Discussiones Mathematicae
Graph Theory 37 (2017) 523–535
doi:10.7151/dmgt.1936

C_7-DECOMPOSITIONS OF THE TENSOR PRODUCT OF COMPLETE GRAPHS

R.S. Manikandan

Department of Mathematics
Bharathidasan University Constituent College
Lalgudi-621 601, India

e-mail: manirs2004@yahoo.co.in

and

P. Paulraja

Department of Mathematics
Kalasalingam University
Krishnankoil-626 126, India

e-mail: ppraja56@gmail.com

Abstract

In this paper we consider a decomposition of $K_m \times K_n$, where $\times$ denotes the tensor product of graphs, into cycles of length seven. We prove that for $m, n \geq 3$, cycles of length seven decompose the graph $K_m \times K_n$ if and only if (1) either m or n is odd and (2) $14 \mid m(m-1)n(n-1)$. The results of this paper together with the results of [C_p-*Decompositions of some regular graphs*, Discrete Math. **306** (2006) 429–451] and [C_5-*Decompositions of the tensor product of complete graphs*, Australasian J. Combinatorics **37** (2007) 285–293], give necessary and sufficient conditions for the existence of a p-cycle decomposition, where $p \geq 5$ is a prime number, of the graph $K_m \times K_n$.

Keywords: cycle decomposition, tensor product.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

All graphs considered here are simple and finite. Let C_n denote the cycle of length n. We write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ if $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$. If the edge set of the graph G can be partitioned into cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_r}$, then we say that $C_{n_1}, C_{n_2}, \ldots, C_{n_r}$ *decompose* G. If $n_1 = n_2 = \cdots = n_r = k$, then we say that G has a C_k-*decomposition* and in this case we write $C_k \mid G$. We may also call a cycle of length k a k-*cycle*. If G has a 2-factorization and each 2-factor of it has only cycles of length k, then we say that G has a C_k-*factorization* (we use the notation $C_k \| G$.) The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. For some positive integer k, the graph kH denotes k disjoint copies of H.

For two graphs G and H their *wreath product*, $G * H$, has the vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Similarly, $G \times H$, the *tensor product* of the graphs G and H has the vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Clearly the tensor product is distributive over edge-disjoint union of graphs; that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. For $h \in V(H)$, $V(G) \times h = \{(v, h) | v \in V(G)\}$ is called the *column* of vertices in $G \times H$ corresponding to h. Further, for $x \in V(G)$, $x \times V(H) = \{(x, v) \mid v \in V(H)\}$ is called the *layer* of vertices in $G \times H$ corresponding to x. Similarly we can define column and layer for wreath product of graphs also. We can easily observe that $K_m * \overline{K}_n$ is isomorphic to the complete m-partite graph in which each partite set has exactly n vertices.

A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \ldots, n\}$, such that each row and each column of the array contains each of the symbols in $\{1, 2, \ldots, n\}$ exactly once. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i, $1 \le i \le n$. Let (X, Y) be the bipartition of the complete bipartite graph $K_{n,n}$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Let $F_i(X, Y) = \{x_1y_{1+i}, x_2y_{2+i}, \ldots, x_ny_{n+i}\}$, $0 \le i \le n-1$, where the addition in the suffixes are taken modulo n with residues $1, 2, \ldots, n$. We call $F_i(X, Y)$ as the 1-factor of distance i from X to Y in $K_{n,n}$. Note that, in general, $F_i(X, Y)$ need not be equal to $F_i(Y, X)$ as $F_i(Y, X) = \{y_1x_{1+i}, y_2x_{2+i}, \ldots, x_ny_{n+i}\}$. From the definition of the tensor product and the wreath product, it is clear that $K_m \times K_n = K_m * \overline{K}_n - \bigcup_{i \ne j} F_0(X_i, X_j)$, where X_j's are the partitite sets of the complete m-partite graph in which each of the partite sets has cardinality n. In fact, $\bigcup_{i \ne j} F_0(X_i, X_j)$ consists of n disjoint copies of K_m.

It is known that if n is odd and $m \mid \binom{n}{2}$ or n is even and $m \mid \left(\binom{n}{2} - \frac{n}{2}\right)$, then $C_m \mid K_n$ or $C_m \mid K_n - I$, where I is a 1-factor of K_n; see [1, 13]. A similar problem can also be considered for regular complete multipartite graphs; Billington and Cavenagh [6] and Mahamoodian and Mirzakhani [9] have considered C_5-decompositions of complete tripartite graphs. Moreover, Billington [3] has studied the decompositions of complete tripartite graphs into cycles of length 3 and 4. Further, Billington and Cavenagh [5] have studied the decompositions of complete multipartite graphs into cycles of length 4, 6 and 8.

The present authors, in [10, 11], have proved that the necessary conditions for the existence of a C_5-decomposition and a C_p-decomposition for $p \geq 11$, where p is a prime number, of $K_m * \overline{K}_n$ are sufficient. Smith, in [14, 15, 16], proved that the obvious necessary conditions for the existence of a C_k-decomposition of $K_m * \overline{K}_n$ are sufficient when $k = 2p, 3p$ and p^2, where $p \geq 3$ is a prime number. Later, in [17], he has obtained the conditions under which $\lambda(K_m * \overline{K}_n)$ can be decomposed into cycles of length p, where p is a prime; his approach is different from [10, 11]. For related work see also [4]. Liu, in [8], studied the C_t-factorization of $K_m * \overline{K}_n$, $m \geq 3$.

One can easily observe that the graph $K_m \times K_n$ is obtained from the regular complete multipartite graph $K_m * \overline{K}_n$, by deleting a suitable set of n disjoint copies of K_m.

In this paper, the obvious necessary conditions for $K_m \times K_n$, $m, n \geq 3$, to admit a C_7-decomposition are proved to be sufficient. In this context, it is pertinent to point out that the existence of C_p, p being a prime, decomposition of $K_m \times K_n$ played a significant role in establishing the existence of C_p-decomposition of $K_m * \overline{K}_n$, see [10, 11]. We give below the main theorem obtained here.

Theorem 1. *For m, $n \geq 3$, $C_7 \mid K_m \times K_n$ if and only if*

(1) $14 \mid nm(m-1)(n-1)$, *and*

(2) *either m or n is odd.*

For our future reference we list below some known results.

Theorem A [2]**.** *Let s be an odd integer and t be a prime so that $3 \leq s \leq t$. Then $C_s * \overline{K}_t$ has a 2-factorization so that each 2-factor is composed of s cycles of length t.*

Theorem B [1]**.** *If $n \equiv 1$ or $7 \pmod{14}$, then $C_7 \mid K_n$.*

Theorem C [7]**.** *Let m be an odd integer, $m \geq 3$.*

(1) *If $m \equiv 1$ or $3 \pmod 6$, then $C_3 \mid K_m$.*

(2) *If $m \equiv 5 \pmod 6$, then K_m can be decomposed into $(m(m-1) - 20)/6$ 3-cycles and two 5-cycles.*

2. C_7-DECOMPOSITIONS OF $C_3 \times K_m$

We quote the following lemma for our future reference.

Lemma 2 [10]**.** *For any odd integer* $t \geq 3$, $C_t \,||\, C_3 \times K_t$.

Lemma 3. $C_7 \mid C_3 \times K_8$.

Proof. Let the partite sets of the tripartite graph $C_3 \times K_8$ be $\{u_1, u_2, \ldots, u_8\}$, $\{v_1, v_2, \ldots, v_8\}$ and $\{w_1, w_2, \ldots, w_8\}$, where we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Now the cycle $(u_1v_3w_6v_2w_7v_1w_8u_1)$ under the permutation $(u_1u_2\cdots u_8)(v_1v_2\cdots v_8)$ $(w_1w_2\cdots w_8)$ and its powers give us eight 7-cycles. These eight 7-cycles under the permutation $(u_1v_1w_1)(u_2v_2w_2)\cdots(u_8v_8w_8)$ and its powers give us the required twenty four 7-cycles. ■

Remark 4. Let the partite sets of the complete tripartite graph $C_3 * \overline{K}_m$, $m \geq 1$, be $\{u_1, u_2, \ldots, u_m\}$, $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_m\}$. Consider a latin square $\mathcal{L}$ of order m. We associate a triangle of $C_3 * \overline{K}_m$ with each entry of $\mathcal{L}$ as follows: if k is the $(i,j)^{\text{th}}$ entry of $\mathcal{L}$, then the triangle of $C_3 * \overline{K}_m$ corresponding to k is $(u_iv_jw_ku_i)$. Clearly the triangles corresponding to the entries of $\mathcal{L}$ decompose $C_3 * \overline{K}_m$, see [3].

The necessary condition for the existence of decomposition of $C_3 \times K_m$, $m \geq 3$, into C_7 is $m \equiv 0$ or $1 \pmod 7$. We prove that it is also sufficient.

Theorem 5. $C_7 \mid C_3 \times K_m$ *if and only if* $m \equiv 0$ *or* $1 \pmod 7$.

Proof. The necessity is obvious. We prove the sufficiency in two cases.

Case 1. $m \equiv 1 \pmod 7$. Let $m = 7k + 1$.

Subcase 1.1. $k \neq 2$. Let partite sets of the tripartite graph $C_3 \times K_m$ be $U = \{u_0\} \cup \left(\bigcup_{i=1}^{k}\{u_1^i, u_2^i, \ldots, u_7^i\}\right)$, $V = \{v_0\} \cup \left(\bigcup_{i=1}^{k}\{v_1^i, v_2^i, \ldots, v_7^i\}\right)$ and $W = \{w_0\} \cup \left(\bigcup_{i=1}^{k}\{w_1^i, w_2^i, \ldots, w_7^i\}\right)$; we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. By the definition of the tensor product,$\{u_0, v_0, w_0\}$ and $\{u_j^i, v_j^i, w_j^i\}$, $1 \leq j \leq 7$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$ and $W \cup U$ is isomorphic to $K_{m,m} - F_0$, where F_0 is the 1-factor of distance zero in $K_{m,m}$.

We obtain a new graph out of $H = (C_3 \times K_m) - \{u_0, v_0, w_0\} \cong C_3 \times K_{7k}$ as follows: for each i, $1 \leq i \leq k$, identify the sets of vertices $\{u_1^i, u_2^i, \ldots, u_7^i\}$, $\{v_1^i, v_2^i, \ldots, v_7^i\}$ and $\{w_1^i, w_2^i, \ldots, w_7^i\}$ into new vertices u^i, v^i and w^i respectively; two new vertices are adjacent if and only if the corresponding sets of vertices in H induce a complete bipartite subgraph $K_{7,7}$ or a $K_{7,7} - F$, where F is a 1-factor of $K_{7,7}$. This defines the graph $C_3 * \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$ and $\{w^1, w^2, \ldots, w^k\}$.

Consider an idempotent latin square $\mathcal{L}$ of order k, $k \neq 2$ (which exists, see [7]). To complete the proof of this subcase, we associate with entries of $\mathcal{L}$ edge-disjoint subgraphs of $C_3 * \overline{K}_m$ which are decomposable by C_7. The i^{th} diagonal entry of $\mathcal{L}$ corresponds to the triangle $(u^i v^i w^i u^i)$, $1 \leq i \leq k$, of $C_3 * \overline{K}_k$, see Remark 4. The subgraph of H corresponding to the triangle of $C_3 * \overline{K}_k$ is isomorphic to $C_3 \times K_7$. For each triangle $(u^i v^i w^i u^i)$, $1 \leq i \leq k$, of $C_3 * \overline{K}_k$ corresponding to the i^{th} diagonal entry of $\mathcal{L}$, associate the subgraph of $C_3 \times K_m$ induced by vertices $\{u_0, u_1^i, u_2^i, \ldots, u_7^i\} \cup \{v_0, v_1^i, v_2^i, \ldots, v_7^i\} \cup \{w_0, w_1^i, w_2^i, \ldots, w_7^i\}$; as this subgraph is isomorphic to $C_3 \times K_8$, it can be decomposed into 7-cycles, by Lemma 3. Again, if we consider the subgraph of H corresponding to the triangle of $C_3 * \overline{K}_k$, which corresponds to a non-diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 * \overline{K}_7$. By Theorem A, $C_3 * \overline{K}_7$ can be decomposed into 7-cycles. Thus we have decomposed $C_3 \times K_m$ into 7-cycles when $k \neq 2$.

Subcase 1.2. $k = 2$. By Theorem B, $C_7 \mid K_{15}$ and hence we write $C_3 \times K_{15} \cong K_{15} \times C_3 = (C_7 \times C_3) \oplus (C_7 \times C_3) \oplus \cdots \oplus (C_7 \times C_3)$. Each copy of $C_7 \times C_3$ can be decomposed into 7-cycles, see Figure 1. This proves that $C_7 \mid C_3 \times K_{15}$.

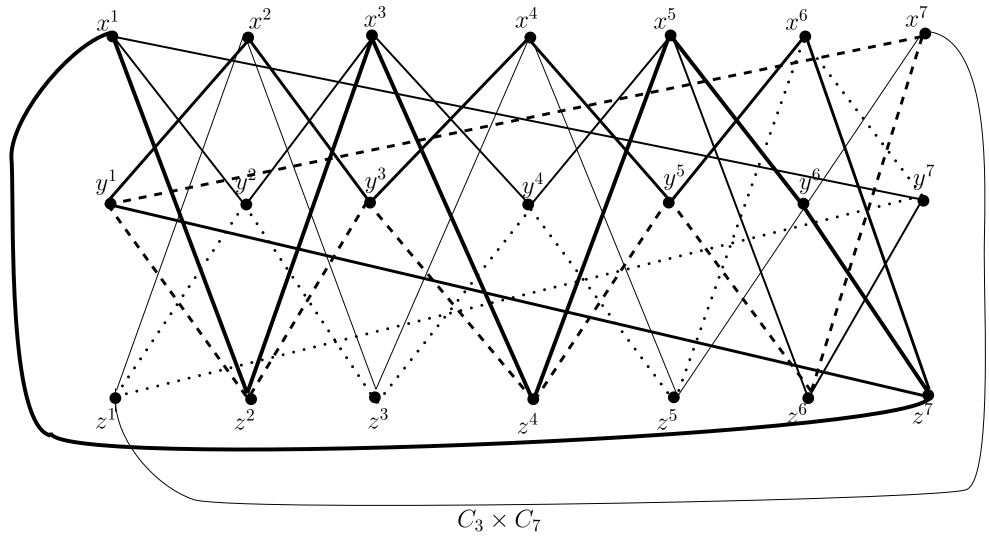

Figure 1. A 7-cycle decomposition of $C_3 \times C_7$. Different types of edges give different 7-cycles.

Case 2. $m \equiv 0 \pmod 7$. Let $m = 7k$.

Subcase 2.1. $k \neq 2$. As in the previous case, let the partite sets of the tripartite graph $C_3 \times K_m$ be $U = \bigcup_{i=1}^{k}\{u_1^i, u_2^i, \ldots, u_7^i\}$, $V = \bigcup_{i=1}^{k}\{v_1^i, v_2^i, \ldots, v_7^i\}$ and $W = \bigcup_{i=1}^{k}\{w_1^i, w_2^i, \ldots, w_7^i\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, from $C_3 \times K_m = C_3 \times K_{7k}$ we obtain the graph $C_3 * \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}, \{v^1, v^2, \ldots, v^k\}$ and $\{w^1, w^2, \ldots, w^k\}$.

Consider an idempotent latin square $\mathcal{L}$ of order k, $k \neq 2$. The diagonal entries of $\mathcal{L}$ correspond to the triangles $(u^i v^i w^i u^i)$, $1 \leq i \leq k$, of $C_3 * \overline{K}_k$. If we consider the subgraph of $C_3 \times K_m$ corresponding to a triangle of $C_3 * \overline{K}_k$, which corresponds to a diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 \times K_7$. By Lemma 2, $C_7 \mid C_3 \times K_7$. Again, as in the previous case, the triangle of $C_3 * \overline{K}_k$ corresponding to a non-diagonal entry of $\mathcal{L}$ gives a subgraph of $C_3 \times K_m$ isomorphic to $C_3 * \overline{K}_7$; by Theorem A, $C_7 \mid C_3 * \overline{K}_7$.

Subcase 2.2. $k = 2$. Let the partite sets of the tripartite graph $C_3 \times K_{14}$ be $X = \{x_1, x_2, \ldots, x_{14}\}, Y = \{y_1, y_2, \ldots, y_{14}\}$ and $Z = \{z_1, z_2, \ldots, z_{14}\}$; we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. By the definition of the tensor product, $\{x_i, y_i, z_i\}$, $1 \leq i \leq 14$, are independent sets and the subgraph induced by each of the subsets of vertices $X \cup Y$, $Y \cup Z$ and $Z \cup X$ are isomorphic to $K_{14,14} - F_0$, where F_0 is the 1-factor of distance zero in $K_{14,14}$.

We obtain a new graph out of $C_3 \times K_{14}$ as follows: for each i, $1 \leq i \leq 7$, identify the subsets of vertices $\{x_{2i-1}, x_{2i}\}, \{y_{2i-1}, y_{2i}\}$ and $\{z_{2i-1}, z_{2i}\}$ into new vertices x^i, y^i and z^i, respectively, and two of these vertices are adjacent if and only if the corresponding subsets of vertices in $C_3 \times K_{14}$ induce a $K_{2,2}$. The resulting graph is isomorphic to $C_3 \times K_7$ with partite sets $X' = \{x^1, x^2, \ldots, x^7\}, Y' = \{y^1, y^2, \ldots, y^7\}$ and $Z' = \{z^1, z^2, \ldots, z^7\}$; note that $\{x^i, y^i, z^i\}$, $1 \leq i \leq 7$, are independent sets of $C_3 \times K_7$. Now $C_3 \times K_7 \cong C_3 \times (C_7 \oplus C_7 \oplus C_7) = (C_3 \times C_7) \oplus (C_3 \times C_7) \oplus (C_3 \times C_7)$. The graph $C_3 \times C_7$ can be decomposed into 7-cycles, see Figure 1, and hence $C_7 \mid C_3 \times K_7$.

By "lifting back" these 7-cycles of $C_3 \times K_7$ to $C_3 \times K_{14}$, we get edge-disjoint subgraphs isomorphic to $C_7 * \overline{K}_2$. But $C_7 * \overline{K}_2$ can be decomposed into cycles of length 7, see [12]. Thus the subgraphs of $C_3 \times K_{14}$ obtained by "lifting back" the 7-cycles of $C_3 \times K_7$ to $C_3 \times K_{14}$ can be decomposed into cycles of length 7. The edges of $C_3 \times K_{14}$ which are not covered by these 7-cycles are shown in Figure 2. To complete the proof we fuse some of the 7-cycles obtained above with the graph of Figure 2 and decompose the resulting graph into cycles of length 7. Let H' be the graph obtained by the union of the graph of Figure 2 and the subgraph of $C_3 \times K_{14}$ which is obtained by "lifting back" two 7-cycles of $C_3 \times K_7$, namely, $(x^1 y^2 x^3 y^4 x^5 z^6 y^7 x^1)$ and $(z^1 y^2 z^3 y^4 z^5 x^6 y^7 z^1)$ shown in Figure 1.

The subgraph H' of $C_3 \times K_{14}$ is shown in Figure 3. A 7-cycle decomposition of H' is given below:

$(x_1 y_2 z_1 x_2 y_4 z_6 y_3 x_1)$, $(x_1 z_2 y_1 x_2 y_3 z_5 y_4 x_1)$, $(x_5 y_6 z_5 y_8 z_9 y_7 z_6 x_5)$,
$(x_6 y_5 z_6 y_8 z_{10} y_7 z_5 x_6)$, $(x_9 y_{10} z_9 x_{11} y_{14} x_{12} z_{10} x_9)$, $(x_{10} y_9 z_{10} x_{11} y_{13} x_{12} z_9 x_{10})$,
$(x_{14} y_{13} z_1 y_4 z_2 y_{14} z_{13} x_{14})$, $(x_{13} y_{14} z_1 y_3 z_2 y_{13} z_{14} x_{13})$, $(x_3 y_4 x_6 y_8 x_5 y_3 z_4 x_3)$,

$(x_4z_3y_4x_5y_7x_6y_3x_4)$, $(x_7y_8x_{10}z_{11}x_9y_7z_8x_7)$, $(x_8y_7x_{10}z_{12}x_9y_8z_7x_8)$,
$(x_{11}y_{12}z_{11}y_{13}x_2y_{14}z_{12}x_{11})$ and $(x_{12}y_{11}z_{12}y_{13}x_1y_{14}z_{11}x_{12})$.
This completes the proof. ■

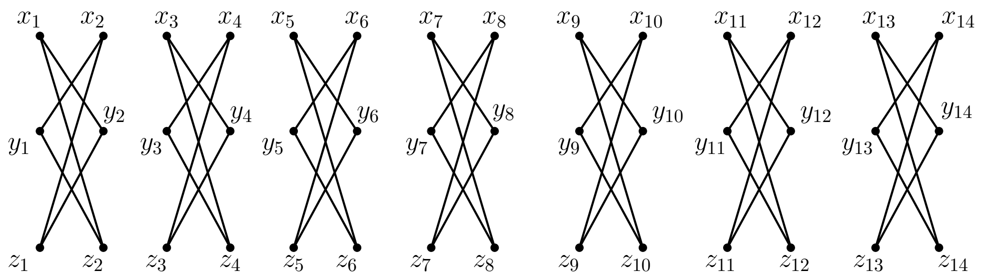

Figure 2

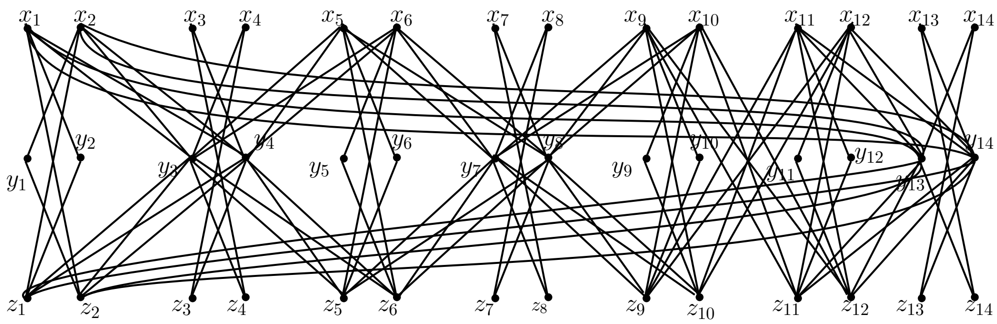

Figure 3

3. C_7-Decomposition of $C_5 \times K_m$

For our future reference we quote the following results.

Theorem 6 [11]**.** *For $m \geq 3$, $k \geq 1$, $C_{2k+1} \mid C_{2k+1} \times K_m$.*

Theorem 7 [11]**.** *For m, $k \geq 1$, $C_{2k+1} \mid C_{2k+1} * \overline{K}_m$.*

Lemma 8 [10]**.** *For any odd integer $t \geq 5$, $C_t \,\|\, C_5 \times K_t$.*

Lemma 9. $C_7 \mid C_5 \times \, K_8$.

Proof. Let the partite sets of the 5-partite graph $C_5 \times K_8$ be $\{u_1, u_2, \ldots, u_8\}$, $\{v_1, v_2, \ldots, v_8\}$, $\{w_1, w_2, \ldots, w_8\}$, $\{x_1, x_2, \ldots, x_8\}$ and $\{y_1, y_2, \ldots, y_8\}$. We assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Now the cycle $(u_1v_3w_6x_2y_7x_1y_8u_1)$ under the permutation

$(u_1u_2\cdots u_8)(v_1v_2\cdots v_8)\ (w_1w_2\cdots w_8)(x_1x_2\cdots x_8)(y_1y_2\cdots y_8)$ and its powers give us eight 7-cycles. These eight 7-cycles under the permutation $(u_1v_1w_1x_1y_1)\ (u_2v_2 w_2x_2y_2)\cdots(u_8v_8w_8x_8y_8)$ and its powers give us the required forty 7-cycles. ■

Remark 10. Let the vertex set of the 5-partite graph $C_5 * \overline{K}_m$, $m \neq 2$, be $\{u_1, u_2, \ldots, u_m\}$, $\{v_1, v_2, \ldots, v_m\}$, $\{w_1, w_2, \ldots, w_m\}$, $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_m\}$. From Theorems 6 and 7, $C_5 * \overline{K}_m$ has a 5-cycle decomposition containing the m 5-cycles $\{(u_iv_iw_ix_iy_iu_i) \,|\, 1 \leq i \leq m\}$, since the edge set of $C_5 * \overline{K}_m$ and $C_5 \times K_m$ differ only by these m disjoint 5-cycles.

Theorem 11. *For $m \geq 3$, $C_7 \mid C_5 \times K_m$ if and only if $m \equiv 0$ or $1 \pmod 7$.*

Proof. The proof of the necessity is obvious. We prove the sufficiency in two cases.

Case 1. $m \equiv 1 \pmod 7$. Let $m = 7k + 1$.

Subcase 1.1. $k \neq 2$. Let the partite sets of the 5-partite graph $C_5 \times K_m$ be $U = \{u_0\} \cup (\bigcup_{i=1}^{k}\{u_1^i, u_2^i, \ldots, u_7^i\})$, $V = \{v_0\} \cup \left(\bigcup_{i=1}^{k}\{v_1^i, v_2^i, \ldots, v_7^i\}\right)$, $W = \{w_0\} \cup \left(\bigcup_{i=1}^{k}\{w_1^i, w_2^i, \ldots, w_7^i\}\right)$, $X = \{x_0\} \cup \left(\bigcup_{i=1}^{k}\{x_1^i, x_2^i, \ldots, x_7^i\}\right)$ and $Y = \{y_0\} \cup \left(\bigcup_{i=1}^{k}\{y_1^i, y_2^i, \ldots, y_7^i\}\right)$, where we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. From the definition of the tensor product, in $C_5 \times K_m$, $\{u_0, v_0, w_0, x_0, y_0\}$ and $\{u_j^i, v_j^i, w_j^i, x_j^i, y_j^i\}$, $1 \leq j \leq 7$, $1 \leq i \leq k$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$, $W \cup X$, $X \cup Y$ and $Y \cup U$ is isomorphic to $K_{m,m} - F_0$, where F_0 is the 1-factor of distance zero.

We obtain a new graph out of $H = (C_5 \times K_m) - \{u_0, v_0, w_0, x_0, y_0\} \cong C_5 \times K_{7k}$ as follows: for each i, $1 \leq i \leq k$, identify the subsets of vertices $\{u_1^i, u_2^i, \ldots, u_7^i\}$, $\{v_1^i, v_2^i, \ldots, v_7^i\}$, $\{w_1^i, w_2^i, \ldots, w_7^i\}$, $\{x_1^i, x_2^i, \ldots, x_7^i\}$ and $\{y_1^i, y_2^i, \ldots, y_7^i\}$ into new vertices u^i, v^i, w^i, x^i and y^i, respectively, and two new vertices are adjacent if and only if the corresponding sets of vertices in H induce a complete bipartite subgraph $K_{7,7}$ or a complete bipartite subgraph minus a 1-factor $K_{7,7} - F$, where F is a 1-factor of $K_{7,7}$. The new graph thus obtained is isomorphic to the graph $C_5 * \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$, $\{w^1, w^2, \ldots, w^k\}$, $\{x^1, x^2, \ldots, x^k\}$ and $\{y^1, y^2, \ldots, y^k\}$. The graph $C_5 * \overline{K}_k$ has a C_5-decomposition containing the 5-cycles $(u^iv^iw^ix^iy^iu^i)$, $1 \leq i \leq k$, by Remark 10. The subgraph of H corresponding to these k 5-cycles of the graph $C_5 * \overline{K}_k$ consists of k vertex disjoint copies of $C_5 \times K_7$. To each of these k 5-cycles $(u^iv^iw^ix^iy^iu^i)$, $1 \leq i \leq k$, associate the 5-partite subgraph of $C_5 \times K_m$ induced by $\{u_0, u_1^i, u_2^i, \ldots, u_7^i\} \cup \{v_0, v_1^i, v_2^i, \ldots, v_7^i\} \cup \{w_0, w_1^i, w_2^i, \ldots, w_7^i\} \cup \{x_0, x_1^i, x_2^i, \ldots, x_7^i\} \cup \{y_0, y_1^i, y_2^i, \ldots, y_7^i\}$; as this induced subgraph is isomorphic to $C_5 \times K_8$, it can be decomposed into 7-cycles, by Lemma 9. Again, the subgraphs of $C_5 \times K_m$ corresponding to the other 5-cycles in the decomposition of $C_5 * \overline{K}_k$ are isomorphic to $C_5 * \overline{K}_7$, and they can be decomposed into 7-cycles, by Theorem A. Thus we have decomposed $C_5 \times K_m$ into 7-cycles when $k \neq 2$.

Subcase 1.2. $k = 2$. By Theorem B, $C_7 \mid K_{15}$ and hence $C_5 \times K_{15} \cong K_{15} \times C_5 \cong (C_7 \times C_5) \oplus (C_7 \times C_5) \oplus \cdots \oplus (C_7 \times C_5)$. Further, $C_7 \times C_5$ can be decomposed into 7-cycles, see Figure 4.

Case 2. $m \equiv 0 \pmod 7$. Let $m = 7k$.

Subcase 2.1. $k \neq 2$. As in the previous case, let the partite sets of the 5-partite graph $C_5 \times K_m$ be $U = \bigcup_{i=1}^{k}\{u_1^i, u_2^i, \ldots, u_7^i\}$, $V = \bigcup_{i=1}^{k}\{v_1^i, v_2^i, \ldots, v_7^i\}$, $W = \bigcup_{i=1}^{k}\{w_1^i, w_2^i, \ldots, w_7^i\}$, $X = \bigcup_{i=1}^{k}\{x_1^i, x_2^i, \ldots, x_7^i\}$ and $Y = \bigcup_{i=1}^{k}\{y_1^i, y_2^i, \ldots, y_7^i\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, we obtain the graph $C_5 * \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$, $\{w^1, w^2, \ldots, w^k\}$, $\{x^1, x^2, \ldots, x^k\}$ and $\{y^1, y^2, \ldots, y^k\}$, by suitable identification of vertices of $C_5 \times K_m$. By Remark 10, the graph $C_5 * \overline{K}_k$ has a C_5-decomposition containing the 5-cycles $(u^i v^i w^i x^i y^i u^i)$, $1 \leq i \leq k$. Corresponding to each of these k 5-cycles, associate the corresponding 5-partite subgraph of $C_5 \times K_m$ induced by $\{u_1^i, u_2^i, \ldots, u_7^i\} \cup \{v_1^i, v_2^i, \ldots, v_7^i\} \cup \{w_1^i, w_2^i, \ldots, w_7^i\} \cup \{x_1^i, x_2^i, \ldots, x_7^i\} \cup \{y_1^i, y_2^i, \ldots, y_7^i\}$; as this subgraph is isomorphic to $C_5 \times K_7$, it can be decomposed into 7-cycles, by Lemma 8. Corresponding to each of the other 5-cycles of the C_5-decomposition of $C_5 * \overline{K}_k$ if we associate the corresponding subgraph of $C_5 \times K_m$, then we get a subgraph isomorphic to $C_5 * \overline{K}_7$, and it can be decomposed into 7-cycles, by Theorem A. Thus we have decomposed $C_5 \times K_m$ into 7-cycles when $k \neq 2$.

Subcase 2.2. $k = 2$. Let the partite sets of the 5-partite graph $C_5 \times K_{14}$ be $U = \{u_1, u_2, \ldots, u_{14}\}$, $V = \{v_1, v_2, \ldots, v_{14}\}$, $W = \{w_1, w_2, \ldots, w_{14}\}$, $X = \{x_1, x_2, \ldots, x_{14}\}$, and $Y = \{y_1, y_2, \ldots, y_{14}\}$; we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. From the definition of the tensor product, in $C_5 \times K_{14}$, $\{u_i, v_i, w_i, x_i, y_i\}$, $1 \leq i \leq 14$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$, $W \cup X$, $X \cup Y$ and $Y \cup U$ is isomorphic to $K_{14,14} - F_0$, where F_0 is the 1-factor of distance zero. As in Subcase 1.1 above, we obtain a new graph out of $C_5 \times K_{14}$ as follows: for each i, $1 \leq i \leq 7$, identify the set of vertices $\{u_{2i-1}, u_{2i}\}$, $\{v_{2i-1}, v_{2i}\}$, $\{w_{2i-1}, w_{2i}\}$, $\{x_{2i-1}, x_{2i}\}$ and $\{y_{2i-1}, y_{2i}\}$ into new vertices u^i, v^i, w^i, x^i and y^i, respectively, and two of these vertices are adjacent if and only if the corresponding sets of vertices in $C_5 \times K_{14}$ induce the subgraph isomorphic to $K_{2,2}$ in $C_5 \times K_{14}$.

The resulting graph is isomorphic to $C_5 \times K_7$ with partite sets $U' = \{u^1, u^2, \ldots, u^7\}$, $V' = \{v^1, v^2, \ldots, v^7\}$, $W' = \{w^1, w^2, \ldots, w^7\}$, $X' = \{x^1, x^2, \ldots, x^7\}$ and $Y' = \{y^1, y^2, \ldots, y^7\}$, where $\{u^i, v^i, w^i, x^i, y^i\}$, $1 \leq i \leq 7$, are independent sets of $C_5 \times K_7$. Clearly, $C_5 \times K_7 = (C_5 \times C_7) \oplus (C_5 \times C_7) \oplus (C_5 \times C_7)$. The graph $C_5 \times C_7$ can be decomposed into 7-cycles, see Figure 4.

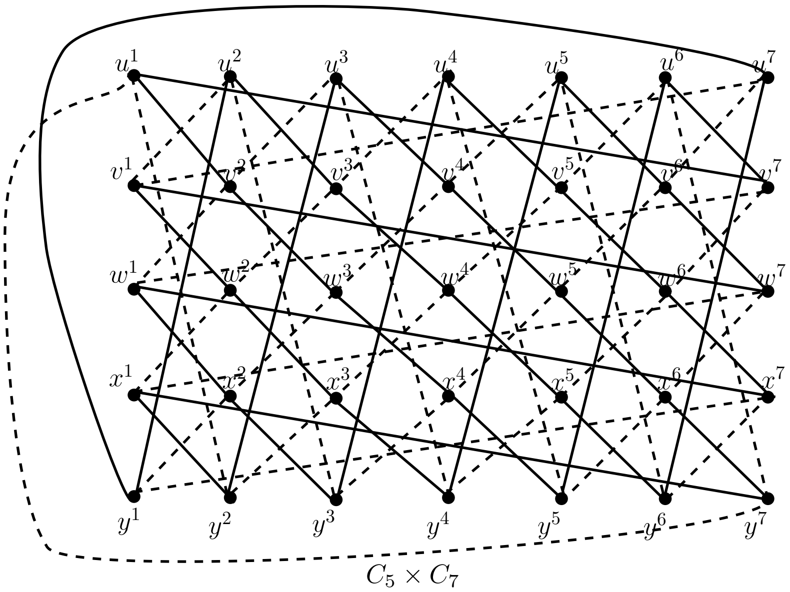

Figure 4. A 7-cycle decomposition of $C_5 \times C_7$.

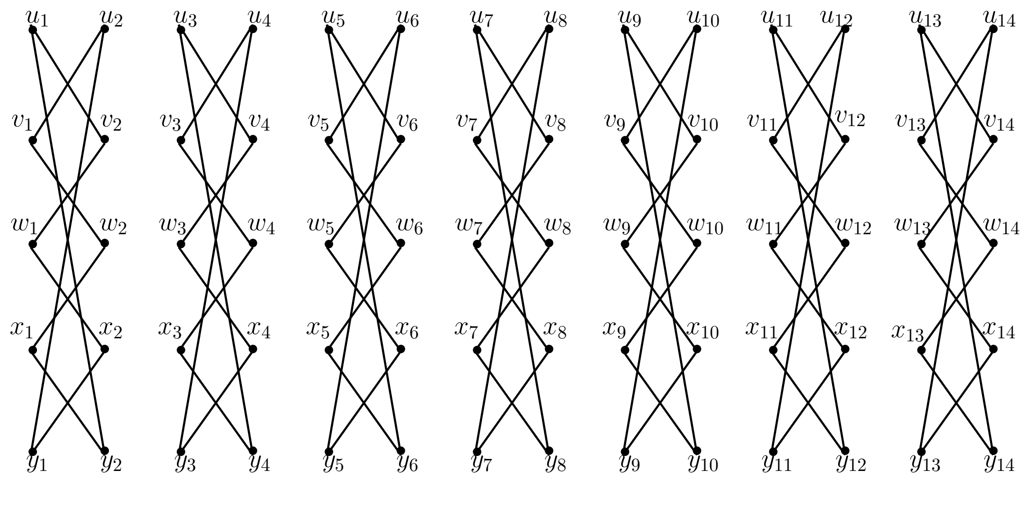

Figure 5

By "lifting back" each of these 7-cycles in a C_7-decomposition of $C_5 \times C_7$ to $C_5 \times K_{14}$, the corresponding subgraph is isomorphic to $C_7 * \overline{K}_2$ and this graph can be decomposed into cycles of length 7, see [12]. Thus the subgraph of $C_5 \times K_{14}$ obtained by the lifting of the 7-cycles of $C_5 \times K_7$ can be decomposed into cycles of length 7. The edges of $C_5 \times K_{14}$ which are not covered by these 7-cycles are shown in Figure 5.

To complete the proof we fuse with the graph of Figure 5 some of the 7-cycles obtained above and decompose the resulting graph, say, H', into 7-cycles. Let H' be the graph obtained by the union of the graph of Figure 5 and the subgraph of $C_5 \times K_{14}$ which corresponds to the 7-cycle of $C_5 \times K_7$, namely, $(u^1v^2w^3x^4y^5u^6v^7u^1)$ shown by solid line in Figure 4. The graph H' is shown in Figure 6.

A 7-cycle decomposition of H' is given below:
$(u_1v_2w_1x_2y_1u_2v_4u_1)$, $(u_1y_2x_1w_2v_1u_2v_3u_1)$, $(u_3y_4x_3w_4v_3w_6v_4u_3)$, $(u_4y_3x_4w_3v_4w_5v_3u_4)$, $(u_6y_5x_6w_5x_8w_6v_5u_6)$, $(u_5y_6x_5w_6x_7w_5v_6u_5)$, $(u_7y_8x_7y_{10}x_8w_7v_8u_7)$, $(u_8y_7x_8y_9x_7w_8v_7u_8)$, $(u_9y_{10}u_{12}y_9x_{10}w_9v_{10}u_9)$, $(u_{10}y_9u_{11}y_{10}x_9w_{10}v_9u_{10})$, $(u_{11}v_{12}w_{11}x_{12}y_{11}u_{12}v_{14}u_{11})$, $(u_{11}y_{12}x_{11}w_{12}v_{11}u_{12}v_{13}u_{11})$, $(u_{13}y_{14}x_{13}w_{14}v_{13}u_2v_{14}u_{13})$ and $(u_{14}y_{13}x_{14}w_{13}v_{14}u_1v_{13}u_{14})$. ■

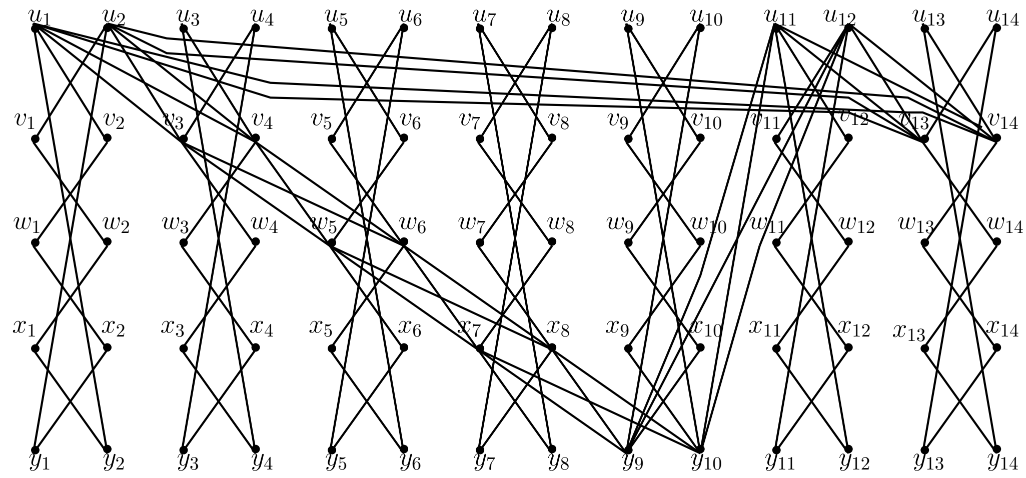

Figure 6

4. PROOF OF THE MAIN THEOREM

***Proof of Theorem* 1.** The proof of the neccessity is obvious and we prove the sufficiency in two cases. Since the tensor product is commutative, we may assume that m is odd and so $m \equiv 1, 3$ or $5 \pmod 6$.

Case 1. $n \equiv 0$ or $1 \pmod 7$.

Subcase 1.1. $m \equiv 1$ or $3 \pmod 6$. By Theorem C, $C_3 \mid K_m$ and hence $K_m \times K_n = (C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)$. As $C_7 \mid C_3 \times K_n$, by Theorem 5, $C_7 \mid K_m \times K_n$.

Subcase 1.2. $m \equiv 5 \pmod 6$. By Theorem C,
$K_m = \underbrace{C_3 \oplus C_3 \oplus \cdots \oplus C_3}_{(m(m-1)-20)/6 \text{ times}} \oplus (C_5 \oplus C_5)$.

Now $K_m \times K_n = ((C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)) \oplus ((C_5 \times K_n) \oplus (C_5 \times K_n))$. As $C_7 \mid C_3 \times K_n$, by Theorem 5, and $C_7 \mid C_5 \times K_n$, by Theorem 11, $C_7 \mid K_m \times K_n$.

Case 2. $n \not\equiv 0$ or $1 \pmod 7$. As $n(n-1) \not\equiv 0 \pmod 7$, condition (1) implies that $m \equiv 0$ or $m \equiv 1 \pmod 7$. As m is odd we have $m \equiv 1$ or $m \equiv 7 \pmod{14}$. As $C_7 \mid K_m$, by Theorem B, $K_m \times K_n = (C_7 \times K_n) \oplus (C_7 \times K_n) \oplus \cdots \oplus (C_7 \times K_n)$. $C_7 \mid C_7 \times K_n$, by Theorem 6, and so $C_7 \mid K_m \times K_n$.

This completes the proof. ∎

References

[1] B. Alspach and H. Gavlas, *Cycle decompositions of K_n and $K_n - I$*, J. Combin. Theory Ser. B **81** (2001) 77–99.
doi:10.1006/jctb.2000.1996

[2] B. Alspach, P.J. Schellenberg, D.R. Stinson and D. Wagner, *The Oberwolfach problem and factors of uniform odd length cycles*, J. Combin. Theory Ser. A **52** (1989) 20–43.
doi:10.1016/0097-3165(89)90059-9

[3] E.J. Billington, *Decomposing complete tripartite graphs into cycles of length* 3 *and* 4, Discrete Math. **197/198** (1999) 123–135.
doi:10.1016/S0012-365X(98)00227-1

[4] E.J. Billington, D.G. Hoffman and B.M. Maenhaut, *Group divisible pentagon systems*, Util. Math. **55** (1999) 211–219.

[5] N.J. Cavenagh and E.J. Billington, *Decompositions of complete multipartite graphs into cycles of even length*, Graphs Combin. **16** (2000) 49–65.
doi:10.1007/s003730050003

[6] N.J. Cavenagh and E.J. Billington, *On decomposing complete tripartite graphs into* 5*-cycles*, Australas. J. Combin. **22** (2000) 41–62.

[7] C.C. Lindner and C.A. Rodger, Design Theory (CRC Press New York, 1997).

[8] J. Liu, *A Generalization of the Oberwolfach problem and C_t-factorizations of complete equipartite graphs*, J. Combin. Designs **8** (2000) 42–49.
doi:10.1002/(SICI)1520-6610(2000)8:1⟨42::AID-JCD6⟩3.0.CO;2-R

[9] E.S. Mahmoodian and M. Mirzakhani, *Decomposition of complete tripartite graphs into* 5*-cycles*, in: Combinatorics and Advances, C.J. Colbourn and E.S. Mahmoodian (Ed(s)), (Kluwer Academic Publ., 1995) 235–241.

[10] R.S. Manikandan and P. Paulraja, *C_p-decompositions of some regular graphs*, Discrete Math. **306** (2006) 429–451.
doi:10.1016/j.disc.2005.08.006

[11] R.S. Manikandan and P. Paulraja, *C_5-decompositions of the tensor product of complete graphs*, Australas. J. Combin. **37** (2007) 285–293.

[12] A. Muthusamy and P. Paulraja, *Factorizations of product graphs into cycles of uniform length*, Graphs Combin. **11** (1995) 69–90. doi:10.1007/BF01787423

[13] M. Šajna, *Cycle decompositions* III: *Complete graphs and fixed length cycles*, J. Combin. Designs **10** (2002) 27–78. doi:10.1002/jcd.1027

[14] B.R. Smith, *Decomposing complete equipartite graphs into cycles of length* $2p$, J. Combin. Designs **16** (2008) 244–252. doi:10.1002/jcd.20173

[15] B.R. Smith, *Complete equipartite* $3p$*-cycle systems*, Australas. J. Combin. **45** (2009) 125–138.

[16] B.R. Smith, *Decomposing complete equipartite graphs into odd square-length cycles: number of parts odd*, J. Combin. Designs **18** (2010) 401–414. doi:10.1002/jcd.20268

[17] B.R. Smith, *Cycle decompositions of λ-fold complete equipartite graphs*, Australas. J. Combin. **47** (2010) 145–156.

Received 14 December 2015
Revised 5 May 2016
Accepted 5 May 2016