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CHARACTERIZATION RESULTS FOR THE $L(2,1,1)$-LABELING PROBLEM ON TREES

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Abstract

An $L(2,1,1)$-labeling of a graph G is an assignment of non-negative integers (labels) to the vertices of G such that adjacent vertices receive labels with difference at least 2, and vertices at distance 2 or 3 receive distinct labels. The span of such a labelling is the difference between the maximum and minimum labels used, and the minimum span over all $L(2,1,1)$-labelings of G is called the $L(2,1,1)$-labeling number of G, denoted by $\lambda_{2,1,1}(G)$. It was shown by King, Ras and Zhou in [*The* $L(h,1,1)$*-labelling problem for trees*, European J. Combin. **31** (2010) 1295–1306] that every tree T has $\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T)$, where $\Delta_2(T) = \max_{uv \in E(T)}(d(u) + d(v))$. And they conjectured that almost all trees have the $L(2,1,1)$-labeling number attain the lower bound. This paper provides some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2(T)$. Furthermore, we show that the sufficient conditions we provide are also necessary for trees with diameter at most 6.

Keywords: $L(2,1,1)$-labeling, tree, diameter.

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1. Introduction

Multilevel distance labeling is a generalization of distance two labeling, which is motivated by the channel assignment problem introduced by Hale [9]. The channel assignment problem is the assignment of frequencies to transmitters subject to satisfy certain distance restrictions to avoid interference between nearby transmitters. If there is high usage of wireless communication networks, we have to find an appropriate channel assignment solution, so that the range of channels used is minimized.

Griggs and Yeh [8] firstly proposed the notation of distance two labeling of a graph, and they generalized it to p-levels of interference. Specifically for given positive integers $k_1, k_2, \ldots, k_p$, an $L(k_1, k_2, \ldots, k_p)$*-labeling* of a graph G is a function f from the vertices of G to non-negative integers (labels), such that for all distinct vertices u, v of G, $|f(u) - f(v)| \geq k_t$ if $dist(u, v) = t$, where $dist(u, v)$ denotes the distance between u and v. The *span* of f is the maximum difference $f(u) - f(v)$ of any pair of vertices u, v of G. The $L(k_1, k_2, \ldots, k_p)$*-labeling number*, denoted by $\lambda_{k_1,k_2,\ldots,k_p}(G)$, is the minimum span of all $L(k_1, k_2, \ldots, k_p)$-labelings of G. In practical terms, the label of the vertex u under f, i.e., $f(u)$ is the channel assigned to the transmitter corresponding to u.

The $L(k_1, k_2, \ldots, k_p)$-labeling problem above is interesting in both theory and practical applications. For instance, when $p = 1$ and $k_1 = 1$ it becomes the ordinary vertex-coloring problem. When $p = 2$, many interesting results (see [8, 3, 6, 13]) have been obtained for various families of finite graphs, especially for the case $(k_1, k_2) = (2, 1)$. For more details, one may refer to the surveys [14, 2]. In [8], it was proved that, for any finite tree T, $\Delta(T)+1 \leq \lambda_{2,1}(T) \leq \Delta(T)+2$, where $\Delta(T)$ is the maximum degree of T. A polynomial time algorithm for determining $\lambda_{2,1}(T)$ was given in [3]. Furthermore, a linear time algorithm for determining $\lambda_{2,1}(T)$ was given by authors of reference [10].

More recently, researchers began to investigate the $L(k_1, k_2, k_3)$-labeling problem [4, 11]. For example, Zhou studied the problem for hypercubes Q_n in [15]. The $L(h, 1, 1)$-labeling problem for outer-planar graphs was investigated in [**?**]. In [12], King *et al.* studied the $L(h, 1, 1)$-labeling problem for trees. They proved that $\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T)$ and proposed the following questions: 1. To characterize finite trees T with diameter at least 3 such that $\lambda_{2,1,1}(T) = \Delta_2(T)$ (Question 10 from [12]); 2. For a fixed integer $h \geq 2$, is the problem of determining $\lambda_{h,1,1}(T)$ for finite trees solvable in polynomial time? (Question 12 from [12]) In addition, they conjectured that almost all trees have the $L(2, 1, 1)$-labeling number attain the lower bound. Recently, the results in [7, 5] assert that deciding whether a given tree has the $L(2, 1, 1)$-labeling number attain the lower bound is *NP*-complete. So the Question 12 in [12] has been solved.

In this paper, we study the $L(2,1,1)$-labeling problem for finite trees with diameter at least 3. We provide some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2(T)$ in Section 3, which gives a partial answer for Question 10 in [12]. Furthermore, in Section 4, we show that the sufficient conditions we provide are also necessary for trees with diameter at most 6. This means that the problem of deciding whether the $L(2,1,1)$-labeling number of a tree T is $\Delta_2(T) - 1$ is polynomial for trees with diameter at most 6.

2. Preliminaries

In this paper, we always suppose that T is a finite tree with diameter at least 3. Define $\Delta_2(T) := \max_{uv \in E(T)}(d(u) + d(v))$, where $d(u)$ is the degree of u in T. Then $\Delta_2(T) \geq 3$ for a tree with diameter at least 3. An edge $e = uv$ is said to be *heavy* if $d(u) + d(v) = \Delta_2(T)$, *light* if $d(u) + d(v) < \Delta_2(T)$. In the following, we abbreviate $\Delta_2(T)$ to Δ_2.

For $u \in V(T)$, let $N(u) = \{w : uw \in E(T)\}$. Let $N_0(u) = \{w : uw \text{ is light}\}$, $d_0(u) = |N_0(u)|$ and $N_1(u) = \{w : uw \text{ is heavy}\}$, $d_1(u) = |N_1(u)|$. Then $N(u) = N_0(u) \cup N_1(u)$ and $d(u) = d_0(u) + d_1(u)$. Furthermore, let $N_{0,1}(u) = \{w \in N_0(u) : d(w) \geq 2, d_0(w) = 1\}$ and $d_{0,1}(u) = |N_{0,1}(u)|$.

Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the *root* of this tree. And we denote by T_u the tree rooted at u. For a rooted tree T_u, define $L_i(u) := \{w \in V(T_u) : dist(u,w) = i\}$ for $i = 0, 1, \ldots$. In particular, $L_0(u) = \{u\}$. Define $E_i(u) := \{xy : x \in L_{i-1}(u), y \in L_i(u)\}$ for $i = 1, 2, \ldots$. For $xy \in E(T_u)$, if $x \in L_{i-1}(u), y \in L_i(u)$, then we call x the *parent* of y, which is denoted by y_p. If x is the parent of y and y is the parent of z, then we call x the *grandparent* of z, which is denoted by z_g.

The *diameter* of T, denoted by $\operatorname{diam}(T)$, is the length of the longest path of T. Note that if $\operatorname{diam}(T)$ is even, then there must exist a vertex, say u, such that every path of length $\operatorname{diam}(T)$ goes through u. Thus if we treat T as a rooted tree T_u, then

$$
\begin{aligned}
V(T) &= \{u\} \cup L_1(u) \cup L_2(u) \cup \cdots \cup L_{\frac{\operatorname{diam}(T)}{2}}(u),\\
E(T) &= E_1(u) \cup E_2(u) \cup \cdots \cup E_{\frac{\operatorname{diam}(T)}{2}}(u).
\end{aligned}
$$

Such a vertex u is called the *crossing vertex* of T.

If $\operatorname{diam}(T)$ is odd, then there must exist an edge, say uv, such that every path of length $\operatorname{diam}(T)$ goes through uv. Such an edge uv is called the *crossing edge* of T. Let T_u and T_v be the two rooted trees obtained from T by deleting the edge uv, respectively. Then

$$\begin{aligned}V(T) = \{u,v\} &\cup L_1(u) \cup L_2(u) \cup \cdots \cup L_{\frac{\operatorname{diam}(T)-1}{2}}(u)\\ &\cup L_1(v) \cup L_2(v) \cup \cdots \cup L_{\frac{\operatorname{diam}(T)-1}{2}}(v),\\ E(T) = \{uv\} &\cup E_1(u) \cup E_2(u) \cup \cdots \cup E_{\frac{\operatorname{diam}(T)-1}{2}}(u)\\ &\cup E_1(v) \cup E_2(v) \cup \cdots \cup E_{\frac{\operatorname{diam}(T)-1}{2}}(v).\end{aligned}$$

3. Some Sufficient Conditions for $\lambda_{2,1,1}(T) = \Delta_2$

King *et al.* [12] studied the $L(2,1,1)$-labeling of trees and gave the following result.

Lemma 1 [12]**.** $\Delta_2 - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2$.

Before providing some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2$, we give two useful lemmas as follows.

For integers i and j with $i \leq j$, we denote $[i,j]$ as the set $\{i, i+1, \ldots, j-1, j\}$. Let $F = [0, \Delta_2 - 1]$.

Lemma 2. *Let f be an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$. Let uv be a heavy edge. Then $f(N(u)) \cup f(N(v)) = F$ and $|f(u) - f(v)| > 2$.*

Proof. Note that all the vertices in $N(u) \cup N(v)$ are of distance no more than 3 from each other. So they receive different labels under f. This implies $|f(N(u) \cup N(v))| = |N(u) \cup N(v)| = \Delta_2$, since uv is a heavy edge. Thus, $f(N(v)) \cup f(N(u)) = F$.

On the other hand, it is easy to see that $|f(u) - f(v)| \geq 2$. Suppose on the contrary that $|f(u) - f(v)| = 2$. Then the integer between $f(u)$ and $f(v)$ cannot be labeled on any vertex in $N(u) \cup N(v)$, a contradiction. So $|f(u) - f(v)| > 2$. ∎

Lemma 3. *Let f be an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$. If there exists a vertex u satisfying $d(u) \geq 2$ and $N_0(u) = \{w\}$, then either* (I) *$f(w) = d(u) - 1$, $f(N_1(u)) = [0, d(u) - 2]$ and $f(N(v)) = [d(u), \Delta_2 - 1]$ for every $v \in N_1(u)$ or* (II) *$f(w) = \Delta_2 - d(u), f(N_1(u)) = [\Delta_2 - d(u) + 1, \Delta_2 - 1]$ and $f(N(v)) = [0, \Delta_2 - d(u) - 1]$ for every $v \in N_1(u)$.*

Proof. By definition, uv is heavy for every $v \in N_1(u)$. Then by Lemma 2, $f(N(v)) = F \setminus f(N(u))$ for every $v \in N_1(u)$. In addition, for every $v \in N_1(u)$, $dist(v,x) = 1$ for all x in $N(v)$. This implies $f(v)$ is at least 2 apart from each integer in $f(N(v)) = F \setminus f(N(u))$. Thus each integer in $f(N_1(u))$ is at least 2 apart from each one in $F \setminus f(N(u))$. Hence, $f(N_1(u))$ and $F \setminus f(N(u))$ are two consecutive integer sets separated by $f(w)$, since $F = f(N_1(u)) \cup [F \setminus f(N(u))] \cup \{f(w)\} = [0, \Delta_2 - 1]$ is a set of consecutive integers. This implies the conclusion. ∎

In view of the above results, we now give some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2$.

Theorem 4. *Let T be a tree with diameter at least 3. If T contains one of the following configurations, then $\lambda_{2,1,1}(T) = \Delta_2$.*

(C1) *There exists a vertex u such that $d_0(u) = 0$.*

(C2) *There exist vertices u, v, w such that $u, v \in N_{0,1}(w)$ and $d(u) \neq d(v)$.*

(C3) *There exist vertices u, v, x, y such that $xy \in E(T)$, $u \in N_{0,1}(x)$, $v \in N_{0,1}(y)$ and one of the following holds:*

(i) *$d(u) + d(v) \geq \Delta_2$.*

(ii) *xy is heavy and $d(u) + d(v) \geq \Delta_2 - 1$.*

(C4) *There exist vertices u, v, x, y such that $dist(x, y) = 2$ or $dist(x, y) = 3$, $u \in N_{0,1}(x)$, $v \in N_{0,1}(y)$ and $d(u) = d(v) = \frac{\Delta_2+1}{2}$.*

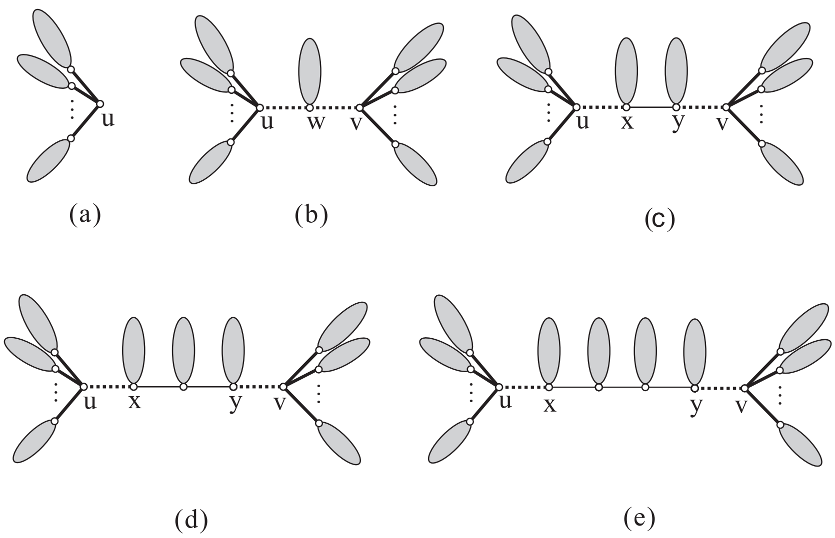

Figure 1. Figures for Theorem 4: (a) for (C1), (b) for (C2), (c) for (C3), (d) and (e) for (C4).

Proof. Suppose that T admits an $L(2,1,1)$-labeling f with span $\Delta_2 - 1$.

(C1) In the case, uv is heavy for each $v \in N(u)$, since $d_0(u) = 0$. Then by Lemma 2, $f(N(v)) = F \setminus f(N(u))$ for each $v \in N(u)$. In addition, for every $v \in N(u)$, $dist(v, x) = 1$ for every $x \in N(v)$. This implies $f(v)$ is at least 2 apart from each integer in $f(N(v)) = F \setminus f(N(u))$. That is, each integer in $f(N(u))$ is at least 2 apart from each one in $F \setminus f(N(u))$. But this is impossible, since $F = f(N(u)) \cup [F \setminus f(N(u))]$ is a set of consecutive integers. Thus, $\lambda_{2,1,1}(T) = \Delta_2$.

(C2) Let x be an arbitrary vertex in $N_1(v)$, and y be an arbitrary vertex in $N_1(u)$. Then vx and uy are heavy. So by Lemma 3 and the hypothesis of $d(u) \neq d(v)$, we have either $f(w) = d(u) - 1 = \Delta_2 - d(v)$ and $f(N_1(u)) = f(N(x)) = [0, d(u) - 2]$, or $f(w) = d(v) - 1 = \Delta_2 - d(u)$ and $f(N_1(v)) = f(N(y)) = [0, d(v) - 2]$. In addition, $v \in N(x)$ and $u \in N(y)$. So $f(v) \in f(N(x)) = f(N_1(u))$ and $f(u) \in f(N(y)) = f(N_1(v))$. On the other hand, $dist(u, x) = 3$ and $dist(v, y) = 3$, which implies $f(u) \neq f(x)$ and $f(v) \neq f(y)$. So $f(u) \notin f(N_1(v))$ and $f(v) \notin f(N_1(u))$, since x is an arbitrary vertex in $N_1(v)$, and y is an arbitrary vertex in $N_1(u)$, which is a contradiction. Therefore, $\lambda_{2,1,1}(T) = \Delta_2$.

(C3) By Lemma 3, $f(x) = d(u) - 1$, $f(N_1(u)) = [0, d(u) - 2]$ or $f(x) = \Delta_2 - d(u)$, $f(N_1(u)) = [\Delta_2 - d(u) + 1, \Delta_2 - 1]$, and $f(y) = d(v) - 1$, $f(N_1(v)) = [0, d(v) - 2]$ or $f(y) = \Delta_2 - d(v)$, $f(N_1(v)) = [\Delta_2 - d(v) + 1, \Delta_2 - 1]$.

(i) If $f(x) = d(u) - 1$, then $f(y) \geq f(x) + 2 = d(u) + 1$, since $dist(x, y) = 1$ and $dist(w_0, y) = 3$ for every $w_0 \in N_1(u)$. In the case, if $f(y) = d(v) - 1$, then $f(x) = d(u) - 1 \leq f(y) - 2 = d(v) - 3$. This implies $f(x) = d(u) - 1 \in [0, d(v) - 2]$, a contradiction with $dist(w_1, x) = 3$ for every $w_1 \in N_1(v)$. If $f(y) = \Delta_2 - d(v)$, then $f(y) = \Delta_2 - d(v) \geq f(x) + 2 = d(u) + 1$, which implies $d(u) + d(v) \leq \Delta_2 - 1$, a contradiction to $d(u) + d(v) \geq \Delta_2$. Similarly, it cannot be that $f(x) = \Delta_2 - d(u)$.

(ii) By Lemma 2, we have $|f(x) - f(y)| > 2$, since xy is heavy. If $f(x) = d(u) - 1$, then $f(y) > f(x) + 2 = d(u) + 1$, since $dist(x, y) = 1$ and $dist(w_0, y) = 3$ for every $w_0 \in N_1(u)$. In the case, if $f(y) = d(v) - 1$, then $f(x) = d(u) - 1 < f(y) - 2 = d(v) - 3$. This implies $f(x) = d(u) - 1 \in [0, d(v) - 2]$, a contradiction with $dist(w_1, x) = 3$ for every $w_1 \in N_1(v)$. If $f(y) = \Delta_2 - d(v)$, then $f(y) = \Delta_2 - d(v) > f(x) + 2 = d(u) + 1$, which implies $d(u) + d(v) < \Delta_2 - 1$, a contradiction to $d(u) + d(v) \geq \Delta_2 - 1$. Similarly, it cannot be that $f(x) = \Delta_2 - d(u)$.

(C4) By Lemma 3, $f(x) = d(u) - 1$ or $f(x) = \Delta_2 - d(u)$, and $f(y) = d(v) - 1$ or $f(y) = \Delta_2 - d(v)$. Then $f(x) = f(y)$, since $d(u) = d(v) = \frac{\Delta_2 + 1}{2}$, which is a contradiction with $dist(x, y) = 2$ or 3. ∎

Based on Theorem 4, we have the following results for subdivisions of trees and complete m-ary trees immediately.

For a tree T, a *subdivision* of T is a tree obtained from T by replacing every edge of T by a path of length greater than 1. A *complete m-ary tree* of height k, denoted by $T_{k;m}$, is a rooted tree such that each vertex other than leaves (degree-one vertices) has m children and all leaves are at distance k apart from the root.

Corollary 5. *Let T_s be a subdivision of a tree T. Then $\lambda_{2,1,1}(T_s) = \Delta_2(T_s)$.*

Proof. Let u be a vertex of T such that $d(u) = \Delta(T)$. Then all the edges incident with u are heavy in T_s, which implies $\lambda_{2,1,1}(T_s) = \Delta_2(T_s)$ by (C1) of Theorem 4. ∎

Corollary 6. *Let $T_{k;m}$ be a complete m-ary tree of height k, where $k = 2$ or $k \geq 4$. Then $\lambda_{2,1,1}(T_{k;m}) = \Delta_2(T_{k;m})$.*

Proof. Let u be the root of $T_{k;m}$. If $k = 2$, then all the edges incident with u are heavy. If $k \geq 4$, then all the edges incident with each vertex in $L_2(u)$ are heavy. Thus $\lambda_{2,1,1}(T_{k;m}) = \Delta_2(T_{k;m})$ by (C1) of Theorem 4. ■

4. RESULTS FOR TREES WITH DIAMETER AT MOST 6

The following results show that the sufficient conditions in Theorem 4 are also necessary for trees with diameter at most 6.

Theorem 7. *Let T be a tree with diameter 3. Then $\lambda_{2,1,1}(T) = \Delta_2 - 1$.*

Proof. Let uv be the crossing edge of T. Then T is the unique heavy edge of T. Define

(i) $f(u) = 0,\ f(v) = \Delta_2 - 1$.
(ii) $f(N(u) \setminus \{v\}) = [d(v), \Delta_2 - 2]$, $f(N(v) \setminus \{u\}) = [1, d(v) - 1]$.

Note that all the vertices of T have different labels. Next, $|f(u) - f(v)| = \Delta_2 - 1 \geq 2$, $\min_{x \in L_1(u)} |f(u) - f(x)| \geq d(v) \geq 2$ and $\min_{y \in L_1(v)} |f(v) - f(y)| \geq \Delta_2 - 1 - (d(v) - 1) = d(u) \geq 2$. So f is an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$, which implies $\lambda_{2,1,1}(T) \leq \Delta_2 - 1$. Thus, $\lambda_{2,1,1}(T) = \Delta_2 - 1$ by Lemma 1. ■

Theorem 8. *Let T be a tree with diameter 4. Then $\lambda_{2,1,1}(T) = \Delta_2 - 1$ if and only if $d_0(x) \geq 1$ for all $x \in V(T)$.*

Proof. The necessity follows from (C1) of Theorem 4. We now prove the sufficiency. Assume that $d_0(x) \geq 1$ for all $x \in V(T)$. It suffices to show that T admits an $L(2,1,1)$-labeling with span $\Delta_2 - 1$.

Let u be the crossing vertex of T. Consider the following labeling f.

(i) $f(u) = 0$.
(ii) $f(N_1(u)) = [\Delta_2 - d_1(u), \Delta_2 - 1]$, $f(N_0(u)) = [\Delta_2 - d(u), \Delta_2 - d_1(u) - 1]$.
(iii) $f(N(x) \setminus \{u\}) = [1, d(x) - 1]$ for each $x \in L_1(u)$.

It is clear that any pair of vertices of distance at most 3 have different labels. Secondly, $\min_{x \in L_1(u)} |f(u) - f(x)| \geq \Delta_2 - d(u) \geq 2$. Finally, since $d_0(u) \geq 1$ by the assumption, $\min_{y \in N(x) \setminus \{u\}} |f(x) - f(y)| \geq \Delta_2 - d(u) - d(x) + 1 \geq 2$ if ux is light; $\min_{y \in N(x) \setminus \{u\}} |f(x) - f(y)| \geq \Delta_2 - d(u) + 1 - d(x) + 1 \geq 2$ if ux is heavy. Thus, $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq 2$. So any pair of adjacent vertices have labels differing at least 2 apart.

Therefore, f is an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$, which implies $\lambda_{2,1,1}(T) \leq \Delta_2 - 1$. Thus, $\lambda_{2,1,1}(T) = \Delta_2 - 1$ by Lemma 1. ■

Theorem 9. *Let T be a tree with diameter* 5. *Then $\lambda_{2,1,1}(T) = \Delta_2 - 1$ if and only if $d_0(x) \geq 1$ for all $x \in V(T)$.*

Proof. The necessity follows from (C1) of Theorem 4. Now we prove the sufficiency. Assume that $d_0(x) \geq 1$ for all $x \in V(T)$.

Let uv be the crossing edge of T. Let T_u and T_v be the two rooted trees obtained from T by deleting uv. We treat the following two cases.

Case 1. If uv is heavy, then we define an $L(2,1,1)$-labeling of T as follows.

(i) $f(u) = 0$, $f(v) = \Delta_2 - 1$.

(ii) $f(N_1(v) \setminus \{u\}) = [1, d_1(v) - 1]$, $f(N_0(v)) = [d_1(v), d(v) - 1]$, $f(N_1(u) \setminus \{v\}) = [d(v) + d_0(u), \Delta_2 - 2]$, $f(N_0(u)) = [d(v), d(v) + d_0(u) - 1]$.

(iii) $f(N(x) \setminus \{u\}) = [1, d(x) - 1]$ for each $x \in L_1(u)$, $f(N(x) \setminus \{v\}) = [\Delta_2 - d(x), \Delta_2 - 2]$ for each $x \in L_1(v)$.

Firstly, it is not difficult to check that any pair of vertices of distance at most 3 have different labels. Next, $f(v) - f(u) = \Delta_2 - 1 \geq 2$, $\min_{x \in L_1(u)} |f(u) - f(x)| = d(v) \geq 2$ and $\min_{x \in L_1(v)} |f(v) - f(x)| = \Delta_2 - d(v) = d(u) \geq 2$. Finally, let $|f(w) - f(w_p)| = \min_{xy \in E_2(u)} |f(x) - f(y)|$, where $w \in L_2(u)$. Then $|f(w) - f(w_p)| \geq d(v) - \max_{x \in L_1(u)} d(x) + 1 \geq 1$. And the equality holds only if $f(w_p) = d(v)$ and $f(w) = d(w_p) - 1 = d(v) - 1$ (so $d(w_p) = d(v)$). Note that $N_0(u) \neq \emptyset$. Then $w_p \in N_0(u)$, since $f(w_p) = d(v) \in f(N_0(u))$. So uw_p is light. On the other hand, $d(u) + d(w_p) = d(u) + d(v) = \Delta_2$, which implies uw_p is heavy, a contradiction. Therefore, $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq 2$. Similarly, $\min_{xy \in E_2(v)} |f(x) - f(y)| \geq 2$. So any pair of adjacent vertices have labels differing at least 2 apart.

Hence, f is an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$, which implies $\lambda_{2,1,1}(T) \leq \Delta_2 - 1$. Thus, $\lambda_{2,1,1}(T) = \Delta_2 - 1$ by Lemma 1.

Case 2. If uv is light, then we define an $L(2,1,1)$-labeling of T as follows.

(i) $f(u) = d(v) - 1$, $f(v) = \Delta_2 - d(u)$.

(ii) $f(N_1(v)) = [0, d_1(v) - 1]$, $f(N_0(v) \setminus \{u\}) = [d_1(v), d(v) - 2]$, $f(N_1(u)) = [\Delta_2 - d_1(u), \Delta_2 - 1]$, $f(N_0(u) \setminus \{v\}) = [\Delta_2 - d(u) + 1, \Delta_2 - d_1(u) - 1]$.

(iii) $f(N(x) \setminus \{u\}) = [0, d(x) - 1] \setminus \{d(v) - 1\}$ for each $x \in L_1(u)$, $f(N(x) \setminus \{v\}) = [\Delta_2 - d(x), \Delta_2 - 1] \setminus \{\Delta_2 - d(u)\}$ for each $x \in L_1(v)$.

Note that any pair of vertices of distance at most 3 have different labels. Secondly, $f(v) - f(u) = \Delta_2 - d(u) - d(v) + 1 \geq 2$, since uv is light. And $\min_{x \in L_1(u)} |f(u) - f(x)| \geq \Delta_2 - d(u) + 1 - (d(v) - 1) > 2$ and $\min_{x \in L_1(v)} |f(v) - f(x)| \geq \Delta_2 - d(u) - (d(v) - 2) > 2$. Finally, $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq \Delta_2 - d(u) + 1 - \max_{x \in L_1(u)} d(x) + 1 \geq 2$. Similarly, $\min_{xy \in E_2(v)} |f(x) - f(y)| \geq 2$. So any pair of adjacent vertices have labels differing at least 2 apart.

Therefore, f is an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$, which implies $\lambda_{2,1,1}(T) \leq \Delta_2 - 1$. Thus, $\lambda_{2,1,1}(T) = \Delta_2 - 1$ by Lemma 1. ■

Given a set $A = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$. Let $S_k(A)$ and $B_k(A)$ denote the sets of the smallest and largest k numbers in A, respectively. That is, $S_k(A) = \{a_1, a_2, \ldots, a_k\}$ and $B_k(A) = \{a_{n-k+1}, a_{n-k+2}, \ldots, a_n\}$. Let $M_{[i,j]}(A)$ denote the set from the i-th to j-th element of A. That is, $M_{[i,j]}(A) = \{a_i, a_{i+1}, \ldots, a_j\}$.

Theorem 10. *Let T be a tree with diameter 6. Then $\lambda_{2,1,1}(T) = \Delta_2 - 1$ if and only if for each $x \in V(T)$, $d_0(x) \geq 1$ and all the vertices in $N_{0,1}(x)$ have the same degree.*

Proof. For each $x \in V(T)$, suppose that $d_0(x) \geq 1$ and all the vertices in $N_{0,1}(x)$ have the same degree. By Theorem 4, it is sufficient to prove $\lambda_{2,1,1}(T) = \Delta_2 - 1$ in this assumption.

Let u be the crossing vertex of T. Now we treat the following two cases.

Case 1. If $d_{0,1}(u) \geq 1$, without loss of generality, let $v \in N_{0,1}(u)$. Then we define an $L(2,1,1)$-labeling of T as follows.

(i) $f(u) = d(v) - 1$.

(ii) $f(N_1(u)) = [\Delta_2 - d_1(u), \Delta_2 - 1]$, $f(N_0(u)) = [\Delta_2 - d(u), \Delta_2 - d_1(u) - 1]$.

(iii) For each $x \in N_1(u)$, $f(N_1(x) \setminus \{u\}) = S_{d_1(x)-1}([0, d(x) - 1] \setminus \{d(v) - 1\})$, $f(N_0(x)) = f(N_0(x)) = M_{[d_1(x), d(x)-1]}([0, d(x) - 1] \setminus \{d(v) - 1\})$. For each $x \in N_0(u)$, $f(N_1(x)) = S_{d_1(x)}([0, d(x) - 1] \setminus \{d(v) - 1\})$, $f(N_0(x) \setminus \{u\}) = M_{[d_1(x)+1, d(x)-1]}([0, d(x) - 1] \setminus \{d(v) - 1\})$.

(iv) For each $x \in L_2(u)$ satisfying $d(x_p) \geq d(v)$, $f(N(x) \setminus \{x_p\}) = B_{d(x)-1}([\Delta_2 - d(x), \Delta_2 - 1] \setminus \{f(x_p)\})$. For each $x \in L_2(u)$ satisfying $d(x_p) < d(v)$, $f(N(x) \setminus \{x_p\}) = B_{d(x)-1}([\Delta_2 - d(x) - 1, \Delta_2 - 1] \setminus \{f(x_p), f(u)\})$.

Now we verify that f is an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$ by the following three steps.

Step 1. $|f(x) - f(y)| \geq 2$ if $dist(x, y) = 1$.

Firstly, $\min_{ux \in E_1(u)} |f(x) - f(u)| \geq \Delta_2 - d(u) - d(v) + 1 \geq 2$, since uv is light.

Secondly, $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq \Delta_2 - d(u) - \max_{x \in L_1(u)} d(x) + 1 \geq 1$. Suppose that there exists some $ww_p \in E_2(u)$ which makes the equality hold, where $w \in L_2(u)$. Then $f(w_p) = \Delta_2 - d(u)$, $f(w) = d(w_p) - 1$ and uw_p is heavy. But by the labeling way of f, $f(w_p) = \Delta_2 - d(u)$ will imply uw_p is light, a contradiction. Thus, $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq 2$.

Thirdly, let $|f(w) - f(w_p)| = \min_{xy \in E_3(u)} |f(x) - f(y)|$, where $w \in L_3(u)$.

If $d(w_g) \geq d(v)$, then $f(w) \geq \Delta_2 - d(w_p)$ and $f(w_p) \leq d(w_g) - 1$. If $f(w_p) \leq d(w_g) - 2$, then $|f(w) - f(w_p)| \geq \Delta_2 - d(w_p) - d(w_g) + 2 \geq 2$. If $f(w_p) = d(w_g) - 1$, then by the labeling way of f, we have $d(v) - 1 \in [0, d(w_g) - 2]$ which implies $d(w_g) \neq d(v)$. So $w_g \notin N_{0,1}(u)$, since $v \in N_{0,1}(u)$ and all the vertices in $N_{0,1}(u)$ have the same degree. Then $N_0(w_g) \setminus \{u\} \neq \emptyset$. Thus $w_p \in N_0(w_g) \setminus \{u\}$, since

$f(w_p) = d(w_g) - 1 \in f(N_0(w_g) \setminus \{u\})$ by the labeling way of f. So w_pw_g is light which also implies $|f(w) - f(w_p)| \geq \Delta_2 - d(w_p) - d(w_g) + 1 \geq 2$.

If $d(w_g) < d(v)$, then $f(w) \geq \Delta_2 - d(w_p) - 1$ and $f(w_p) \leq d(w_g) - 2$. In the case, $w_g \notin N_{0,1}(u)$, since $v \in N_{0,1}(u)$ and all the vertices in $N_{0,1}(u)$ have the same degree. Thus $N_0(w_g) \setminus \{u\} \neq \emptyset$. If $f(w_p) \leq d(w_g) - 3$, then $|f(w) - f(w_p)| \geq \Delta_2 - d(w_p) - d(w_g) + 2 \geq 2$. If $f(w_p) = d(w_g) - 2$, then $w_p \in N_0(w_g) \setminus \{u\}$, since $f(w_p) = d(w_g) - 2 \in f(N_0(w_g) \setminus \{u\})$ by the labeling way of f. So w_pw_g is light. This implies $|f(w) - f(w_p)| \geq \Delta_2 - d(w_p) - d(w_g) + 1 \geq 2$.

Thus, $\min_{xy \in E_3(u)} |f(x) - f(y)| \geq 2$.

Step 2. $|f(x) - f(y)| \geq 1$ if $dist(x, y) = 2$.

By the labeling way of f, we know that $|f(x) - f(y)| \geq 1$ for any two vertices x, y with $dist(x, y) = 2$.

Step 3. $|f(x) - f(y)| \geq 1$ if $dist(x, y) = 3$.

Firstly, $\min_{x \in L_1(u), y \in L_2(u)} |f(x) - f(y)| = \Delta_2 - d(u) - \max_{x \in L_1(u)} d(x) + 1 \geq 1$. Secondly, $\min_{w \in L_3(u), x \in N(w_g) \setminus \{u\}} |f(w) - f(x)| \geq \Delta_2 - d(w_p) - d(w_g) + 1 \geq 1$. Thirdly, for each $w \in L_3(u)$ with $d(w_g) < d(v)$, w has a different label from u; for each $w \in L_3(u)$ with $d(w_g) \geq d(v)$, $f(w) \geq \Delta_2 - d(w_p) \geq \Delta_2 - (\Delta_2 - d(w_g)) = d(w_g) \geq d(v)$. So $f(w) > f(u)$. Therefore, $|f(x) - f(y)| \geq 1$ for any two vertices x, y with $dist(x, y) = 3$.

Thus, f is an $L(2, 1, 1)$-labeling of T with span $\Delta_2 - 1$. So $\lambda_{2,1,1}(T) \leq \Delta_2 - 1$. And by Lemma 1, we have $\lambda_{2,1,1}(T) = \Delta_2 - 1$.

Case 2. If $d_{0,1}(u) = 0$, then we define an $L(2, 1, 1)$-labeling of T as follows.

(i) $f(u) = 0$.

(ii) $f(N_1(u)) = [\Delta_2 - d_1(u), \Delta_2 - 1]$, $f(N_0(u)) = [\Delta_2 - d(u), \Delta_2 - d_1(u) - 1]$.

(iii) For each $x \in N_1(u)$, $f(N_1(x) \setminus \{u\}) = [1, d_1(x) - 1]$, $f(N_0(x)) = [d_1(x), d(x) - 1]$. For each $x \in N_0(u)$, $f(N_1(x)) = [1, d_1(x)]$, $f(N_0(x) \setminus \{u\}) = [d_1(x) + 1, d(x) - 1]$.

(iv) For each $x \in L_2(u)$, $f(N(x) \setminus \{x_p\}) = B_{d_0(x)-1}([\Delta_2 - d(x), \Delta_2 - 1] \setminus \{f(x_p)\})$.

Now we will verify that f is an $L(2, 1, 1)$-labeling of T with span $\Delta_2 - 1$ by the following three steps.

Step 1. $|f(x) - f(y)| \geq 2$ if $dist(x, y) = 1$.

Firstly, $\min_{ux \in E_1(u)} |f(x) - f(u)| \geq \Delta_2 - d(u) \geq 2$.

Secondly, $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq \Delta_2 - d(u) - \max_{x \in L_1(u)} d(x) + 1 \geq 1$. Suppose that there exists some $ww_p \in E_2(u)$ which makes the equality hold, where $w \in L_2(u)$. Then $f(w_p) = \Delta_2 - d(u)$, $f(w) = d(w_p) - 1$ and ww_p is heavy. But by the labeling way of f, $f(w_p) = \Delta_2 - d(u)$ will imply ww_p is light, a contradiction. So $\min_{xy \in E_2(u)} |f(x) - f(y)| \geq 2$.

Thirdly, let $|f(w) - f(w_p)| = \min_{xy \in E_3(u)} |f(x) - f(y)|$, where $w \in L_3(u)$.

Then $f(w) \geq \Delta_2 - d(w_p)$ and $f(w_p) \leq d(w_g) - 1$. If $f(w_p) \leq d(w_g) - 2$, then $|f(w) - f(w_p)| \geq \Delta_2 - d(w_p) - d(w_g) + 2 \geq 2$. Note that $N_0(w_g) \setminus \{u\} \neq \emptyset$ in view of $d_{0,1}(u) = 0$. Thus, if $f(w_p) = d(w_g) - 1$, then $w_p \in N_0(w_g) \setminus \{u\}$, since $f(w_p) = d(w_g) - 1 \in f(N_0(w_g) \setminus \{u\})$ by the labeling way of f. So $w_p w_g$ is light, which also implies $|f(w) - f(w_p)| \geq \Delta_2 - d(w_p) - d(w_g) + 1 \geq 2$. Thus, $\min_{xy \in E_3(u)} |f(x) - f(y)| \geq 2$.

Step 2. $|f(x) - f(y)| \geq 1$ if $dist(x, y) = 2$.

By the labeling way of f, we know that $|f(x) - f(y)| \geq 1$ for any two vertices x, y with $dist(x, y) = 2$.

Step 3. $|f(x) - f(y)| \geq 1$ if $dist(x, y) = 3$.

Firstly, $\min_{x \in L_1(u), y \in L_2(u)} |f(x) - f(y)| \geq \Delta_2 - d(u) - \max_{x \in L_1(u)} d(x) - 1 \geq 1$. Secondly, $\min_{w \in L_3(u), x \in N(w_g) \setminus \{u\}} |f(w) - f(x)| \geq \Delta_2 - d(w_p) - d(w_g) + 1 \geq 1$. Thirdly, $\min_{w \in L_3(u)} f(w) \geq \Delta_2 - d(w_p) \geq \Delta_2 - (\Delta_2 - d(w_g)) = d(w_g) > 0 = f(u)$. So $|f(x) - f(y)| \geq 1$ for any two vertices x, y with $dist(x, y) = 3$.

Therefore, f is an $L(2,1,1)$-labeling of T with span $\Delta_2 - 1$. So $\lambda_{2,1,1}(T) \leq \Delta_2 - 1$. And by Lemma 1, we have $\lambda_{2,1,1}(T) = \Delta_2 - 1$. ■

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