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Note

ON DOUBLE-STAR DECOMPOSITION OF GRAPHS

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Abstract

A tree containing exactly two non-pendant vertices is called a double-star. A double-star with degree sequence $(k_1+1, k_2+1, 1, \ldots, 1)$ is denoted by S_{k_1,k_2}. We study the edge-decomposition of graphs into double-stars. It was proved that every double-star of size k decomposes every $2k$-regular graph. In this paper, we extend this result by showing that every graph in which every vertex has degree $2k+1$ or $2k+2$ and containing a 2-factor is decomposed into S_{k_1,k_2} and S_{k_1-1,k_2}, for all positive integers k_1 and k_2 such that $k_1+k_2=k$.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. We denote the set of all neighbors of v by $N(v)$. The degree of a vertex v in G is denoted by $d_G(v)$ (by $d(v)$ when no confusion can arise). By *size* and *order* of G we mean $|E(G)|$ and

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$|V(G)|$, respectively. A subset $M \subseteq E(G)$ is called a *matching* if no two edges of M are adjacent. A matching M is called a *perfect matching*, if every vertex of G is incident with an edge of M.

A *factor* of G is a spanning subgraph of G. A subgraph H is called an *r-factor* if H is a factor of G and $d_H(v) = r$, for every $v \in V(G)$.

If $d(v) = 1$, then v is called a *pendant vertex*. A tree containing exactly two non-pendant vertices is called a *double-star*. A double-star with degree sequence $(k_1 + 1, k_2 + 1, 1, \ldots, 1)$ is denoted by S_{k_1,k_2}. Suppose that $u_1, u_2 \in V(S_{k_1,k_2})$ and $d(u_i) = k_i + 1$, for $i = 1, 2$. Then $e = u_1u_2$ is called the *central edge* of the double-star.

For a graph H, the graph G has an *H-decomposition*, if all edges of G can be partitioned into subgraphs isomorphic to H. Also, we say that G has an *$\{H_1, \ldots, H_t\}$-decomposition* if all edges of G can be partitioned into subgraphs, each of them isomorphic to some H_i, for $1 \le i \le t$. If G has an H-decomposition, we say that G is *H-decomposable*. A graph G is *k-factorable* if it can be decomposed into k-factors.

Let G be a directed graph and $v \in V(G)$. We define $N^+(v) = \{u \in V(G) : (v, u) \in E(G)\}$, where (v, u) denotes the edge from v to u. By *out-degree* of v we mean $|N^+(v)|$ and denote it by $d^+_G(v)$. Similarly, we define $N^-(v) = \{u \in V(G) : (u, v) \in E(G)\}$ and denote $|N^-(v)|$ by $d^-_G(v)$. An orientation O is called *Eulerian* if $d^+_G(v) = d^-_G(v)$, for every $v \in V(G)$. A *k-orientation* is an orientation such that $d^+_G(v) = k$, for every $v \in V(G)$.

In 1979, Kötzig conjectured that every $(2k + 1)$-regular graph can be decomposed into $S_{k,k}$ if and only if it has a perfect matching [6]. Jaeger, Payan and Kouider in 1983 proved this conjecture, see [5]. El-Zanati *et al.* proved that every $2k$-regular graph containing a perfect matching is $S_{k,k-1}$-decomposable, see [3]. The following interesting conjecture was proposed by Ringel, see [7].

Conjecture 1. *Every tree of size k decomposes the complete graph K_{2k+1}.*

A broadening of Ringel's conjecture is due to Graham and Häggkvist.

Conjecture 2. *Every tree of size k decomposes every $2k$-regular graph.*

El-Zanati *et al.* proved the following theorem in [3].

Theorem 3. *Every double-star of size k decomposes every $2k$-regular graph.*

Jacobson *et al.* in 1991 proposed the following conjecture about the tree decomposition of regular bipartite graphs, see [4].

Conjecture 4. *Let T be a tree of size r. Then every r-regular bipartite graph is T-decomposable.*

They proved that the conjecture holds for double-stars. In this paper, we study double-star decomposition of graphs. First, we prove some results about the double-star decomposition of regular bipartite graphs. We present a short proof for Conjecture 4, when T is a double-star. Then we present a theorem which indicates that every graph in which every vertex has degree $2k+1$ or $2k+2$ which contains a 2-factor is $\{S_{k_1,k_2}, S_{k_1-1,k_2}\}$-decomposable, for any double-star S_{k_1,k_2} of size $k+1$. This theorem generalizes Theorem 3.

2. Results

In this section, we prove some results about the double-star decomposition of graphs. The following theorem was proved in [4]. We present a short proof for this result.

Theorem 5. *For $r \geq 3$, let G be an r-regular bipartite graph. Then every double-star of size r decomposes G.*

Proof. Let A and B be two parts of G. Then König's Theorem [1, Theorem 2.2] implies that G has a 1-factorization with 1-factors $M_1, \ldots, M_r$. Suppose that S_{k_1,k_2} is a double-star of size r. Now, let G_1 and G_2 be two induced subgraphs of G with the edges $M_1 \cup M_2 \cup \cdots \cup M_{k_1}$ and $M_{k_1+1} \cup \cdots \cup M_{r-1}$, respectively. Suppose that $e = u_1u_2 \in M_r$, where $u_1 \in A$ and $u_2 \in B$. Now, define S_e to be the double-star containing the central edge e, $E_1(u_1)$ and $E_2(u_2)$, where $E_i(u_i)$ is the set of all edges incident with u_i in G_i. Clearly, S_e is isomorphic to S_{k_1,k_2}. On the other hand, S_e and $S_{e'}$ are edge disjoint, for every two distinct edges $e, e' \in M_r$. Hence, $E(G) = \bigcup_{e \in M_1} S_e$, and this completes the proof. ■

Now, we have the following corollaries.

Corollary 6. *Let $r, s \geq 3$ be positive integers and $s \mid r$. Then every r-regular bipartite graph can be decomposed into any double-star of size s.*

Proof. Let $r = sk$ and S_{k_1,k_2} be a double-star of size s. Since G is 1-factorable, G can be decomposed into spanning subgraphs $G_1, \ldots, G_k$, each of them is s-regular. Now, Theorem 5 implies that each G_i can be decomposed into S_{k_1,k_2}, and this completes the proof. ■

Corollary 7. *Let r, s, k and t be positive integers such that $r = sk + t$ and $r, s, t \geq 3$. Moreover, suppose that S_1 and S_2 are two double-stars of size s and t, respectively. Then every r-regular bipartite graph G is $\{S_1, S_2\}$-decomposable.*

Proof. Similarly to the proof of the previous corollary, G can be decomposed into $G_1, \ldots, G_{k+1}$, where $G_1, \ldots, G_k$ are s-regular and G_{k+1} is t-regular. Now, Theorem 5 implies that $G_1, \ldots, G_k$ and G_{k+1} can be decomposed into S_1 and S_2, respectively. This completes the proof. ■

Another generalization of Theorem 5 is as follows.

Theorem 8. *Let $r \geq 3$ be an integer and $G = (A, B)$ be a bipartite graph such that for every $v \in V(G)$, $r \mid d(v)$. Then every double-star of size r decomposes G.*

Proof. Let $v \in A$ and $d(v) = rk$, for some positive integer k, and S be a double-star of size r. Suppose that $N(v) = \{u_1, \ldots, u_{rk}\}$. Let G' be the graph obtained from G by removing v and adding $v_1, \ldots, v_k$ to A. For $i = 1, \ldots, k$, join v_i to every vertex of the set $\{u_{(i-1)r+1}, \ldots, u_{ir}\}$. It is not hard to see that if G' is S-decomposable, then G is S-decomposable, too.

By repeating this procedure for all vertices of G, one can obtain an r-regular bipartite graph, say H. Now, Theorem 5 implies that H is S-decomposable and hence G is S-decomposable. ■

Now, we generalize Theorem 3. We prove the following.

Theorem 9. *Let k be an integer and G be a graph in which every vertex has degree $2k+1$ or $2k+2$. Also, suppose that k_1 and k_2 are two positive integers such that $k_1 + k_2 = k$. If G contains a 2-factor, then G is $\{S_{k_1,k_2}, S_{k_1-1,k_2}\}$-decomposable.*

Proof. We will use the following structure given in [3]. Let G be a $2k$-regular graph. Then Petersen Theorem [1, Theorem 3.1] implies that G is 2-factorable. Let F be a 2-factor of G with cycles $C_1, \ldots, C_l$. Note that $G \setminus F$ has an Eulerian orientation. Also, orient C_i clockwise, for $i = 1, \ldots, l$, to obtain an Eulerian orientation of G. We define G_{C_i} as the subgraph of G with the edge set $E = \{(u, v) : u \in V(C_i)\}$. Clearly, $\{G_{C_1}, \ldots, G_{C_l}\}$ partitions $E(G)$. So, if we show that each G_{C_i} is H-decomposable, for some H, then G is H-decomposable too.

As mentioned, it suffices to decompose G_{C_i}, for $i = 1, \ldots, l$, into S_{k_1,k_2} and S_{k_1-1,k_2}. Add a new vertex z adjacent to all vertices of degree $2k+1$ and call the resulting graph H. Now, we use the method given in the proof of Theorem 3. Note that all vertices in H have even degree. So, H has an Eulerian orientation in which cycles of F are directed. Let $C : v_0, e_1, v_1, \ldots, v_{t-1}, e_t, v_t = v_0$ be a cycle in F. Then $d^+(v_i) = k_1 + k_2 + 1$, for $i = 1, \ldots, t$. We use $e_1, \ldots, e_t$ as the central edges of double-stars.

Start with an edge e_i in which (u_i, z) is a directed edge (if such an edge exists). With no loss of generality assume that $i = 1$. Choose k_1 edges directed out from v_1 such that (v_1, z) is chosen. Note that since e_2 is going to be a central edge, it cannot be one of these k_1 edges. Call the set of end vertices of these edges Y_1. Now, we start constructing double-stars. In the first step, use e_2 as the central edge. Choose the remaining k_2 edges directed out from v_1 and call them X_2. We choose k_1 edges directed out from v_2 such that:

(1) no edge (v_2, u) is chosen in which $u \in X_2$;

(2) include all the edges (v_2, u) in which $u \in Y_1$.

Note that since e_3 is going to be a central edge, it cannot be one of these k_1 edges. Call the set of these vertices Y_2. Our first double-star consists of central edge e_2 together with k_2 edges from v_1 to X_2, and k_1 or $k_1 - 1$ edges from v_2 to Y_2 (excluding $v_2 z$ if this is an edge). We repeat this procedure for $i = 3, \ldots, t, 1$. Note that this can be done because in the i-th step, all edges directed out from v_{i+1} to Y_1 are chosen in Y_i and hence $X_{i+1} \cap Y_1 = \emptyset$. Also, it is clear that if a double-star contains z, then z is adjacent to a vertex of degree k_1. Hence, G is $\{S_{k_1-1,k_2}, S_{k_1,k_2}\}$-decomposable. ■

We have an immediate corollary for $(2k+1)$-regular graphs.

Corollary 10. *Let k be a positive integer and G be a $(2k+1)$-regular graph containing a 2-factor. Then G is $\{S_{k_1,k_2}, S_{k_1-1,k_2}\}$-decomposable, for any double-star S_{k_1,k_2} of size k.*

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