

C_7 -DECOMPOSITIONS OF THE TENSOR PRODUCT OF COMPLETE GRAPHS

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Abstract

In this paper we consider a decomposition of $K_m \times K_n$, where $\times$ denotes the tensor product of graphs, into cycles of length seven. We prove that for $m, n \geq 3$, cycles of length seven decompose the graph $K_m \times K_n$ if and only if (1) either m or n is odd and (2) $14 \mid m(m-1)n(n-1)$. The results of this paper together with the results of [C_p -Decompositions of some regular graphs, Discrete Math. **306** (2006) 429–451] and [C_5 -Decompositions of the tensor product of complete graphs, Australasian J. Combinatorics **37** (2007) 285–293], give necessary and sufficient conditions for the existence of a p -cycle decomposition, where $p \geq 5$ is a prime number, of the graph $K_m \times K_n$.

Keywords: cycle decomposition, tensor product.

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1. INTRODUCTION

All graphs considered here are simple and finite. Let C_n denote the cycle of length n . We write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ if $H_1, H_2, \dots, H_k$ are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$. If the edge set of

the graph G can be partitioned into cycles $C_{n_1}, C_{n_2}, \dots, C_{n_r}$, then we say that $C_{n_1}, C_{n_2}, \dots, C_{n_r}$ *decompose* G . If $n_1 = n_2 = \dots = n_r = k$, then we say that G has a C_k -*decomposition* and in this case we write $C_k \mid G$. We may also call a cycle of length k a k -*cycle*. If G has a 2-factorization and each 2-factor of it has only cycles of length k , then we say that G has a C_k -*factorization* (we use the notation $C_k \parallel G$.) The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. For some positive integer k , the graph kH denotes k disjoint copies of H .

For two graphs G and H their *wreath product*, $G * H$, has the vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$. Similarly, $G \times H$, the *tensor product* of the graphs G and H has the vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. Clearly the tensor product is distributive over edge-disjoint union of graphs; that is, if $G = H_1 \oplus H_2 \oplus \dots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \dots \oplus (H_k \times H)$. For $h \in V(H)$, $V(G) \times h = \{(v, h) | v \in V(G)\}$ is called the *column* of vertices in $G \times H$ corresponding to h . Further, for $x \in V(G)$, $x \times V(H) = \{(x, v) | v \in V(H)\}$ is called the *layer* of vertices in $G \times H$ corresponding to x . Similarly we can define column and layer for wreath product of graphs also. We can easily observe that $K_m * \overline{K}_n$ is isomorphic to the complete m -partite graph in which each partite set has exactly n vertices.

A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \dots, n\}$, such that each row and each column of the array contains each of the symbols in $\{1, 2, \dots, n\}$ exactly once. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i , $1 \leq i \leq n$. Let (X, Y) be the bipartition of the complete bipartite graph $K_{n,n}$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. Let $F_i(X, Y) = \{x_1 y_{1+i}, x_2 y_{2+i}, \dots, x_n y_{n+i}\}$, $0 \leq i \leq n-1$, where the addition in the suffixes are taken modulo n with residues $1, 2, \dots, n$. We call $F_i(X, Y)$ as the 1-factor of distance i from X to Y in $K_{n,n}$. Note that, in general, $F_i(X, Y)$ need not be equal to $F_i(Y, X)$ as $F_i(Y, X) = \{y_1 x_{1+i}, y_2 x_{2+i}, \dots, y_n x_{n+i}\}$. From the definition of the tensor product and the wreath product, it is clear that $K_m \times K_n = K_m * \overline{K}_n - \bigcup_{i \neq j} F_0(X_i, X_j)$, where X_j 's are the partite sets of the complete m -partite graph in which each of the partite sets has cardinality n . In fact, $\bigcup_{i \neq j} F_0(X_i, X_j)$ consists of n disjoint copies of K_m .

It is known that if n is odd and $m \mid \binom{n}{2}$ or n is even and $m \mid \left(\binom{n}{2} - \frac{n}{2}\right)$, then $C_m \mid K_n$ or $C_m \mid K_n - I$, where I is a 1-factor of K_n ; see [1, 13]. A similar problem can also be considered for regular complete multipartite graphs; Billington and Cavenagh [6] and Mahamoodian and Mirzakhani [9] have considered C_5 -decompositions of complete tripartite graphs. Moreover, Billington [3] has studied the decompositions of complete tripartite graphs into cycles of length 3

and 4. Further, Billington and Cavenagh [5] have studied the decompositions of complete multipartite graphs into cycles of length 4, 6 and 8.

The present authors, in [10, 11], have proved that the necessary conditions for the existence of a C_5 -decomposition and a C_p -decomposition for $p \geq 11$, where p is a prime number, of $K_m * \overline{K}_n$ are sufficient. Smith, in [14, 15, 16], proved that the obvious necessary conditions for the existence of a C_k -decomposition of $K_m * \overline{K}_n$ are sufficient when $k = 2p, 3p$ and p^2 , where $p \geq 3$ is a prime number. Later, in [17], he has obtained the conditions under which $\lambda(K_m * \overline{K}_n)$ can be decomposed into cycles of length p , where p is a prime; his approach is different from [10, 11]. For related work see also [4]. Liu, in [8], studied the C_t -factorization of $K_m * \overline{K}_n$, $m \geq 3$.

One can easily observe that the graph $K_m \times K_n$ is obtained from the regular complete multipartite graph $K_m * \overline{K}_n$, by deleting a suitable set of n disjoint copies of K_m .

In this paper, the obvious necessary conditions for $K_m \times K_n$, $m, n \geq 3$, to admit a C_7 -decomposition are proved to be sufficient. In this context, it is pertinent to point out that the existence of C_p , p being a prime, decomposition of $K_m \times K_n$ played a significant role in establishing the existence of C_p -decomposition of $K_m * \overline{K}_n$, see [10, 11]. We give below the main theorem obtained here.

Theorem 1. *For $m, n \geq 3$, $C_7 \mid K_m \times K_n$ if and only if*

- (1) $14 \mid nm(m-1)(n-1)$, and
- (2) *either m or n is odd.*

For our future reference we list below some known results.

Theorem A [2]. *Let s be an odd integer and t be a prime so that $3 \leq s \leq t$. Then $C_s * \overline{K}_t$ has a 2-factorization so that each 2-factor is composed of s cycles of length t .*

Theorem B [1]. *If $n \equiv 1$ or $7 \pmod{14}$, then $C_7 \mid K_n$.*

Theorem C [7]. *Let m be an odd integer, $m \geq 3$.*

- (1) *If $m \equiv 1$ or $3 \pmod{6}$, then $C_3 \mid K_m$.*
- (2) *If $m \equiv 5 \pmod{6}$, then K_m can be decomposed into $(m(m-1)-20)/6$ 3-cycles and two 5-cycles.*

2. C_7 -DECOMPOSITIONS OF $C_3 \times K_m$

We quote the following lemma for our future reference.

Lemma 2 [10]. For any odd integer $t \geq 3$, $C_t \parallel C_3 \times K_t$.

Lemma 3. $C_7 \mid C_3 \times K_8$.

Proof. Let the partite sets of the tripartite graph $C_3 \times K_8$ be $\{u_1, u_2, \dots, u_8\}$, $\{v_1, v_2, \dots, v_8\}$ and $\{w_1, w_2, \dots, w_8\}$, where we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Now the cycle $(u_1 v_3 w_6 v_2 w_7 v_1 w_8 u_1)$ under the permutation $(u_1 u_2 \cdots u_8)(v_1 v_2 \cdots v_8)(w_1 w_2 \cdots w_8)$ and its powers give us eight 7-cycles. These eight 7-cycles under the permutation $(u_1 v_1 w_1)(u_2 v_2 w_2) \cdots (u_8 v_8 w_8)$ and its powers give us the required twenty four 7-cycles. ■

Remark 4. Let the partite sets of the complete tripartite graph $C_3 * \overline{K}_m$, $m \geq 1$, be $\{u_1, u_2, \dots, u_m\}$, $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_m\}$. Consider a latin square $\mathcal{L}$ of order m . We associate a triangle of $C_3 * \overline{K}_m$ with each entry of $\mathcal{L}$ as follows: if k is the $(i, j)^{\text{th}}$ entry of $\mathcal{L}$, then the triangle of $C_3 * \overline{K}_m$ corresponding to k is $(u_i v_j w_k u_i)$. Clearly the triangles corresponding to the entries of $\mathcal{L}$ decompose $C_3 * \overline{K}_m$, see [3].

The necessary condition for the existence of decomposition of $C_3 \times K_m$, $m \geq 3$, into C_7 is $m \equiv 0$ or $1 \pmod{7}$. We prove that it is also sufficient.

Theorem 5. $C_7 \mid C_3 \times K_m$ if and only if $m \equiv 0$ or $1 \pmod{7}$.

Proof. The necessity is obvious. We prove the sufficiency in two cases.

Case 1. $m \equiv 1 \pmod{7}$. Let $m = 7k + 1$.

Subcase 1.1. $k \neq 2$. Let partite sets of the tripartite graph $C_3 \times K_m$ be $U = \{u_0\} \cup \left(\bigcup_{i=1}^k \{u_1^i, u_2^i, \dots, u_7^i\}\right)$, $V = \{v_0\} \cup \left(\bigcup_{i=1}^k \{v_1^i, v_2^i, \dots, v_7^i\}\right)$ and $W = \{w_0\} \cup \left(\bigcup_{i=1}^k \{w_1^i, w_2^i, \dots, w_7^i\}\right)$; we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. By the definition of the tensor product, $\{u_0, v_0, w_0\}$ and $\{u_j^i, v_j^i, w_j^i\}$, $1 \leq j \leq 7$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$ and $W \cup U$ is isomorphic to $K_{m,m} - F_0$, where F_0 is the 1-factor of distance zero in $K_{m,m}$.

We obtain a new graph out of $H = (C_3 \times K_m) - \{u_0, v_0, w_0\} \cong C_3 \times K_{7k}$ as follows: for each i , $1 \leq i \leq k$, identify the sets of vertices $\{u_1^i, u_2^i, \dots, u_7^i\}$, $\{v_1^i, v_2^i, \dots, v_7^i\}$ and $\{w_1^i, w_2^i, \dots, w_7^i\}$ into new vertices u^i, v^i and w^i respectively; two new vertices are adjacent if and only if the corresponding sets of vertices in H induce a complete bipartite subgraph $K_{7,7}$ or a $K_{7,7} - F$, where F is a 1-factor of $K_{7,7}$. This defines the graph $C_3 * \overline{K}_k$ with partite sets $\{u^1, u^2, \dots, u^k\}$, $\{v^1, v^2, \dots, v^k\}$ and $\{w^1, w^2, \dots, w^k\}$.

Consider an idempotent latin square $\mathcal{L}$ of order k , $k \neq 2$ (which exists, see [7]). To complete the proof of this subcase, we associate with entries of $\mathcal{L}$ edge-disjoint subgraphs of $C_3 * \overline{K}_m$ which are decomposable by C_7 . The i^{th} diagonal entry of $\mathcal{L}$ corresponds to the triangle $(u^i v^i w^i u^i)$, $1 \leq i \leq k$, of $C_3 * \overline{K}_k$, see Remark 4. The subgraph of H corresponding to the triangle of $C_3 * \overline{K}_k$ is isomorphic to $C_3 \times K_7$. For each triangle $(u^i v^i w^i u^i)$, $1 \leq i \leq k$, of $C_3 * \overline{K}_k$ corresponding to the i^{th} diagonal entry of $\mathcal{L}$, associate the subgraph of $C_3 \times K_m$ induced by vertices $\{u_0, u_1^i, u_2^i, \dots, u_7^i\} \cup \{v_0, v_1^i, v_2^i, \dots, v_7^i\} \cup \{w_0, w_1^i, w_2^i, \dots, w_7^i\}$; as this subgraph is isomorphic to $C_3 \times K_8$, it can be decomposed into 7-cycles, by Lemma 3. Again, if we consider the subgraph of H corresponding to the triangle of $C_3 * \overline{K}_k$, which corresponds to a non-diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 * \overline{K}_7$. By Theorem A, $C_3 * \overline{K}_7$ can be decomposed into 7-cycles. Thus we have decomposed $C_3 \times K_m$ into 7-cycles when $k \neq 2$.

Subcase 1.2. $k = 2$. By Theorem B, $C_7 \mid K_{15}$ and hence we write $C_3 \times K_{15} \cong K_{15} \times C_3 = (C_7 \times C_3) \oplus (C_7 \times C_3) \oplus \dots \oplus (C_7 \times C_3)$. Each copy of $C_7 \times C_3$ can be decomposed into 7-cycles, see Figure 1. This proves that $C_7 \mid C_3 \times K_{15}$.

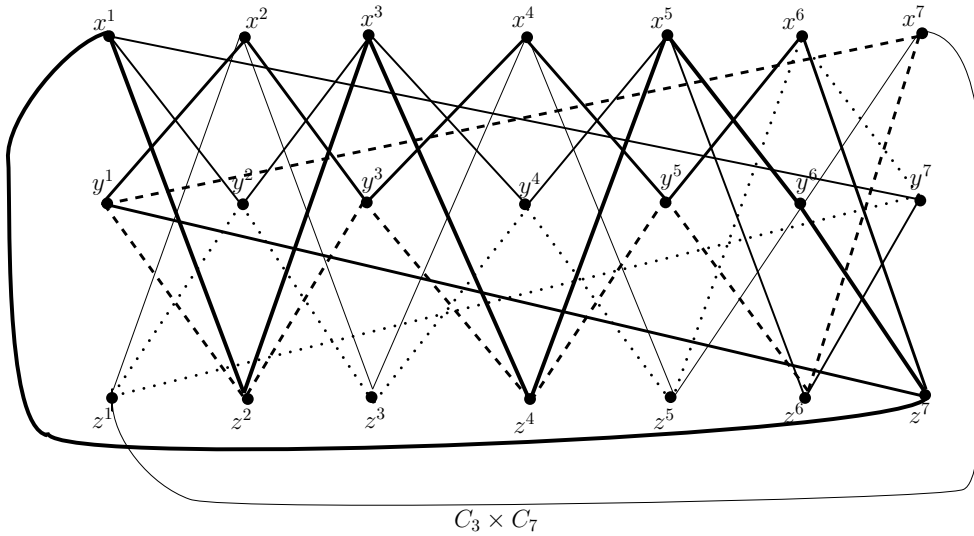


Figure 1. A 7-cycle decomposition of $C_3 \times C_7$. Different types of edges give different 7-cycles.

Case 2. $m \equiv 0 \pmod{7}$. Let $m = 7k$.

Subcase 2.1. $k \neq 2$. As in the previous case, let the partite sets of the tripartite graph $C_3 \times K_m$ be $U = \bigcup_{i=1}^k \{u_1^i, u_2^i, \dots, u_7^i\}$, $V = \bigcup_{i=1}^k \{v_1^i, v_2^i, \dots, v_7^i\}$ and $W = \bigcup_{i=1}^k \{w_1^i, w_2^i, \dots, w_7^i\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in

the proof of Subcase 1.1, from $C_3 \times K_m = C_3 \times K_{7k}$ we obtain the graph $C_3 * \overline{K}_k$ with partite sets $\{u^1, u^2, \dots, u^k\}, \{v^1, v^2, \dots, v^k\}$ and $\{w^1, w^2, \dots, w^k\}$.

Consider an idempotent latin square $\mathcal{L}$ of order k , $k \neq 2$. The diagonal entries of $\mathcal{L}$ correspond to the triangles $(u^i v^i w^i u^i)$, $1 \leq i \leq k$, of $C_3 * \overline{K}_k$. If we consider the subgraph of $C_3 \times K_m$ corresponding to a triangle of $C_3 * \overline{K}_k$, which corresponds to a diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 \times K_7$. By Lemma 2, $C_7 \mid C_3 \times K_7$. Again, as in the previous case, the triangle of $C_3 * \overline{K}_k$ corresponding to a non-diagonal entry of $\mathcal{L}$ gives a subgraph of $C_3 \times K_m$ isomorphic to $C_3 * \overline{K}_7$; by Theorem A, $C_7 \mid C_3 * \overline{K}_7$.

Subcase 2.2. $k = 2$. Let the partite sets of the tripartite graph $C_3 \times K_{14}$ be $X = \{x_1, x_2, \dots, x_{14}\}, Y = \{y_1, y_2, \dots, y_{14}\}$ and $Z = \{z_1, z_2, \dots, z_{14}\}$; we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. By the definition of the tensor product, $\{x_i, y_i, z_i\}$, $1 \leq i \leq 14$, are independent sets and the subgraph induced by each of the subsets of vertices $X \cup Y$, $Y \cup Z$ and $Z \cup X$ are isomorphic to $K_{14,14} - F_0$, where F_0 is the 1-factor of distance zero in $K_{14,14}$.

We obtain a new graph out of $C_3 \times K_{14}$ as follows: for each i , $1 \leq i \leq 7$, identify the subsets of vertices $\{x_{2i-1}, x_{2i}\}, \{y_{2i-1}, y_{2i}\}$ and $\{z_{2i-1}, z_{2i}\}$ into new vertices x^i, y^i and z^i , respectively, and two of these vertices are adjacent if and only if the corresponding subsets of vertices in $C_3 \times K_{14}$ induce a $K_{2,2}$. The resulting graph is isomorphic to $C_3 \times K_7$ with partite sets $X' = \{x^1, x^2, \dots, x^7\}, Y' = \{y^1, y^2, \dots, y^7\}$ and $Z' = \{z^1, z^2, \dots, z^7\}$; note that $\{x^i, y^i, z^i\}$, $1 \leq i \leq 7$, are independent sets of $C_3 \times K_7$. Now $C_3 \times K_7 \cong C_3 \times (C_7 \oplus C_7 \oplus C_7) = (C_3 \times C_7) \oplus (C_3 \times C_7) \oplus (C_3 \times C_7)$. The graph $C_3 \times C_7$ can be decomposed into 7-cycles, see Figure 1, and hence $C_7 \mid C_3 \times K_7$.

By “lifting back” these 7-cycles of $C_3 \times K_7$ to $C_3 \times K_{14}$, we get edge-disjoint subgraphs isomorphic to $C_7 * \overline{K}_2$. But $C_7 * \overline{K}_2$ can be decomposed into cycles of length 7, see [12]. Thus the subgraphs of $C_3 \times K_{14}$ obtained by “lifting back” the 7-cycles of $C_3 \times K_7$ to $C_3 \times K_{14}$ can be decomposed into cycles of length 7. The edges of $C_3 \times K_{14}$ which are not covered by these 7-cycles are shown in Figure 2. To complete the proof we fuse some of the 7-cycles obtained above with the graph of Figure 2 and decompose the resulting graph into cycles of length 7. Let H' be the graph obtained by the union of the graph of Figure 2 and the subgraph of $C_3 \times K_{14}$ which is obtained by “lifting back” two 7-cycles of $C_3 \times K_7$, namely, $(x^1 y^2 x^3 y^4 x^5 z^6 y^7 x^1)$ and $(z^1 y^2 z^3 y^4 z^5 x^6 y^7 z^1)$ shown in Figure 1.

The subgraph H' of $C_3 \times K_{14}$ is shown in Figure 3. A 7-cycle decomposition of H' is given below:

$$\begin{aligned} & (x_1 y_2 z_1 x_2 y_4 z_6 y_3 x_1), (x_1 z_2 y_1 x_2 y_3 z_5 y_4 x_1), (x_5 y_6 z_5 y_8 z_9 y_7 z_6 x_5), \\ & (x_6 y_5 z_6 y_8 z_{10} y_7 z_5 x_6), (x_9 y_{10} z_9 x_{11} y_{14} x_{12} z_{10} x_9), (x_{10} y_9 z_{10} x_{11} y_{13} x_{12} z_9 x_{10}), \\ & (x_{14} y_{13} z_{11} y_4 z_2 y_{14} z_{13} x_{14}), (x_{13} y_{14} z_{11} y_3 z_2 y_{13} z_{14} x_{13}), (x_3 y_4 x_6 y_8 x_5 y_3 z_4 x_3), \end{aligned}$$

$$(x_4z_3y_4x_5y_7x_6y_3x_4), (x_7y_8x_{10}z_{11}x_9y_7z_8x_7), (x_8y_7x_{10}z_{12}x_9y_8z_7x_8), \\ (x_{11}y_{12}z_{11}y_{13}x_2y_{14}z_{12}x_{11}) \text{ and } (x_{12}y_{11}z_{12}y_{13}x_1y_{14}z_{11}x_{12}).$$

This completes the proof. ■

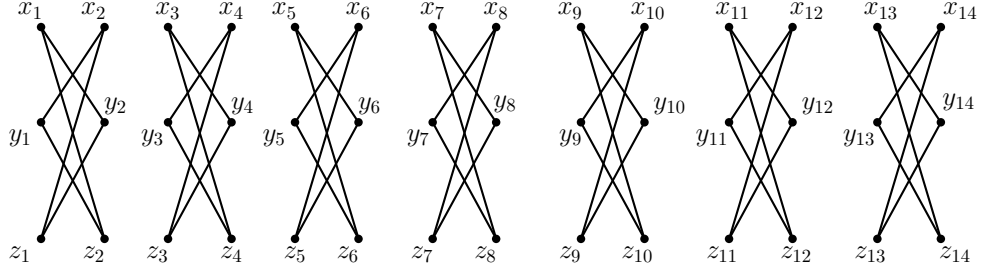


Figure 2

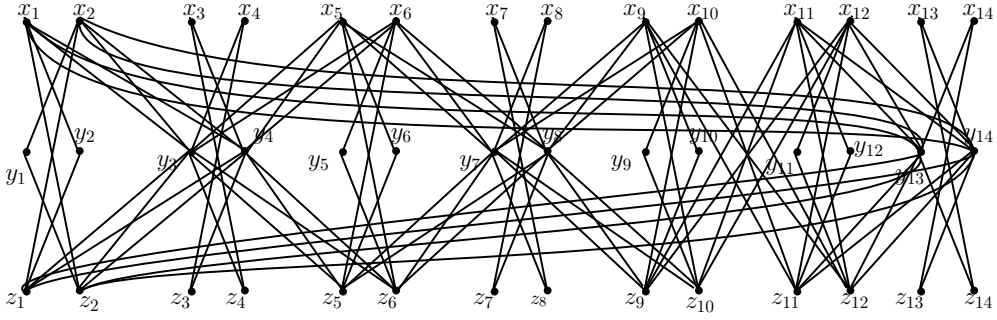


Figure 3

3. C_7 -DECOMPOSITION OF $C_5 \times K_m$

For our future reference we quote the following results.

Theorem 6 [11]. For $m \geq 3$, $k \geq 1$, $C_{2k+1} \mid C_{2k+1} \times K_m$.

Theorem 7 [11]. For $m, k \geq 1$, $C_{2k+1} \mid C_{2k+1} * \overline{K}_m$.

Lemma 8 [10]. For any odd integer $t \geq 5$, $C_t \parallel C_5 \times K_t$.

Lemma 9. $C_7 \mid C_5 \times K_8$.

Proof. Let the partite sets of the 5-partite graph $C_5 \times K_8$ be $\{u_1, u_2, \dots, u_8\}$, $\{v_1, v_2, \dots, v_8\}$, $\{w_1, w_2, \dots, w_8\}$, $\{x_1, x_2, \dots, x_8\}$ and $\{y_1, y_2, \dots, y_8\}$. We assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Now the cycle $(u_1v_3w_6x_2y_7x_1y_8u_1)$ under the permutation

$(u_1u_2 \cdots u_8)(v_1v_2 \cdots v_8)(w_1w_2 \cdots w_8)(x_1x_2 \cdots x_8)(y_1y_2 \cdots y_8)$ and its powers give us eight 7-cycles. These eight 7-cycles under the permutation $(u_1v_1w_1x_1y_1)(u_2v_2w_2x_2y_2) \cdots (u_8v_8w_8x_8y_8)$ and its powers give us the required forty 7-cycles. ■

Remark 10. Let the vertex set of the 5-partite graph $C_5 * \overline{K}_m$, $m \neq 2$, be $\{u_1, u_2, \dots, u_m\}$, $\{v_1, v_2, \dots, v_m\}$, $\{w_1, w_2, \dots, w_m\}$, $\{x_1, x_2, \dots, x_m\}$ and $\{y_1, y_2, \dots, y_m\}$. From Theorems 6 and 7, $C_5 * \overline{K}_m$ has a 5-cycle decomposition containing the m 5-cycles $\{(u_i v_i w_i x_i y_i u_i) \mid 1 \leq i \leq m\}$, since the edge set of $C_5 * \overline{K}_m$ and $C_5 \times K_m$ differ only by these m disjoint 5-cycles.

Theorem 11. For $m \geq 3$, $C_7 \mid C_5 \times K_m$ if and only if $m \equiv 0$ or $1 \pmod{7}$.

Proof. The proof of the necessity is obvious. We prove the sufficiency in two cases.

Case 1. $m \equiv 1 \pmod{7}$. Let $m = 7k + 1$.

Subcase 1.1. $k \neq 2$. Let the partite sets of the 5-partite graph $C_5 \times K_m$ be $U = \{u_0\} \cup \left(\bigcup_{i=1}^k \{u_1^i, u_2^i, \dots, u_7^i\}\right)$, $V = \{v_0\} \cup \left(\bigcup_{i=1}^k \{v_1^i, v_2^i, \dots, v_7^i\}\right)$, $W = \{w_0\} \cup \left(\bigcup_{i=1}^k \{w_1^i, w_2^i, \dots, w_7^i\}\right)$, $X = \{x_0\} \cup \left(\bigcup_{i=1}^k \{x_1^i, x_2^i, \dots, x_7^i\}\right)$ and $Y = \{y_0\} \cup \left(\bigcup_{i=1}^k \{y_1^i, y_2^i, \dots, y_7^i\}\right)$, where we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. From the definition of the tensor product, in $C_5 \times K_m$, $\{u_0, v_0, w_0, x_0, y_0\}$ and $\{u_j^i, v_j^i, w_j^i, x_j^i, y_j^i\}$, $1 \leq j \leq 7$, $1 \leq i \leq k$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$, $W \cup X$, $X \cup Y$ and $Y \cup U$ is isomorphic to $K_{m,m} - F_0$, where F_0 is the 1-factor of distance zero.

We obtain a new graph out of $H = (C_5 \times K_m) - \{u_0, v_0, w_0, x_0, y_0\} \cong C_5 \times K_{7k}$ as follows: for each i , $1 \leq i \leq k$, identify the subsets of vertices $\{u_1^i, u_2^i, \dots, u_7^i\}$, $\{v_1^i, v_2^i, \dots, v_7^i\}$, $\{w_1^i, w_2^i, \dots, w_7^i\}$, $\{x_1^i, x_2^i, \dots, x_7^i\}$ and $\{y_1^i, y_2^i, \dots, y_7^i\}$ into new vertices u^i, v^i, w^i, x^i and y^i , respectively, and two new vertices are adjacent if and only if the corresponding sets of vertices in H induce a complete bipartite subgraph $K_{7,7}$ or a complete bipartite subgraph minus a 1-factor $K_{7,7} - F$, where F is a 1-factor of $K_{7,7}$. The new graph thus obtained is isomorphic to the graph $C_5 * \overline{K}_k$ with partite sets $\{u^1, u^2, \dots, u^k\}$, $\{v^1, v^2, \dots, v^k\}$, $\{w^1, w^2, \dots, w^k\}$, $\{x^1, x^2, \dots, x^k\}$ and $\{y^1, y^2, \dots, y^k\}$. The graph $C_5 * \overline{K}_k$ has a C_5 -decomposition containing the 5-cycles $(u^i v^i w^i x^i y^i u^i)$, $1 \leq i \leq k$, by Remark 10. The subgraph of H corresponding to these k 5-cycles of the graph $C_5 * \overline{K}_k$ consists of k vertex disjoint copies of $C_5 \times K_7$. To each of these k 5-cycles $(u^i v^i w^i x^i y^i u^i)$, $1 \leq i \leq k$, associate the 5-partite subgraph of $C_5 \times K_m$ induced by $\{u_0, u_1^i, u_2^i, \dots, u_7^i\} \cup \{v_0, v_1^i, v_2^i, \dots, v_7^i\} \cup \{w_0, w_1^i, w_2^i, \dots, w_7^i\} \cup \{x_0, x_1^i, x_2^i, \dots, x_7^i\} \cup \{y_0, y_1^i, y_2^i, \dots, y_7^i\}$; as this induced subgraph is isomorphic to $C_5 \times K_8$, it can be decomposed into 7-cycles, by Lemma 9. Again, the subgraphs of $C_5 \times K_m$ corresponding to the

other 5-cycles in the decomposition of $C_5 * \overline{K}_k$ are isomorphic to $C_5 * \overline{K}_7$, and they can be decomposed into 7-cycles, by Theorem A. Thus we have decomposed $C_5 \times K_m$ into 7-cycles when $k \neq 2$.

Subcase 1.2. $k = 2$. By Theorem B, $C_7 \mid K_{15}$ and hence $C_5 \times K_{15} \cong K_{15} \times C_5 \cong (C_7 \times C_5) \oplus (C_7 \times C_5) \oplus \cdots \oplus (C_7 \times C_5)$. Further, $C_7 \times C_5$ can be decomposed into 7-cycles, see Figure 4.

Case 2. $m \equiv 0 \pmod{7}$. Let $m = 7k$.

Subcase 2.1. $k \neq 2$. As in the previous case, let the partite sets of the 5-partite graph $C_5 \times K_m$ be $U = \bigcup_{i=1}^k \{u_1^i, u_2^i, \dots, u_7^i\}$, $V = \bigcup_{i=1}^k \{v_1^i, v_2^i, \dots, v_7^i\}$, $W = \bigcup_{i=1}^k \{w_1^i, w_2^i, \dots, w_7^i\}$, $X = \bigcup_{i=1}^k \{x_1^i, x_2^i, \dots, x_7^i\}$ and $Y = \bigcup_{i=1}^k \{y_1^i, y_2^i, \dots, y_7^i\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, we obtain the graph $C_5 * \overline{K}_k$ with partite sets $\{u^1, u^2, \dots, u^k\}$, $\{v^1, v^2, \dots, v^k\}$, $\{w^1, w^2, \dots, w^k\}$, $\{x^1, x^2, \dots, x^k\}$ and $\{y^1, y^2, \dots, y^k\}$, by suitable identification of vertices of $C_5 \times K_m$. By Remark 10, the graph $C_5 * \overline{K}_k$ has a C_5 -decomposition containing the 5-cycles $(u^i v^i w^i x^i y^i u^i)$, $1 \leq i \leq k$. Corresponding to each of these k 5-cycles, associate the corresponding 5-partite subgraph of $C_5 \times K_m$ induced by $\{u_1^i, u_2^i, \dots, u_7^i\} \cup \{v_1^i, v_2^i, \dots, v_7^i\} \cup \{w_1^i, w_2^i, \dots, w_7^i\} \cup \{x_1^i, x_2^i, \dots, x_7^i\} \cup \{y_1^i, y_2^i, \dots, y_7^i\}$; as this subgraph is isomorphic to $C_5 \times K_7$, it can be decomposed into 7-cycles, by Lemma 8. Corresponding to each of the other 5-cycles of the C_5 -decomposition of $C_5 * \overline{K}_k$ if we associate the corresponding subgraph of $C_5 \times K_m$, then we get a subgraph isomorphic to $C_5 * \overline{K}_7$, and it can be decomposed into 7-cycles, by Theorem A. Thus we have decomposed $C_5 \times K_m$ into 7-cycles when $k \neq 2$.

Subcase 2.2. $k = 2$. Let the partite sets of the 5-partite graph $C_5 \times K_{14}$ be $U = \{u_1, u_2, \dots, u_{14}\}$, $V = \{v_1, v_2, \dots, v_{14}\}$, $W = \{w_1, w_2, \dots, w_{14}\}$, $X = \{x_1, x_2, \dots, x_{14}\}$, and $Y = \{y_1, y_2, \dots, y_{14}\}$; we assume that the vertices having the same subscript are the corresponding vertices of the partite sets. From the definition of the tensor product, in $C_5 \times K_{14}$, $\{u_i, v_i, w_i, x_i, y_i\}$, $1 \leq i \leq 14$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$, $W \cup X$, $X \cup Y$ and $Y \cup U$ is isomorphic to $K_{14,14} - F_0$, where F_0 is the 1-factor of distance zero. As in Subcase 1.1 above, we obtain a new graph out of $C_5 \times K_{14}$ as follows: for each i , $1 \leq i \leq 7$, identify the set of vertices $\{u_{2i-1}, u_{2i}\}$, $\{v_{2i-1}, v_{2i}\}$, $\{w_{2i-1}, w_{2i}\}$, $\{x_{2i-1}, x_{2i}\}$ and $\{y_{2i-1}, y_{2i}\}$ into new vertices u^i, v^i, w^i, x^i and y^i , respectively, and two of these vertices are adjacent if and only if the corresponding sets of vertices in $C_5 \times K_{14}$ induce the subgraph isomorphic to $K_{2,2}$ in $C_5 \times K_{14}$.

The resulting graph is isomorphic to $C_5 \times K_7$ with partite sets $U' = \{u^1, u^2, \dots, u^7\}$, $V' = \{v^1, v^2, \dots, v^7\}$, $W' = \{w^1, w^2, \dots, w^7\}$, $X' = \{x^1, x^2, \dots, x^7\}$ and $Y' = \{y^1, y^2, \dots, y^7\}$, where $\{u^i, v^i, w^i, x^i, y^i\}$, $1 \leq i \leq 7$, are independent sets

of $C_5 \times K_7$. Clearly, $C_5 \times K_7 = (C_5 \times C_7) \oplus (C_5 \times C_7) \oplus (C_5 \times C_7)$. The graph $C_5 \times C_7$ can be decomposed into 7-cycles, see Figure 4.

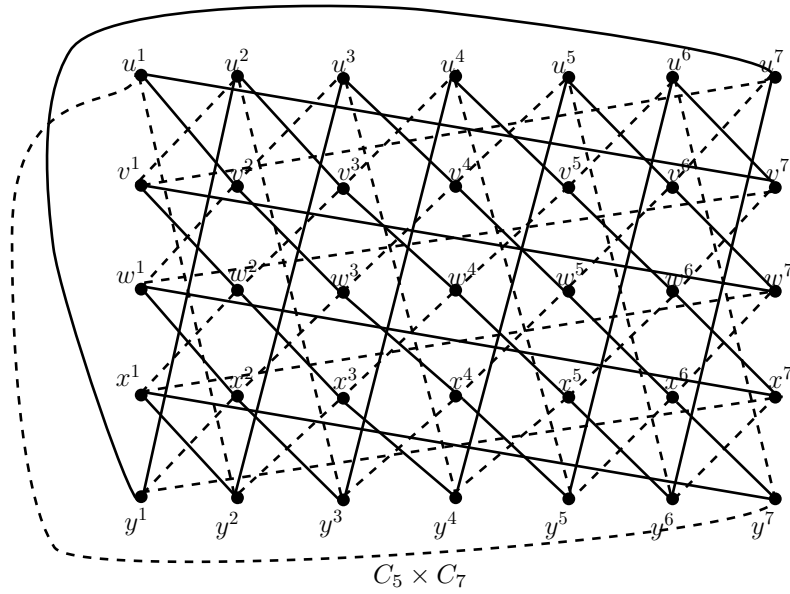


Figure 4. A 7-cycle decomposition of $C_5 \times C_7$.

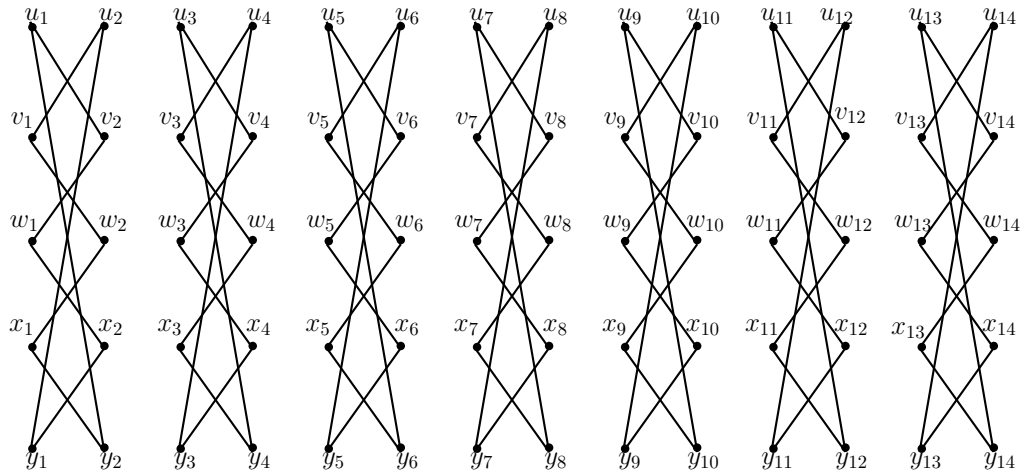


Figure 5

By “lifting back” each of these 7-cycles in a C_7 -decomposition of $C_5 \times C_7$ to $C_5 \times K_{14}$, the corresponding subgraph is isomorphic to $C_7 * \overline{K}_2$ and this graph can be decomposed into cycles of length 7, see [12]. Thus the subgraph of $C_5 \times K_{14}$ obtained by the lifting of the 7-cycles of $C_5 \times K_7$ can be decomposed into cycles

of length 7. The edges of $C_5 \times K_{14}$ which are not covered by these 7-cycles are shown in Figure 5.

To complete the proof we fuse with the graph of Figure 5 some of the 7-cycles obtained above and decompose the resulting graph, say, H' , into 7-cycles. Let H' be the graph obtained by the union of the graph of Figure 5 and the subgraph of $C_5 \times K_{14}$ which corresponds to the 7-cycle of $C_5 \times K_7$, namely, $(u^1 v^2 w^3 x^4 y^5 u^6 v^7 u^1)$ shown by solid line in Figure 4. The graph H' is shown in Figure 6.

A 7-cycle decomposition of H' is given below:

$(u_1 v_2 w_1 x_2 y_1 u_2 v_4 u_1)$, $(u_1 y_2 x_1 w_2 v_1 u_2 v_3 u_1)$, $(u_3 y_4 x_3 w_4 v_3 w_6 v_4 u_3)$,
 $(u_4 y_3 x_4 w_3 v_4 w_5 v_3 u_4)$, $(u_6 y_5 x_6 w_5 x_8 w_6 v_5 u_6)$, $(u_5 y_6 x_5 w_6 x_7 w_5 v_6 u_5)$,
 $(u_7 y_8 x_7 y_{10} x_8 w_7 v_8 u_7)$, $(u_8 y_7 x_8 y_9 x_7 w_8 v_7 u_8)$, $(u_9 y_{10} u_{12} y_9 x_{10} w_9 v_{10} u_9)$,
 $(u_{10} y_9 u_{11} y_{10} x_9 w_{10} v_9 u_{10})$, $(u_{11} v_{12} w_{11} x_{12} y_{11} u_{12} v_{14} u_{11})$, $(u_{11} y_{12} x_{11} w_{12} v_{11} u_{12} v_{13} u_{11})$,
 $(u_{13} y_{14} x_{13} w_{14} v_{13} u_{12} v_{14} u_{13})$ and $(u_{14} y_{13} x_{14} w_{13} v_{14} u_1 v_{13} u_{14})$. ■

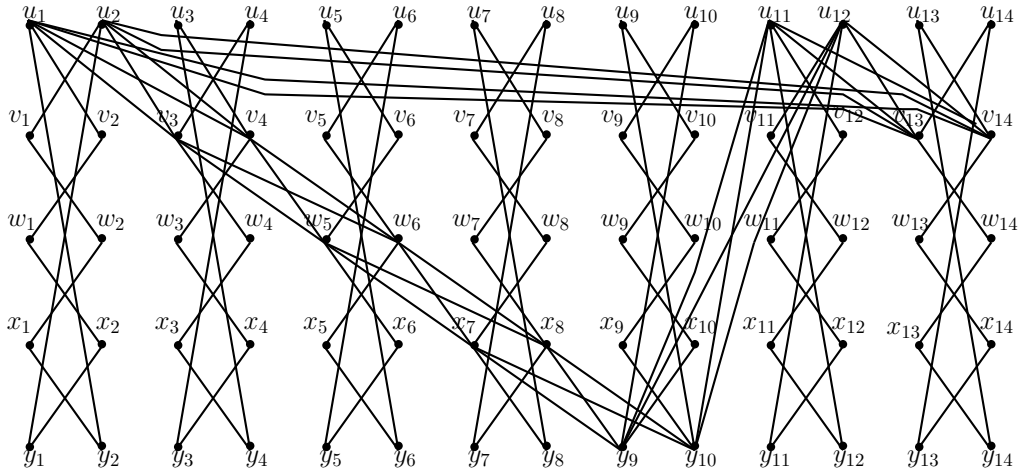


Figure 6

4. PROOF OF THE MAIN THEOREM

Proof of Theorem 1. The proof of the necessity is obvious and we prove the sufficiency in two cases. Since the tensor product is commutative, we may assume that m is odd and so $m \equiv 1, 3$ or $5 \pmod{6}$.

Case 1. $n \equiv 0$ or $1 \pmod{7}$.

Subcase 1.1. $m \equiv 1$ or $3 \pmod{6}$. By Theorem C, $C_3 \mid K_m$ and hence $K_m \times K_n = (C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)$. As $C_7 \mid C_3 \times K_n$, by Theorem 5, $C_7 \mid K_m \times K_n$.

Subcase 1.2. $m \equiv 5 \pmod{6}$. By Theorem C,

$$K_m = \underbrace{C_3 \oplus C_3 \oplus \cdots \oplus C_3}_{(m(m-1)-20)/6 \text{ times}} \oplus (C_5 \oplus C_5).$$

Now $K_m \times K_n = ((C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)) \oplus ((C_5 \times K_n) \oplus (C_5 \times K_n))$. As $C_7 \mid C_3 \times K_n$, by Theorem 5, and $C_7 \mid C_5 \times K_n$, by Theorem 11, $C_7 \mid K_m \times K_n$.

Case 2. $n \not\equiv 0$ or $1 \pmod{7}$. As $n(n-1) \not\equiv 0 \pmod{7}$, condition (1) implies that $m \equiv 0$ or $m \equiv 1 \pmod{7}$. As m is odd we have $m \equiv 1$ or $m \equiv 7 \pmod{14}$. As $C_7 \mid K_m$, by Theorem B, $K_m \times K_n = (C_7 \times K_n) \oplus (C_7 \times K_n) \oplus \cdots \oplus (C_7 \times K_n)$. $C_7 \mid C_7 \times K_n$, by Theorem 6, and so $C_7 \mid K_m \times K_n$.

This completes the proof. ■

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