 - (8406n-7752i-32520)x^7 \\
&\quad + (12385n-10696i-52659)x^6 - (12202n-9517i-57998)x^5 \\
&\quad + (7994n-5353i-43016)x^4 - (3418n-1824i-20960)x^3 \\
&\quad + (913n-342i-6387)x^2 - (138n-27i-1098)x + (9n-81)].
\end{aligned}$$

Similarly, we have $T_2^3 \prec T_3^3 \prec T_4^3$. So we have $T_1^3 \prec T_2^3 \prec T_3^3 \prec T_4^3$. ■

Theorem 3.3. *For $G \in \mathscr{G}_{n,n+2}^3(i)$, $c_k(G) \ge c_k(T_1^3), k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. Let G^* be the minimal element in $\mathscr{G}_{n,n+2}^3(i)$ under the partial order $\preceq$. Now we only need to prove $G^* \cong T_1^3$.

Let $M(G^*)$ be a maximum matching of G^* containing the most pendent edges. By Lemma 3.1, we have $\widehat{G^*} \cong G_1^3$ or $\widehat{G^*} \cong G_7^3$ and $a = b = c = 3$.

Case 1. If $\widehat{G^*} \cong G_1^3$, let $H = C_b = xyz$, G_1 be the component of $G^* - \{xy, xz, yz\}$ containing y and G_2 be the component of $G^* - \{xy, xz, yz\}$ containing x. If there exist pendent edges at x, by the choice of $M(G^*)$, we know that there is a pendent edge xx' belonging to $M(G^*)$; let $M'(G^*) = M(G^*) - xx' + xz$. By an α_3-transform of G^* from x to y, we can obtain a graph $\widetilde{G}$. Obviously, $N_H(x) - \{y\} \subseteq N_H(y) - \{x\}$, by Lemma 2.7, we have $\widetilde{G} \prec G^*$, it is contradict to the choice of G^*.

Case 2. If $\widehat{G^*} \cong G_7^3$, then $G^* \cong T_i^3$ for some $i \in \{1, 2, 3, 4\}$ (as shown in Figure 5). Further by Lemma 3.2, we have $G^* \cong T_1^3$. ■

Lemma 3.4. *Let T_i^4 $(i = 1, 2, \ldots, 8)$ be the graphs as shown in Figure 6. Then $T_2^4 \prec T_i^4$ for $i = 1, 3, \ldots, 8$.*

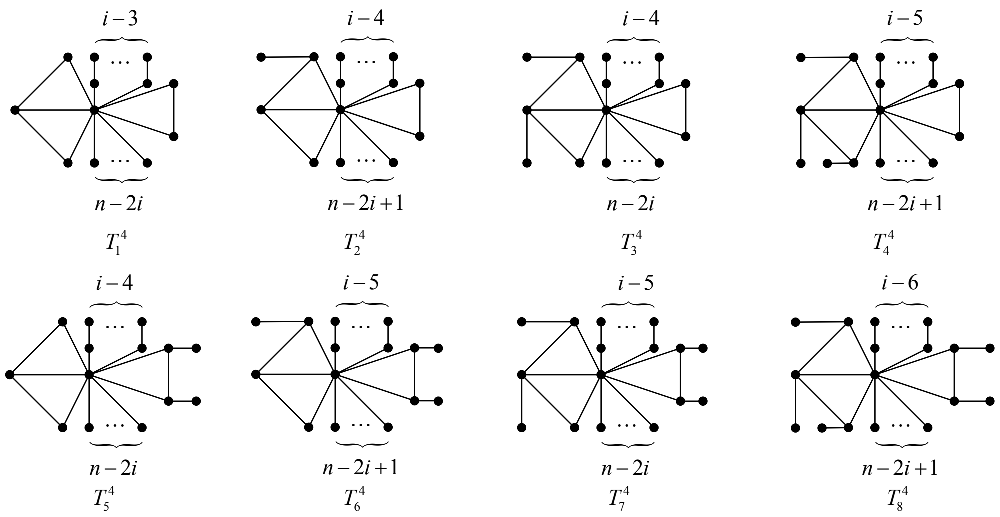

Figure 6. The graphs T_i^4 $(i = 1, 2, \ldots, 8)$.

Proof. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\phi(L(T_2^4)) =\ & x(x^2-3x+1)^{i-5}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(168-584x\\ &+728x^2-424x^3+125x^4-18x^5+x^6)\\ &-(n-2i+1)(x^2-3x+1)(24-113x+194x^2-158x^3+65x^4\\ &-13x^5+x^6)-(i-4)(x-2)(x-1)(24-113x+194x^2-158x^3\\ &+65x^4-13x^5+x^6)]\\ =\ & x(x^2-3x+1)^{i-5}(x-1)^{n-2i}h(x)\end{aligned} \tag{2}$$

where

$$\begin{aligned}h(x) =\ & (x-1)(x^2-3x+1)(168-584x+728x^2-424x^3+125x^4-18x^5+x^6)\\ &-(n-2i+1)(x^2-3x+1)(24-113x+194x^2-158x^3\\ &+65x^4-13x^5+x^6)-(i-4)(x-2)(x-1)(24-113x+194x^2\\ &-158x^3+65x^4-13x^5+x^6).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\phi(L(T_1^4))\\ &=(x^2-3x+1)^{i-4}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)\\ &\quad(-144x+324x^2-260x^3+95x^4-16x^5+x^6)\\ &\quad-(n-2i)x(x^2-3x+1)(x^5-11x^4+45x^3-85x^2+74x-24)\\ &\quad-(i-3)(x^2-2x)(x-1)(x^5-11x^4+45x^3-85x^2+74x-24)],\end{aligned}$$

$$
\begin{aligned}
&\phi(L(T_3^4))\\
&= (x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(-192x+889x^2\\
&\quad -1574x^3+1366x^4-632x^5+158x^6-20x^7+x^8)-(n-2i)x(x^2-3x+1)\\
&\quad (-24+149x-353x^2+414x^3-260x^4+88x^5-15x^6+x^7)\\
&\quad -(i-4)(x^2-2x)(x-1)(-24+149x-353x^2+414x^3-260x^4\\
&\quad +88x^5-15x^6+x^7)],\\
&\phi(L(T_4^4))\\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(216x-1284x^2+3026x^3\\
&\quad -3634x^4+2411x^5-914x^6+196x^7-22x^8+x^9)-(n-2i+1)x\\
&\quad (x^2-3x+1)(24-188x+582x^2-924x^3+817x^4-411x^5\\
&\quad +116x^6-17x^7+x^8)-(i-5)(x^2-2x)(x-1)(24-188x+582x^2-924x^3\\
&\quad +817x^4-411x^5+116x^6-17x^7+x^8)],\\
&\phi(L(T_5^4))\\
&= (x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(-192x+920x^2-1646x^3\\
&\quad +1413x^4-644x^5+159x^6-20x^7+x^8)-(n-2i)x(x^2-3x+1)\\
&\quad (-24+154x-369x^2+431x^3-267x^4+89x^5-15x^6+x^7)-(i-4)\\
&\quad (x^2-2x)(x-1)(-24+154x-369x^2+431x^3-267x^4+89x^5-15x^6+x^7)],\\
&\phi(L(T_6^4))\\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(216x-1338x^2+3184x^3\\
&\quad -3792x^4+2481x^5-928x^6+197x^7-22x^8+x^9)\\
&\quad -(n-2i+1)x(x^2-3x+1)(24-193x+608x^2-968x^3+847x^4-420x^5\\
&\quad +117x^6-17x^7+x^8)-(i-5)(x^2-2x)(x-1)(24-193x+608x^2\\
&\quad -968x^3+847x^4-420x^5+117x^6-17x^7+x^8)],\\
&\phi(L(T_7^4))\\
&= (x^2-3x+1)^{i-6}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(-240x+1795x^2\\
&\quad -5354x^3+8332x^4-7436x^5+3959x^6-1268x^7+238x^8-24x^9+x^{10})\\
&\quad -(n-2i)x(x^2-3x+1)(-24+229x-887x^2+1810x^3-2124x^4+1479x^5\\
&\quad -614x^6+148x^7-19x^8+x^9)\\
&\quad -(i-5)(x^2-2x)(x-1)(-24+229x-887x^2+1810x^3-2124x^4\\
&\quad +1479x^5-614x^6+148x^7-19x^8+x^9)],\\
&\phi(L(T_8^4))\\
&= (x^2-3x+1)^{i-7}(x-1)^{n-2i}[(x-1)(x^2-3x+1)(88x-900x^2+3762x^3\\
&\quad -8370x^4+10891x^5-8646x^6+4270x^7-1308x^8+240x^9\\
&\quad -24x^{10}+x^{11})-(n-2i+1)x(x^2-3x+1)
\end{aligned}
$$

$$(8-100x+522x^2-1480x^3+2491x^4-2571x^5+1640x^6-643x^7+150x^8 - 19x^9+x^{10})-(i-6)(x^2-2x)(x-1)(8-100x+522x^2 - 1480x^3+2491x^4-2571x^5+1640x^6-643x^7+150x^8-19x^9+x^{10})].$$

Then

$$\begin{aligned}
&\phi(L(T_1^4))-\phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[x^7-(n+14-i)x^6\\
&\quad +(11n+80-11i)x^5-(47n+235-46i)x^4+(98n+365-90i)x^3\\
&\quad -(103n+272-81i)x^2+(51n+66-27i)x-(9n-9)],\\
&\phi(L(T_3^4))-\phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(n-i-2)x^7-(13n-13i-26)x^6\\
&\quad +(68n-67i-140)x^5-(183n-173i-406)x^4+(269n-232i-686)x^3\\
&\quad -(212n-150i-672)x^2+(82n-36i-348)x-(12n-72)],\\
&\phi(L(T_4^4))-\phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i}[(n-i-3)x^8-(14n-14i-46)x^7\\
&\quad +(79n-78i-295)x^6-(230n-219i-1026)x^5+(368n-2094-323i)x^4\\
&\quad -(322n-2528-238i)x^3+(149n-1727-78i)x^2-(34n-600-9i)x\\
&\quad +(3n-81)],\\
&\phi(L(T_5^4))-\phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(n-i-1)x^7-(14n-14i-14)x^6\\
&\quad +(78n-77i-81)x^5-(222n-257-211i)x^4+(343n-491-299i)x^3\\
&\quad -(282n-203i-561)x^2+(113n-340-51k)x-(17n-81)],\\
&\phi(L(T_6^4))-\phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i}[(n-i-2)x^8-(15n-32-15i)x^7\\
&\quad +(91n-90i-213)x^6-((288n-276i-768)x^5+(511n-457i-1627)x^4\\
&\quad -(510n-396i-2042)x^3+(276n-161i-1449)x^2-(75n-24i-520)x\\
&\quad +(8n-72)],\\
&\phi(L(T_7^4))-\phi(L(T_2^4))\\
&= x^2(x^2-3x+1)^{i-6}(x-1)^{n-2i-1}[(2n-2i-5)x^9-(33n-33i-85)x^8\\
&\quad +(226n-224i-608)x^7-(836n-809i-2394)x^6\\
&\quad +(1812n-1678i-5692)x^5-(2394n-2014i-8424)x^4\\
&\quad +(1883n-1345i-7709)x^3-(855n-455i-4183)x^2\\
&\quad +(205n-60i-1216)x-(20n-144)],
\end{aligned}$$

$$\begin{aligned}
&\phi(L(T_8^4)) - \phi(L(T_2^4))\\
&= x(x^2-3x+1)^{i-7}(x-1)^{n-2i}[2x^{12} - 48x^{11} + (4n-4i+492)x^{10}\\
&\quad - (66n+2846-66i)x^9 + (462n+10302-458i)x^8\\
&\quad - (1789n+24395-1735i)x^7 + (4195n-3899i+38323)x^6\\
&\quad - (6150n-5303i+39656)x^5 + (5659n-4301i+26317)x^4\\
&\quad - (3228n+10626-1999i)x^3 + (1101n-487i+2351)x^2\\
&\quad - (205n+217-48i)x + 16n].
\end{aligned}$$

Similarly to the procedure of Lemma 3.2, we have $T_2^4 \prec T_i^4$ for $i = 1, 3, \dots, 8$. ■

Theorem 3.5. *For $G \in \mathscr{G}_{n,n+2}^4(i)$, $c_k(G) \geq c_k(T_2^4)$, $k = 0, 1, \dots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. Let G^* be the minimal element in $\mathscr{G}_{n,n+2}^4(i)$ under the partial order $\preceq$. Repeated by Lemmas 2.7 and 3.1, we have $G^* \cong T_i^4$ for some $i \in \{1, 2, \dots, 8\}$. Further by Lemma 3.4, we have our desirable results. ■

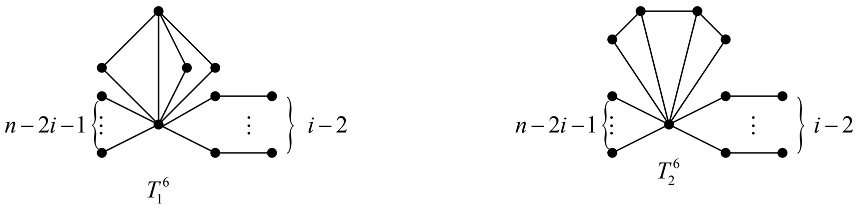

Figure 7. The graphs T_i^6 $(i = 1, 2)$.

Lemma 3.6. *Let T_i^6 $(i = 1, 2)$ be the graphs as shown in Figure 7. Then $T_2^6 \prec T_1^6$.*

Proof. By direct calculation, we have

$$\begin{aligned}
\phi(L(T_1^6)) = {}& (x^2-3x+1)^{i-3}(x-1)^{n-2i-2}[(x-1)(x^2-3x+1)(x^5-14x^4\\
&+ 69x^3 - 140x^2 + 100x)\\
&- (n-2i-1)x(x^2-3x+1)(x^4-10x^3+33x^2-44x+20)\\
&- (i-2)(x^2-2x)(x-1)(x^4-10x^3+33x^2-44x+20)]
\end{aligned}$$

and

$$\begin{aligned}
\phi(L(T_2^6)) = {}& (x^2-3x+1)^{i-3}(x-1)^{n-2i-2}[(x-1)(x^2-3x+1)(x^5-14x^4\\
&+ 70x^3 - 146x^2 + 105x)\\
&- (n-2i-1)x(x^2-3x+1)(-10x^3+x^4+34x^2-46x+21)\\
&- (i-2)(x^2-2x)(x-1)(x^4-10x^3+34x^2-46x+21)].
\end{aligned} \tag{3}$$

Then

$$\begin{aligned}&\phi(L(T_1^6)) - \phi(L(T_2^6))\\&= x^2(x^2-3x+1)^{i-3}(x-1)^{n-2i-2}[x^8 - 19x^7 + 148x^6 - 613x^5 + 1465x^4\\&\quad - (i+2050)x^3 + (5i+1622)x^2 - (7i+652)x + (3i+98)].\end{aligned}$$

Hence $T_2^6 \prec T_1^6$. ■

Theorem 3.7. *For $G \in \mathscr{G}^6_{n,n+2}(i)$, $c_k(G) \geq c_k(T_1^6)$, $k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. Let G^* be the minimal element in $\mathscr{G}^6_{n,n+2}(\beta)$ under the partial order $\preceq$. Repeated by Lemmas 2.7 and 3.1, we have $G^* \cong T_i^6$ for some $i \in \{1, 2\}$. Further by Lemma 3.6, we have our desirable results. ■

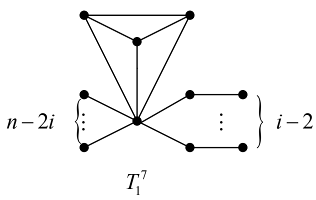

Figure 8. The graph T_1^7.

Theorem 3.8. *For $G \in \mathscr{G}^7_{n,n+2}(i)$, $c_k(G) \geq c_k(T_1^7)$, $k = 0, 1, \ldots, n$. The equality holds if and only if $k \in \{0, n-1, n\}$.*

Proof. By Lemma 3.1, it is easy to obtain our desirable results. ■

Theorem 3.9. *T_1^3, T_2^4, T_1^7 are the only three minimal elements in the partial set $(\mathscr{G}_{n,n+2}(i), \preceq)$.*

Proof. For any graph $G \in \mathscr{G}_{n,n+2}(i)$, by Theorems 3.3, 3.5, 3.7 and 3.8, we have

$$c_k(G) \geq \min\{c_k(T_1^3), c_k(T_2^4), c_k(T_2^6), c_k(T_1^7)\}$$

for $k = 0, 1, \ldots, n$. By direct calculation, we have

$$\begin{aligned}\phi(L(T_1^7)) &= x(x^2-3x+1)^{i-3}(x-1)^{n-2i-1}[(x-1)(x^2-3x+1)(x^3-12x^2\\&\quad + 48x - 64) - (n-2i)(x^2-3x+1)(x^3-9x^2+24x-16)\\&\quad - (i-2)(x-2)(x-1)(x^3-9x^2+24x-16)]\\&= x(x^2-3x+1)^{i-3}(x-1)^{n-2i-1}r(x),\end{aligned} \tag{4}$$

where

$$\begin{aligned} r(x) = &(x-1)(x^2-3x+1)(x^3-12x^2+48x-64) \\ &-(n-2i)(x^2-3x+1)(x^3-9x^2+24x-16) \\ &-(i-2)(x-2)(x-1)(x^3-9x^2+24x-16). \end{aligned}$$

By equations (3) and (4), we have

$$\begin{aligned} &\phi(L(T_2^6))-\phi(L(T_1^7)) \\ = &\ x(x^2-3x+1)^{i-3}(x-1)^{n-2i-1}[5n-(16n-15i+35)x \\ &+(8n-8i+32)x^2-(n-i+10)x^3+x^4], \end{aligned}$$

hence $T_1^7 \prec T_2^6$.

Further by equations (1)–(4), we have

$$\begin{aligned} &\phi(L(T_2^4))-\phi(L(T_1^3)) \\ &= x(x^2-3x+1)^{i-5}(x-1)^{n-2i}[(12+3n-3i)-(4n-4i-453)x \\ &\quad -(35n-35i+1928)x^2+(96n-96i-1871)x^3-(97n-97i+352)x^4 \\ &\quad +(47n-47i-68)-(11n-11i-13)x^6+(n-i-1)x^7], \\ &\phi(L(T_2^4))-\phi(L(T_1^7)) \\ &= x(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[(432-40n)+(257n-120i-3047)x \\ &\quad -(654n-451i-9277)x^2+(905n-746i+15877)x^3 \\ &\quad -(745n-680i-16666)x^4+(367n-354i-11128)x^5 \\ &\quad -(105n-104i-4803)x^6+(16n-16i-1336)x^7-(n-i-232)x^8 \\ &\quad -23x^9+x^{10}], \\ &\phi(L(T_1^7))-\phi(L(T_1^3)) \\ &= x(x^2-3x+1)^{i-5}(x-1)^{n-2i-1}[-11n+(48n+63i-488)x \\ &\quad -(6n+552i-2867)x^2-(243n-1605i+5540)x^3 \\ &\quad +(446n-2201i+49200)x^4-(344n-1622i+2453)x^5 \\ &\quad +(134n-679i+902)x^6-(26n-161i+240)x^7 \\ &\quad +(2n-20i+34)x^8-(2-i)x^9]. \end{aligned}$$

Obviously, T_1^3, T_2^4, T_1^7 are incomparable, thus we obtain our desirable results. ■

4. The Laplacian-Like Energy of Tricyclic Graphs with Prescribed Matching Number

Let G be a graph. The Laplacian matrix $L(G)$ has non-negative eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. The *Laplacian-like energy* of graph G,

$LEL(G)$ for short, is defined as follows:

$$LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k(G)}.$$

Stevanović [11] proved a connection between Laplacian-like energy and Laplacian coefficients of a graph G.

Theorem 4.1 [11]**.** *Let G and H be two n-vertex graphs. If $c_k(G) \leq c_k(H)$ for $k = 1, 2, \ldots, n-1$, then $LEL(G) \leq LEL(H)$. Furthermore, if a strict inequality $c_k(G) < c_k(H)$ holds for some $1 \leq k \leq n-1$, then $LEL(G) < LEL(H)$.*

By Theorems 3.9 and 4.1, we have the following result.

Theorem 4.2. *For $G \in \mathscr{G}_{n,n+2}(i)$, we have $LEL(G) \geq \min\{LEL(T_1^3), LEL(T_2^4), LEL(T_1^7)\}$. The equality holds if and only if $G \cong T_1^3$, $G \cong T_2^4$ or $G \cong T_1^7$.*

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Received 27 November 2015
Revised 1 April 2016
Accepted 4 May 2016