# $C_{7}$-DECOMPOSITIONS OF THE TENSOR PRODUCT OF COMPLETE GRAPHS 

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#### Abstract

In this paper we consider a decomposition of $K_{m} \times K_{n}$, where $\times$ denotes the tensor product of graphs, into cycles of length seven. We prove that for $m, n \geq 3$, cycles of length seven decompose the graph $K_{m} \times K_{n}$ if and only if (1) either $m$ or $n$ is odd and (2) $14 \mid m(m-1) n(n-1)$. The results of this paper together with the results of [ $C_{p}$-Decompositions of some regular graphs, Discrete Math. 306 (2006) 429-451] and [ $C_{5}$-Decompositions of the tensor product of complete graphs, Australasian J. Combinatorics 37 (2007) 285-293], give necessary and sufficient conditions for the existence of a $p$ cycle decomposition, where $p \geq 5$ is a prime number, of the graph $K_{m} \times K_{n}$.


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## 1. Introduction

All graphs considered here are simple and finite. Let $C_{n}$ denote the cycle of length $n$. We write $G=H_{1} \oplus H_{2} \oplus \cdots \oplus H_{k}$ if $H_{1}, H_{2}, \ldots, H_{k}$ are edge-disjoint subgraphs of $G$ and $E(G)=E\left(H_{1}\right) \cup E\left(H_{2}\right) \cup \cdots \cup E\left(H_{k}\right)$. If the edge set of
the graph $G$ can be partitioned into cycles $C_{n_{1}}, C_{n_{2}}, \ldots, C_{n_{r}}$, then we say that $C_{n_{1}}, C_{n_{2}}, \ldots, C_{n_{r}}$ decompose $G$. If $n_{1}=n_{2}=\cdots=n_{r}=k$, then we say that $G$ has a $C_{k}$-decomposition and in this case we write $C_{k} \mid G$. We may also call a cycle of length $k$ a $k$-cycle. If $G$ has a 2 -factorization and each 2 -factor of it has only cycles of length $k$, then we say that $G$ has a $C_{k}$-factorization (we use the notation $C_{k} \| G$.) The complete graph on $m$ vertices is denoted by $K_{m}$ and its complement is denoted by $\bar{K}_{m}$. For some positive integer $k$, the graph $k H$ denotes $k$ disjoint copies of $H$.

For two graphs $G$ and $H$ their wreath product, $G * H$, has the vertex set $V(G) \times V(H)$ in which $\left(g_{1}, h_{1}\right)$ and $\left(g_{2}, h_{2}\right)$ are adjacent whenever $g_{1} g_{2} \in E(G)$ or $g_{1}=g_{2}$ and $h_{1} h_{2} \in E(H)$. Similarly, $G \times H$, the tensor product of the graphs $G$ and $H$ has the vertex set $V(G) \times V(H)$ in which two vertices $\left(g_{1}, h_{1}\right)$ and $\left(g_{2}, h_{2}\right)$ are adjacent whenever $g_{1} g_{2} \in E(G)$ and $h_{1} h_{2} \in E(H)$. Clearly the tensor product is distributive over edge-disjoint union of graphs; that is, if $G=H_{1} \oplus H_{2} \oplus \cdots \oplus H_{k}$, then $G \times H=\left(H_{1} \times H\right) \oplus\left(H_{2} \times H\right) \oplus \cdots \oplus\left(H_{k} \times H\right)$. For $h \in V(H), V(G) \times h=\{(v, h) \mid v \in V(G)\}$ is called the column of vertices in $G \times H$ corresponding to $h$. Further, for $x \in V(G), x \times V(H)=\{(x, v) \mid v \in V(H)\}$ is called the layer of vertices in $G \times H$ corresponding to $x$. Similarly we can define column and layer for wreath product of graphs also. We can easily observe that $K_{m} * \bar{K}_{n}$ is isomorphic to the complete $m$-partite graph in which each partite set has exactly $n$ vertices.

A latin square of order $n$ is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1,2, \ldots, n\}$, such that each row and each column of the array contains each of the symbols in $\{1,2, \ldots, n\}$ exactly once. A latin square is said to be idempotent if the cell $(i, i)$ contains the symbol $i$, $1 \leq i \leq n$. Let $(X, Y)$ be the bipartition of the complete bipartite graph $K_{n, n}$, where $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ and $Y=\left\{y_{1}, y_{2}, \ldots, y_{n}\right\}$. Let $F_{i}(X, Y)=\left\{x_{1} y_{1+i}\right.$, $\left.x_{2} y_{2+i}, \ldots, x_{n} y_{n+i}\right\}, 0 \leq i \leq n-1$, where the addition in the suffixes are taken modulo $n$ with residues $1,2, \ldots, n$. We call $F_{i}(X, Y)$ as the 1 -factor of distance $i$ from $X$ to $Y$ in $K_{n, n}$. Note that, in general, $F_{i}(X, Y)$ need not be equal to $F_{i}(Y, X)$ as $F_{i}(Y, X)=\left\{y_{1} x_{1+i}, y_{2} x_{2+i}, \ldots, x_{n} y_{n+i}\right\}$. From the definition of the tensor product and the wreath product, it is clear that $K_{m} \times K_{n}=K_{m} * \bar{K}_{n}-$ $\bigcup_{i \neq j} F_{0}\left(X_{i}, X_{j}\right)$, where $X_{j}$ 's are the partitite sets of the complete $m$-partite graph in which each of the partite sets has cardinality $n$. In fact, $\bigcup_{i \neq j} F_{0}\left(X_{i}, X_{j}\right)$ consists of $n$ disjoint copies of $K_{m}$.

It is known that if $n$ is odd and $m \left\lvert\,\binom{ n}{2}\right.$ or $n$ is even and $m \left\lvert\,\left(\binom{n}{2}-\frac{n}{2}\right)\right.$, then $C_{m} \mid K_{n}$ or $C_{m} \mid K_{n}-I$, where $I$ is a 1 -factor of $K_{n}$; see [1, 13]. A similar problem can also be considered for regular complete multipartite graphs; Billington and Cavenagh [6] and Mahamoodian and Mirzakhani [9] have considered $C_{5}$-decompositions of complete tripartite graphs. Moreover, Billington [3] has studied the decompositions of complete tripartite graphs into cycles of length 3
and 4. Further, Billington and Cavenagh [5] have studied the decompositions of complete multipartite graphs into cycles of length 4,6 and 8 .

The present authors, in $[10,11]$, have proved that the necessary conditions for the existence of a $C_{5}$-decomposition and a $C_{p}$-decomposition for $p \geq 11$, where $p$ is a prime number, of $K_{m} * \bar{K}_{n}$ are sufficient. Smith, in [14, 15, 16], proved that the obvious necessary conditions for the existence of a $C_{k}$-decomposition of $K_{m} * \bar{K}_{n}$ are sufficient when $k=2 p, 3 p$ and $p^{2}$, where $p \geq 3$ is a prime number. Later, in [17], he has obtained the conditions under which $\lambda\left(K_{m} * \bar{K}_{n}\right)$ can be decomposed into cycles of length $p$, where $p$ is a prime; his approach is different from [10, 11]. For related work see also [4]. Liu, in [8], studied the $C_{t}$-factorization of $K_{m} * \bar{K}_{n}, m \geq 3$.

One can easily observe that the graph $K_{m} \times K_{n}$ is obtained from the regular complete multipartite graph $K_{m} * \bar{K}_{n}$, by deleting a suitable set of $n$ disjoint copies of $K_{m}$.

In this paper, the obvious necessary conditions for $K_{m} \times K_{n}, m, n \geq 3$, to admit a $C_{7}$-decomposition are proved to be sufficient. In this context, it is pertinent to point out that the existence of $C_{p}, p$ being a prime, decomposition of $K_{m} \times K_{n}$ played a significant role in establishing the existence of $C_{p^{-}}$ decomposition of $K_{m} * \bar{K}_{n}$, see $[10,11]$. We give below the main theorem obtained here.

Theorem 1. For $m, n \geq 3, C_{7} \mid K_{m} \times K_{n}$ if and only if
(1) $14 \mid n m(m-1)(n-1)$, and
(2) either $m$ or $n$ is odd.

For our future reference we list below some known results.
Theorem A [2]. Let $s$ be an odd integer and $t$ be a prime so that $3 \leq s \leq t$. Then $C_{s} * \bar{K}_{t}$ has a 2-factorization so that each 2 -factor is composed of $s$ cycles of length $t$.

Theorem B [1]. If $n \equiv 1$ or $7(\bmod 14)$, then $C_{7} \mid K_{n}$.
Theorem C [7]. Let $m$ be an odd integer, $m \geq 3$.
(1) If $m \equiv 1$ or $3(\bmod 6)$, then $C_{3} \mid K_{m}$.
(2) If $m \equiv 5(\bmod 6)$, then $K_{m}$ can be decomposed into $(m(m-1)-20) / 6$ 3-cycles and two 5-cycles.

## 2. $\quad C_{7}$-Decompositions of $C_{3} \times K_{m}$

We quote the following lemma for our future reference.

Lemma 2 [10]. For any odd integer $t \geq 3, C_{t} \| C_{3} \times K_{t}$.
Lemma 3. $C_{7} \mid C_{3} \times K_{8}$.
Proof. Let the partite sets of the tripartite graph $C_{3} \times K_{8}$ be $\left\{u_{1}, u_{2}, \ldots, u_{8}\right\}$, $\left\{v_{1}, v_{2}, \ldots, v_{8}\right\}$ and $\left\{w_{1}, w_{2}, \ldots, w_{8}\right\}$, where we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Now the cycle $\left(u_{1} v_{3} w_{6} v_{2} w_{7} v_{1} w_{8} u_{1}\right)$ under the permutation $\left(u_{1} u_{2} \cdots u_{8}\right)\left(v_{1} v_{2} \cdots v_{8}\right)$ $\left(w_{1} w_{2} \cdots w_{8}\right)$ and its powers give us eight 7 -cycles. These eight 7 -cycles under the permutation $\left(u_{1} v_{1} w_{1}\right)\left(u_{2} v_{2} w_{2}\right) \cdots\left(u_{8} v_{8} w_{8}\right)$ and its powers give us the required twenty four 7 -cycles.

Remark 4. Let the partite sets of the complete tripartite graph $C_{3} * \bar{K}_{m}, m \geq$ 1, be $\left\{u_{1}, u_{2}, \ldots, u_{m}\right\},\left\{v_{1}, v_{2}, \ldots, v_{m}\right\}$ and $\left\{w_{1}, w_{2}, \ldots, w_{m}\right\}$. Consider a latin square $\mathcal{L}$ of order $m$. We associate a triangle of $C_{3} * \bar{K}_{m}$ with each entry of $\mathcal{L}$ as follows: if $k$ is the $(i, j)^{\text {th }}$ entry of $\mathcal{L}$, then the triangle of $C_{3} * \bar{K}_{m}$ corresponding to $k$ is $\left(u_{i} v_{j} w_{k} u_{i}\right)$. Clearly the triangles corresponding to the entries of $\mathcal{L}$ decompose $C_{3} * \bar{K}_{m}$, see [3].

The necessary condition for the existence of decomposition of $C_{3} \times K_{m}, m \geq 3$, into $C_{7}$ is $m \equiv 0$ or $1(\bmod 7)$. We prove that it is also sufficient.

Theorem 5. $C_{7} \mid C_{3} \times K_{m}$ if and only if $m \equiv 0$ or $1(\bmod 7)$.
Proof. The necessity is obvious. We prove the sufficiency in two cases.
Case $1 . m \equiv 1(\bmod 7)$. Let $m=7 k+1$.
Subcase 1.1. $k \neq 2$. Let partite sets of the tripartite graph $C_{3} \times K_{m}$ be $U=\left\{u_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\}\right), V=\left\{v_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\}\right)$ and $W=$ $\left\{w_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\}\right)$; we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. By the definition of the tensor product, $\left\{u_{0}, v_{0}, w_{0}\right\}$ and $\left\{u_{j}^{i}, v_{j}^{i}, w_{j}^{i}\right\}, 1 \leq j \leq 7$, are independent sets and the subgraph induced by each of the sets $U \cup V, V \cup W$ and $W \cup U$ is isomorphic to $K_{m, m}-F_{0}$, where $F_{0}$ is the 1-factor of distance zero in $K_{m, m}$.

We obtain a new graph out of $H=\left(C_{3} \times K_{m}\right)-\left\{u_{0}, v_{0}, w_{0}\right\} \cong C_{3} \times K_{7 k}$ as follows: for each $i, 1 \leq i \leq k$, identify the sets of vertices $\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\}$, $\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\}$ and $\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\}$ into new vertices $u^{i}, v^{i}$ and $w^{i}$ respectively; two new vertices are adjacent if and only if the corresponding sets of vertices in $H$ induce a complete bipartite subgraph $K_{7,7}$ or a $K_{7,7}-F$, where $F$ is a 1 -factor of $K_{7,7}$. This defines the graph $C_{3} * \bar{K}_{k}$ with partite sets $\left\{u^{1}, u^{2}, \ldots, u^{k}\right\}$, $\left\{v^{1}, v^{2}, \ldots, v^{k}\right\}$ and $\left\{w^{1}, w^{2}, \ldots, w^{k}\right\}$.

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Consider an idempotent latin square $\mathcal{L}$ of order $k, k \neq 2$ (which exists, see [7]). To complete the proof of this subcase, we associate with entries of $\mathcal{L}$ edge-disjoint subgraphs of $C_{3} * \bar{K}_{m}$ which are decomposable by $C_{7}$. The $i^{\text {th }}$ diagonal entry of $\mathcal{L}$ corresponds to the triangle $\left(u^{i} v^{i} w^{i} u^{i}\right), 1 \leq i \leq k$, of $C_{3} * \bar{K}_{k}$, see Remark 4. The subgraph of $H$ corresponding to the triangle of $C_{3} * \bar{K}_{k}$ is isomorphic to $C_{3} \times K_{7}$. For each triangle ( $u^{i} v^{i} w^{i} u^{i}$ ), $1 \leq i \leq k$, of $C_{3} * \bar{K}_{k}$ corresponding to the $i^{\text {th }}$ diagonal entry of $\mathcal{L}$, associate the subgraph of $C_{3} \times K_{m}$ induced by vertices $\left\{u_{0}, u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\} \cup\left\{v_{0}, v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\} \cup\left\{w_{0}, w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\}$; as this subgraph is isomorphic to $C_{3} \times K_{8}$, it can be decomposed into 7 -cycles, by Lemma 3. Again, if we consider the subgraph of $H$ corresponding to the triangle of $C_{3} * \bar{K}_{k}$, which corresponds to a non-diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_{3} * \bar{K}_{7}$. By Theorem A, $C_{3} * \bar{K}_{7}$ can be decomposed into 7-cycles. Thus we have decomposed $C_{3} \times K_{m}$ into 7 -cycles when $k \neq 2$.

Subcase 1.2. $k=2$. By Theorem B, $C_{7} \mid K_{15}$ and hence we write $C_{3} \times K_{15} \cong$ $K_{15} \times C_{3}=\left(C_{7} \times C_{3}\right) \oplus\left(C_{7} \times C_{3}\right) \oplus \cdots \oplus\left(C_{7} \times C_{3}\right)$. Each copy of $C_{7} \times C_{3}$ can be decomposed into 7 -cycles, see Figure 1. This proves that $C_{7} \mid C_{3} \times K_{15}$.


Figure 1. A 7-cycle decomposition of $C_{3} \times C_{7}$. Different types of edges give different 7-cycles.

Case 2. $m \equiv 0(\bmod 7)$. Let $m=7 k$.
Subcase 2.1. $k \neq 2$. As in the previous case, let the partite sets of the tripartite graph $C_{3} \times K_{m}$ be $U=\bigcup_{i=1}^{k}\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\}, V=\bigcup_{i=1}^{k}\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\}$ and $W=\bigcup_{i=1}^{k}\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in
the proof of Subcase 1.1, from $C_{3} \times K_{m}=C_{3} \times K_{7 k}$ we obtain the graph $C_{3} * \bar{K}_{k}$ with partite sets $\left\{u^{1}, u^{2}, \ldots, u^{k}\right\},\left\{v^{1}, v^{2}, \ldots, v^{k}\right\}$ and $\left\{w^{1}, w^{2}, \ldots, w^{k}\right\}$.

Consider an idempotent latin square $\mathcal{L}$ of order $k, k \neq 2$. The diagonal entries of $\mathcal{L}$ correspond to the triangles $\left(u^{i} v^{i} w^{i} u^{i}\right), 1 \leq i \leq k$, of $C_{3} * \bar{K}_{k}$. If we consider the subgraph of $C_{3} \times K_{m}$ corresponding to a triangle of $C_{3} * \bar{K}_{k}$, which corresponds to a diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_{3} \times K_{7}$. By Lemma 2, $C_{7} \mid C_{3} \times K_{7}$. Again, as in the previous case, the triangle of $C_{3} * \bar{K}_{k}$ corresponding to a non-diagonal entry of $\mathcal{L}$ gives a subgraph of $C_{3} \times K_{m}$ isomorphic to $C_{3} * \bar{K}_{7}$; by Theorem A, $C_{7} \mid C_{3} * \bar{K}_{7}$.

Subcase 2.2. $k=2$. Let the partite sets of the tripartite graph $C_{3} \times K_{14}$ be $X=\left\{x_{1}, x_{2}, \ldots, x_{14}\right\}, Y=\left\{y_{1}, y_{2}, \ldots, y_{14}\right\}$ and $Z=\left\{z_{1}, z_{2}, \ldots, z_{14}\right\}$; we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. By the definition of the tensor product, $\left\{x_{i}, y_{i}, z_{i}\right\}, 1 \leq i \leq 14$, are independent sets and the subgraph induced by each of the subsets of vertices $X \cup Y, Y \cup Z$ and $Z \cup X$ are isomorphic to $K_{14,14}-F_{0}$, where $F_{0}$ is the 1-factor of distance zero in $K_{14,14}$.

We obtain a new graph out of $C_{3} \times K_{14}$ as follows: for each $i, 1 \leq i \leq 7$, identify the subsets of vertices $\left\{x_{2 i-1}, x_{2 i}\right\},\left\{y_{2 i-1}, y_{2 i}\right\}$ and $\left\{z_{2 i-1}, z_{2 i}\right\}$ into new vertices $x^{i}, y^{i}$ and $z^{i}$, respectively, and two of these vertices are adjacent if and only if the corresponding subsets of vertices in $C_{3} \times K_{14}$ induce a $K_{2,2}$. The resulting graph is isomorphic to $C_{3} \times K_{7}$ with partite sets $X^{\prime}=\left\{x^{1}, x^{2}, \ldots, x^{7}\right\}, Y^{\prime}=$ $\left\{y^{1}, y^{2}, \ldots, y^{7}\right\}$ and $Z^{\prime}=\left\{z^{1}, z^{2}, \ldots, z^{7}\right\}$; note that $\left\{x^{i}, y^{i}, z^{i}\right\}, 1 \leq i \leq 7$, are independent sets of $C_{3} \times K_{7}$. Now $C_{3} \times K_{7} \cong C_{3} \times\left(C_{7} \oplus C_{7} \oplus C_{7}\right)=$ $\left(C_{3} \times C_{7}\right) \oplus\left(C_{3} \times C_{7}\right) \oplus\left(C_{3} \times C_{7}\right)$. The graph $C_{3} \times C_{7}$ can be decomposed into 7-cycles, see Figure 1, and hence $C_{7} \mid C_{3} \times K_{7}$.

By "lifting back" these 7-cycles of $C_{3} \times K_{7}$ to $C_{3} \times K_{14}$, we get edge-disjoint subgraphs isomorphic to $C_{7} * \bar{K}_{2}$. But $C_{7} * \bar{K}_{2}$ can be decomposed into cycles of length 7 , see [12]. Thus the subgraphs of $C_{3} \times K_{14}$ obtained by "lifting back" the 7 -cycles of $C_{3} \times K_{7}$ to $C_{3} \times K_{14}$ can be decomposed into cycles of length 7 . The edges of $C_{3} \times K_{14}$ which are not covered by these 7 -cycles are shown in Figure 2. To complete the proof we fuse some of the 7 -cycles obtained above with the graph of Figure 2 and decompose the resulting graph into cycles of length 7 . Let $H^{\prime}$ be the graph obtained by the union of the graph of Figure 2 and the subgraph of $C_{3} \times K_{14}$ which is obtained by "lifting back" two 7 -cycles of $C_{3} \times K_{7}$, namely, $\left(x^{1} y^{2} x^{3} y^{4} x^{5} z^{6} y^{7} x^{1}\right)$ and $\left(z^{1} y^{2} z^{3} y^{4} z^{5} x^{6} y^{7} z^{1}\right)$ shown in Figure 1.

The subgraph $H^{\prime}$ of $C_{3} \times K_{14}$ is shown in Figure 3. A 7-cycle decomposition of $H^{\prime}$ is given below:

$$
\begin{aligned}
& \left(x_{1} y_{2} z_{1} x_{2} y_{4} z_{6} y_{3} x_{1}\right),\left(x_{1} z_{2} y_{1} x_{2} y_{3} z_{5} y_{4} x_{1}\right),\left(x_{5} y_{6} z_{5} y_{8} z_{9} y_{7} z_{6} x_{5}\right) \\
& \left(x_{6} y_{5} z_{6} y_{8} z_{10} y_{7} z_{5} x_{6}\right),\left(x_{9} y_{10} z_{9} x_{11} y_{14} x_{12} z_{10} x_{9}\right),\left(x_{10} y_{9} z_{10} x_{11} y_{13} x_{12} z_{9} x_{10}\right) \\
& \left(x_{14} y_{13} z_{1} y_{4} z_{2} y_{14} z_{13} x_{14}\right),\left(x_{13} y_{14} z_{1} y_{3} z_{2} y_{13} z_{14} x_{13}\right),\left(x_{3} y_{4} x_{6} y_{8} x_{5} y_{3} z_{4} x_{3}\right)
\end{aligned}
$$

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(x4}\mp@subsup{z}{3}{}\mp@subsup{y}{4}{}\mp@subsup{x}{5}{}\mp@subsup{y}{7}{}\mp@subsup{x}{6}{}\mp@subsup{y}{3}{}\mp@subsup{x}{4}{}),(\mp@subsup{x}{7}{}\mp@subsup{y}{8}{}\mp@subsup{x}{10}{}\mp@subsup{z}{11}{}\mp@subsup{x}{9}{}\mp@subsup{y}{7}{}\mp@subsup{z}{8}{}\mp@subsup{x}{7}{}),(\mp@subsup{x}{8}{}\mp@subsup{y}{7}{}\mp@subsup{x}{10}{}\mp@subsup{z}{12}{}\mp@subsup{x}{9}{}\mp@subsup{y}{8}{}\mp@subsup{z}{7}{}\mp@subsup{x}{8}{})
( }\mp@subsup{x}{11}{}\mp@subsup{y}{12}{}\mp@subsup{z}{11}{}\mp@subsup{y}{13}{}\mp@subsup{x}{2}{}\mp@subsup{y}{14}{}\mp@subsup{z}{12}{}\mp@subsup{x}{11}{})\mathrm{ and ( }\mp@subsup{x}{12}{}\mp@subsup{y}{11}{}\mp@subsup{z}{12}{}\mp@subsup{y}{13}{}\mp@subsup{x}{1}{}\mp@subsup{y}{14}{}\mp@subsup{z}{11}{}\mp@subsup{x}{12}{})
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This completes the proof.


Figure 2


Figure 3
3. $C_{7}$-Decomposition of $C_{5} \times K_{m}$

For our future reference we quote the following results.
Theorem 6 [11]. For $m \geq 3, k \geq 1, C_{2 k+1} \mid C_{2 k+1} \times K_{m}$.
Theorem 7 [11]. For $m, k \geq 1, C_{2 k+1} \mid C_{2 k+1} * \bar{K}_{m}$.
Lemma 8 [10]. For any odd integer $t \geq 5, C_{t} \| C_{5} \times K_{t}$.
Lemma 9. $C_{7} \mid C_{5} \times K_{8}$.
Proof. Let the partite sets of the 5 -partite graph $C_{5} \times K_{8}$ be $\left\{u_{1}, u_{2}, \ldots, u_{8}\right\}$, $\left\{v_{1}, v_{2}, \ldots, v_{8}\right\},\left\{w_{1}, w_{2}, \ldots, w_{8}\right\},\left\{x_{1}, x_{2}, \ldots, x_{8}\right\}$ and $\left\{y_{1}, y_{2}, \ldots, y_{8}\right\}$. We assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Now the cycle $\left(u_{1} v_{3} w_{6} x_{2} y_{7} x_{1} y_{8} u_{1}\right)$ under the permutation
$\left(u_{1} u_{2} \cdots u_{8}\right)\left(v_{1} v_{2} \cdots v_{8}\right)\left(w_{1} w_{2} \cdots w_{8}\right)\left(x_{1} x_{2} \cdots x_{8}\right)\left(y_{1} y_{2} \cdots y_{8}\right)$ and its powers give us eight 7 -cycles. These eight 7 -cycles under the permutation $\left(u_{1} v_{1} w_{1} x_{1} y_{1}\right)\left(u_{2} v_{2}\right.$ $\left.w_{2} x_{2} y_{2}\right) \cdots\left(u_{8} v_{8} w_{8} x_{8} y_{8}\right)$ and its powers give us the required forty 7 -cycles.

Remark 10. Let the vertex set of the 5-partite graph $C_{5} * \bar{K}_{m}, m \neq 2$, be $\left\{u_{1}, u_{2}, \ldots, u_{m}\right\},\left\{v_{1}, v_{2}, \ldots, v_{m}\right\},\left\{w_{1}, w_{2}, \ldots, w_{m}\right\},\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}$ and $\left\{y_{1}, y_{2}\right.$, $\left.\ldots, y_{m}\right\}$. From Theorems 6 and $7, C_{5} * \bar{K}_{m}$ has a 5 -cycle decomposition containing the $m 5$-cycles $\left\{\left(u_{i} v_{i} w_{i} x_{i} y_{i} u_{i}\right) \mid 1 \leq i \leq m\right\}$, since the edge set of $C_{5} * \bar{K}_{m}$ and $C_{5} \times K_{m}$ differ only by these $m$ disjoint 5 -cycles.

Theorem 11. For $m \geq 3, C_{7} \mid C_{5} \times K_{m}$ if and only if $m \equiv 0$ or $1(\bmod 7)$.
Proof. The proof of the necessity is obvious. We prove the sufficiency in two cases.

Case $1 . m \equiv 1(\bmod 7)$. Let $m=7 k+1$.
Subcase 1.1. $k \neq 2$. Let the partite sets of the 5 -partite graph $C_{5} \times K_{m}$ be $U=\left\{u_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\}\right), V=\left\{v_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\}\right), W=$ $\left\{w_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\}\right), X=\left\{x_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{x_{1}^{i}, x_{2}^{i}, \ldots, x_{7}^{i}\right\}\right)$ and $Y=$ $\left\{y_{0}\right\} \cup\left(\bigcup_{i=1}^{k}\left\{y_{1}^{i}, y_{2}^{i}, \ldots, y_{7}^{i}\right\}\right)$, where we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. From the definition of the tensor product, in $C_{5} \times K_{m},\left\{u_{0}, v_{0}, w_{0}, x_{0}, y_{0}\right\}$ and $\left\{u_{j}^{i}, v_{j}^{i}, w_{j}^{i}, x_{j}^{i}, y_{j}^{i}\right\}, 1 \leq j \leq 7,1 \leq i \leq k$, are independent sets and the subgraph induced by each of the sets $U \cup V, V \cup W, W \cup X, X \cup Y$ and $Y \cup U$ is isomorphic to $K_{m, m}-F_{0}$, where $F_{0}$ is the 1 -factor of distance zero.

We obtain a new graph out of $H=\left(C_{5} \times K_{m}\right)-\left\{u_{0}, v_{0}, w_{0}, x_{0}, y_{0}\right\} \cong C_{5} \times K_{7 k}$ as follows: for each $i, 1 \leq i \leq k$, identify the subsets of vertices $\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\}$, $\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\},\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\},\left\{x_{1}^{i}, x_{2}^{i}, \ldots, x_{7}^{i}\right\}$ and $\left\{y_{1}^{i}, y_{2}^{i}, \ldots, y_{7}^{i}\right\}$ into new vertices $u^{i}, v^{i}, w^{i}, x^{i}$ and $y^{i}$, respectively, and two new vertices are adjacent if and only if the corresponding sets of vertices in $H$ induce a complete bipartite subgraph $K_{7,7}$ or a complete bipartite subgraph minus a 1-factor $K_{7,7}-F$, where $F$ is a 1 -factor of $K_{7,7}$. The new graph thus obtained is isomorphic to the graph $C_{5} * \bar{K}_{k}$ with partite sets $\left\{u^{1}, u^{2}, \ldots, u^{k}\right\},\left\{v^{1}, v^{2}, \ldots, v^{k}\right\},\left\{w^{1}, w^{2}, \ldots, w^{k}\right\}$, $\left\{x^{1}, x^{2}, \ldots, x^{k}\right\}$ and $\left\{y^{1}, y^{2}, \ldots, y^{k}\right\}$. The graph $C_{5} * \bar{K}_{k}$ has a $C_{5}$-decomposition containing the 5 -cycles $\left(u^{i} v^{i} w^{i} x^{i} y^{i} u^{i}\right), 1 \leq i \leq k$, by Remark 10 . The subgraph of $H$ corresponding to these $k 5$-cycles of the graph $C_{5} * \bar{K}_{k}$ consists of $k$ vertex disjoint copies of $C_{5} \times K_{7}$. To each of these $k 5$-cycles $\left(u^{i} v^{i} w^{i} x^{i} y^{i} u^{i}\right), 1 \leq i \leq k$, associate the 5-partite subgraph of $C_{5} \times K_{m}$ induced by $\left\{u_{0}, u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\} \cup$ $\left\{v_{0}, v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\} \cup\left\{w_{0}, w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\} \cup\left\{x_{0}, x_{1}^{i}, x_{2}^{i}, \ldots, x_{7}^{i}\right\} \cup\left\{y_{0}, y_{1}^{i}, y_{2}^{i}, \ldots, y_{7}^{i}\right\} ;$ as this induced subgraph is isomorphic to $C_{5} \times K_{8}$, it can be decomposed into 7 -cycles, by Lemma 9. Again, the subgraphs of $C_{5} \times K_{m}$ corresponding to the

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other 5-cycles in the decomposition of $C_{5} * \bar{K}_{k}$ are isomorphic to $C_{5} * \bar{K}_{7}$, and they can be decomposed into 7 -cycles, by Theorem A. Thus we have decomposed $C_{5} \times K_{m}$ into 7 -cycles when $k \neq 2$.

Subcase 1.2. $k=2$. By Theorem B, $C_{7} \mid K_{15}$ and hence $C_{5} \times K_{15} \cong K_{15} \times$ $C_{5} \cong\left(C_{7} \times C_{5}\right) \oplus\left(C_{7} \times C_{5}\right) \oplus \cdots \oplus\left(C_{7} \times C_{5}\right)$. Further, $C_{7} \times C_{5}$ can be decomposed into 7 -cycles, see Figure 4.

Case $2 . m \equiv 0(\bmod 7)$. Let $m=7 k$.
Subcase 2.1. $k \neq 2$. As in the previous case, let the partite sets of the 5 -partite graph $C_{5} \times K_{m}$ be $U=\bigcup_{i=1}^{k}\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\}, V=\bigcup_{i=1}^{k}\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\}$, $W=$ $\bigcup_{i=1}^{k}\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\}, X=\bigcup_{i=1}^{k}\left\{x_{1}^{i}, x_{2}^{i}, \ldots, x_{7}^{i}\right\}$ and $Y=\bigcup_{i=1}^{k}\left\{y_{1}^{i}, y_{2}^{i}, \ldots, y_{7}^{i}\right\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, we obtain the graph $C_{5} * \bar{K}_{k}$ with partite sets $\left\{u^{1}, u^{2}, \ldots, u^{k}\right\},\left\{v^{1}, v^{2}, \ldots, v^{k}\right\}$, $\left\{w^{1}, w^{2}, \ldots, w^{k}\right\},\left\{x^{1}, x^{2}, \ldots, x^{k}\right\}$ and $\left\{y^{1}, y^{2}, \ldots, y^{k}\right\}$, by suitable identification of vertices of $C_{5} \times K_{m}$. By Remark 10, the graph $C_{5} * \bar{K}_{k}$ has a $C_{5}$-decomposition containing the 5 -cycles $\left(u^{i} v^{i} w^{i} x^{i} y^{i} u^{i}\right), 1 \leq i \leq k$. Corresponding to each of these $k 5$-cycles, associate the corresponding 5 -partite subgraph of $C_{5} \times K_{m}$ induced by $\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{7}^{i}\right\} \cup\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{7}^{i}\right\} \cup\left\{w_{1}^{i}, w_{2}^{i}, \ldots, w_{7}^{i}\right\} \cup\left\{x_{1}^{i}, x_{2}^{i}, \ldots, x_{7}^{i}\right\} \cup$ $\left\{y_{1}^{i}, y_{2}^{i}, \ldots, y_{7}^{i}\right\}$; as this subgraph is isomorphic to $C_{5} \times K_{7}$, it can be decomposed into 7 -cycles, by Lemma 8 . Corresponding to each of the other 5 -cycles of the $C_{5}$ decomposition of $C_{5} * \bar{K}_{k}$ if we associate the corresponding subgraph of $C_{5} \times K_{m}$, then we get a subgraph isomorphic to $C_{5} * \bar{K}_{7}$, and it can be decomposed into 7 -cycles, by Theorem A. Thus we have decomposed $C_{5} \times K_{m}$ into 7 -cycles when $k \neq 2$.

Subcase 2.2. $k=2$. Let the partite sets of the 5 -partite graph $C_{5} \times K_{14}$ be $U=\left\{u_{1}, u_{2}, \ldots, u_{14}\right\}, V=\left\{v_{1}, v_{2}, \ldots, v_{14}\right\}, W=\left\{w_{1}, w_{2}, \ldots, w_{14}\right\}, X=\left\{x_{1}, x_{2}\right.$, $\left.\ldots, x_{14}\right\}$, and $Y=\left\{y_{1}, y_{2}, \ldots, y_{14}\right\}$; we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. From the definition of the tensor product, in $C_{5} \times K_{14},\left\{u_{i}, v_{i}, w_{i}, x_{i}, y_{i}\right\}, 1 \leq i \leq 14$, are independent sets and the subgraph induced by each of the sets $U \cup V, V \cup W, W \cup X, X \cup Y$ and $Y \cup U$ is isomorphic to $K_{14,14}-F_{0}$, where $F_{0}$ is the 1-factor of distance zero. As in Subcase 1.1 above, we obtain a new graph out of $C_{5} \times K_{14}$ as follows: for each $i, 1 \leq i \leq 7$, identify the set of vertices $\left\{u_{2 i-1}, u_{2 i}\right\},\left\{v_{2 i-1}, v_{2 i}\right\},\left\{w_{2 i-1}, w_{2 i}\right\}$, $\left\{x_{2 i-1}, x_{2 i}\right\}$ and $\left\{y_{2 i-1}, y_{2 i}\right\}$ into new vertices $u^{i}, v^{i}, w^{i}, x^{i}$ and $y^{i}$, respectively, and two of these vertices are adjacent if and only if the corresponding sets of vertices in $C_{5} \times K_{14}$ induce the subgraph isomorphic to $K_{2,2}$ in $C_{5} \times K_{14}$.

The resulting graph is isomorphic to $C_{5} \times K_{7}$ with partite sets $U^{\prime}=\left\{u^{1}, u^{2}\right.$, $\left.\ldots, u^{7}\right\}, V^{\prime}=\left\{v^{1}, v^{2}, \ldots, v^{7}\right\}, W^{\prime}=\left\{w^{1}, w^{2}, \ldots, w^{7}\right\}, X^{\prime}=\left\{x^{1}, x^{2}, \ldots, x^{7}\right\}$ and $Y^{\prime}=\left\{y^{1}, y^{2}, \ldots, y^{7}\right\}$, where $\left\{u^{i}, v^{i}, w^{i}, x^{i}, y^{i}\right\}, 1 \leq i \leq 7$, are independent sets
of $C_{5} \times K_{7}$. Clearly, $C_{5} \times K_{7}=\left(C_{5} \times C_{7}\right) \oplus\left(C_{5} \times C_{7}\right) \oplus\left(C_{5} \times C_{7}\right)$. The graph $C_{5} \times C_{7}$ can be decomposed into 7-cycles, see Figure 4.


Figure 4. A 7 -cycle decomposition of $C_{5} \times C_{7}$.


Figure 5

By "lifting back" each of these 7-cycles in a $C_{7}$-decomposition of $C_{5} \times C_{7}$ to $C_{5} \times K_{14}$, the corresponding subgraph is isomorphic to $C_{7} * \bar{K}_{2}$ and this graph can be decomposed into cycles of length 7 , see [12]. Thus the subgraph of $C_{5} \times K_{14}$ obtained by the lifting of the 7 -cycles of $C_{5} \times K_{7}$ can be decomposed into cycles

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of length 7 . The edges of $C_{5} \times K_{14}$ which are not covered by these 7 -cycles are shown in Figure 5.

To complete the proof we fuse with the graph of Figure 5 some of the 7 -cycles obtained above and decompose the resulting graph, say, $H^{\prime}$, into 7 -cycles. Let $H^{\prime}$ be the graph obtained by the union of the graph of Figure 5 and the subgraph of $C_{5} \times K_{14}$ which corresponds to the 7 -cycle of $C_{5} \times K_{7}$, namely, $\left(u^{1} v^{2} w^{3} x^{4} y^{5} u^{6} v^{7} u^{1}\right)$ shown by solid line in Figure 4. The graph $H^{\prime}$ is shown in Figure 6.

A 7-cycle decomposition of $H^{\prime}$ is given below:
$\left(u_{1} v_{2} w_{1} x_{2} y_{1} u_{2} v_{4} u_{1}\right),\left(u_{1} y_{2} x_{1} w_{2} v_{1} u_{2} v_{3} u_{1}\right),\left(u_{3} y_{4} x_{3} w_{4} v_{3} w_{6} v_{4} u_{3}\right)$, $\left(u_{4} y_{3} x_{4} w_{3} v_{4} w_{5} v_{3} u_{4}\right),\left(u_{6} y_{5} x_{6} w_{5} x_{8} w_{6} v_{5} u_{6}\right),\left(u_{5} y_{6} x_{5} w_{6} x_{7} w_{5} v_{6} u_{5}\right)$, $\left(u_{7} y_{8} x_{7} y_{10} x_{8} w_{7} v_{8} u_{7}\right),\left(u_{8} y_{7} x_{8} y_{9} x_{7} w_{8} v_{7} u_{8}\right),\left(u_{9} y_{10} u_{12} y_{9} x_{10} w_{9} v_{10} u_{9}\right)$, $\left(u_{10} y_{9} u_{11} y_{10} x_{9} w_{10} v_{9} u_{10}\right),\left(u_{11} v_{12} w_{11} x_{12} y_{11} u_{12} v_{14} u_{11}\right),\left(u_{11} y_{12} x_{11} w_{12} v_{11} u_{12} v_{13} u_{11}\right)$, $\left(u_{13} y_{14} x_{13} w_{14} v_{13} u_{2} v_{14} u_{13}\right)$ and $\left(u_{14} y_{13} x_{14} w_{13} v_{14} u_{1} v_{13} u_{14}\right)$.


Figure 6

## 4. Proof of the Main Theorem

Proof of Theorem 1. The proof of the neccessity is obvious and we prove the sufficiency in two cases. Since the tensor product is commutative, we may assume that $m$ is odd and so $m \equiv 1,3$ or $5(\bmod 6)$.

Case 1. $n \equiv 0$ or $1(\bmod 7)$.
Subcase 1.1. $m \equiv 1$ or $3(\bmod 6)$. By Theorem C, $C_{3} \mid K_{m}$ and hence $K_{m} \times K_{n}=\left(C_{3} \times K_{n}\right) \oplus\left(C_{3} \times K_{n}\right) \oplus \cdots \oplus\left(C_{3} \times K_{n}\right)$. As $C_{7} \mid C_{3} \times K_{n}$, by Theorem $5, C_{7} \mid K_{m} \times K_{n}$.

Subcase 1.2. $m \equiv 5(\bmod 6)$. By Theorem C, $K_{m}=\underbrace{C_{3} \oplus C_{3} \oplus \cdots \oplus C_{3}}_{(m(m-1)-20) / 6 \text { times }} \oplus\left(C_{5} \oplus C_{5}\right)$.

Now $K_{m} \times K_{n}=\left(\left(C_{3} \times K_{n}\right) \oplus\left(C_{3} \times K_{n}\right) \oplus \cdots \oplus\left(C_{3} \times K_{n}\right)\right) \oplus\left(\left(C_{5} \times K_{n}\right)\right.$ $\left.\oplus\left(C_{5} \times K_{n}\right)\right)$. As $C_{7} \mid C_{3} \times K_{n}$, by Theorem 5 , and $C_{7} \mid C_{5} \times K_{n}$, by Theorem 11, $C_{7} \mid K_{m} \times K_{n}$.

Case 2. $n \not \equiv 0$ or $1(\bmod 7)$. As $n(n-1) \not \equiv 0(\bmod 7)$, condition (1) implies that $m \equiv 0$ or $m \equiv 1(\bmod 7)$. As $m$ is odd we have $m \equiv 1$ or $m \equiv 7(\bmod 14)$. As $C_{7} \mid K_{m}$, by Theorem B, $K_{m} \times K_{n}=\left(C_{7} \times K_{n}\right) \oplus\left(C_{7} \times K_{n}\right) \oplus \cdots \oplus\left(C_{7} \times K_{n}\right)$. $C_{7} \mid C_{7} \times K_{n}$, by Theorem 6, and so $C_{7} \mid K_{m} \times K_{n}$.

This completes the proof.

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