## Note

# ON DOUBLE-STAR DECOMPOSITION OF GRAPHS 

Saieed Akbari ${ }^{a}$, Shahab Haghi ${ }^{b}$<br>Hamidreza Maimani ${ }^{b}$ and Abbas Seify ${ }^{1, b}$<br>${ }^{a}$ Department of Mathematical Sciences Sharif University of Technology<br>Tehran, Iran, P.O. Box 11365-11155<br>${ }^{b}$ Mathematics Section, Department of Basic Sciences<br>Shahid Rajaee Teacher Training University<br>Tehran, Iran, P.O. Box 16783-163<br>e-mail: s_akbari@sharif.edu<br>sh.haghi@ipm.ir<br>maimani@ipm.ir<br>abbas.seify@gmail.com


#### Abstract

A tree containing exactly two non-pendant vertices is called a doublestar. A double-star with degree sequence $\left(k_{1}+1, k_{2}+1,1, \ldots, 1\right)$ is denoted by $S_{k_{1}, k_{2}}$. We study the edge-decomposition of graphs into double-stars. It was proved that every double-star of size $k$ decomposes every $2 k$-regular graph. In this paper, we extend this result by showing that every graph in which every vertex has degree $2 k+1$ or $2 k+2$ and containing a 2 -factor is decomposed into $S_{k_{1}, k_{2}}$ and $S_{k_{1}-1, k_{2}}$, for all positive integers $k_{1}$ and $k_{2}$ such that $k_{1}+k_{2}=k$.


Keywords: graph decomposition, double-stars, bipartite graph.
2010 Mathematics Subject Classification: 05C51, 05 C 05.

## 1. Introduction

Let $G=(V(G), E(G))$ be a graph and $v \in V(G)$. We denote the set of all neighbors of $v$ by $N(v)$. The degree of a vertex $v$ in $G$ is denoted by $d_{G}(v)$ (by $d(v)$ when no confusion can arise). By size and order of $G$ we mean $|E(G)|$ and

[^0]$|V(G)|$, respectively. A subset $M \subseteq E(G)$ is called a matching if no two edges of $M$ are adjacent. A matching $M$ is called a perfect matching, if every vertex of $G$ is incident with an edge of $M$.

A factor of $G$ is a spanning subgraph of $G$. A subgraph $H$ is called an $r$-factor if $H$ is a factor of $G$ and $d_{H}(v)=r$, for every $v \in V(G)$.

If $d(v)=1$, then $v$ is called a pendant vertex. A tree containing exactly two non-pendant vertices is called a double-star. A double-star with degree sequence $\left(k_{1}+1, k_{2}+1,1, \ldots, 1\right)$ is denoted by $S_{k_{1}, k_{2}}$. Suppose that $u_{1}, u_{2} \in V\left(S_{k_{1}, k_{2}}\right)$ and $d\left(u_{i}\right)=k_{i}+1$, for $i=1,2$. Then $e=u_{1} u_{2}$ is called the central edge of the double-star.

For a graph $H$, the graph $G$ has an $H$-decomposition, if all edges of $G$ can be partitioned into subgraphs isomorphic to $H$. Also, we say that $G$ has an $\left\{H_{1}, \ldots, H_{t}\right\}$-decomposition if all edges of $G$ can be partitioned into subgraphs, each of them isomorphic to some $H_{i}$, for $1 \leq i \leq t$. If $G$ has an $H$-decomposition, we say that $G$ is $H$-decomposable. A graph $G$ is $k$-factorable if it can be decomposed into $k$-factors.

Let $G$ be a directed graph and $v \in V(G)$. We define $N^{+}(v)=\{u \in V(G)$ : $(v, u) \in E(G)\}$, where $(v, u)$ denotes the edge from $v$ to $u$. By out-degree of $v$ we mean $\left|N^{+}(v)\right|$ and denote it by $d_{G}^{+}(v)$. Similarly, we define $N^{-}(v)=\{u \in$ $V(G):(u, v) \in E(G)\}$ and denote $\left|N^{-}(v)\right|$ by $d_{G}^{-}(v)$. An orientation $O$ is called Eulerian if $d_{G}^{+}(v)=d_{G}^{-}(v)$, for every $v \in V(G)$. A $k$-orientation is an orientation such that $d_{G}^{+}(v)=k$, for every $v \in V(G)$.

In 1979, Kötzig conjectured that every $(2 k+1)$-regular graph can be decomposed into $S_{k, k}$ if and only if it has a perfect matching [6]. Jaeger, Payan and Kouider in 1983 proved this conjecture, see [5]. El-Zanati et al. proved that every $2 k$-regular graph containing a perfect matching is $S_{k, k-1}$-decomposable, see [3]. The following interesting conjecture was proposed by Ringel, see [7].

Conjecture 1. Every tree of size $k$ decomposes the complete graph $K_{2 k+1}$.
A broadening of Ringel's conjecture is due to Graham and Häggkvist.
Conjecture 2. Every tree of size $k$ decomposes every $2 k$-regular graph.
El-Zanati et al. proved the following theorem in [3].
Theorem 3. Every double-star of size $k$ decomposes every $2 k$-regular graph.
Jacobson et al. in 1991 proposed the following conjecture about the tree decomposition of regular bipartite graphs, see [4].

Conjecture 4. Let $T$ be a tree of size $r$. Then every $r$-regular bipartite graph is $T$-decomposable.

They proved that the conjecture holds for double-stars. In this paper, we study double-star decomposition of graphs. First, we prove some results about the double-star decomposition of regular bipartite graphs. We present a short proof for Conjecture 4, when $T$ is a double-star. Then we present a theorem which indicates that every graph in which every vertex has degree $2 k+1$ or $2 k+2$ which contains a 2 -factor is $\left\{S_{k_{1}, k_{2}}, S_{k_{1}-1, k_{2}}\right\}$-decomposable, for any double-star $S_{k_{1}, k_{2}}$ of size $k+1$. This theorem generalizes Theorem 3 .

## 2. Results

In this section, we prove some results about the double-star decomposition of graphs. The following theorem was proved in [4]. We present a short proof for this result.

Theorem 5. For $r \geq 3$, let $G$ be an $r$-regular bipartite graph. Then every doublestar of size $r$ decomposes $G$.

Proof. Let $A$ and $B$ be two parts of $G$. Then König's Theorem [1, Theorem 2.2] implies that $G$ has a 1-factorization with 1-factors $M_{1}, \ldots, M_{r}$. Suppose that $S_{k_{1}, k_{2}}$ is a double-star of size $r$. Now, let $G_{1}$ and $G_{2}$ be two induced subgraphs of $G$ with the edges $M_{1} \cup M_{2} \cup \cdots \cup M_{k_{1}}$ and $M_{k_{1}+1} \cup \cdots \cup M_{r-1}$, respectively. Suppose that $e=u_{1} u_{2} \in M_{r}$, where $u_{1} \in A$ and $u_{2} \in B$. Now, define $S_{e}$ to be the double-star containing the central edge $e, E_{1}\left(u_{1}\right)$ and $E_{2}\left(u_{2}\right)$, where $E_{i}\left(u_{i}\right)$ is the set of all edges incident with $u_{i}$ in $G_{i}$. Clearly, $S_{e}$ is isomorphic to $S_{k_{1}, k_{2}}$. On the other hand, $S_{e}$ and $S_{e^{\prime}}$ are edge disjoint, for every two distinct edges $e, e^{\prime} \in M_{r}$. Hence, $E(G)=\bigcup_{e \in M_{1}} S_{e}$, and this completes the proof.

Now, we have the following corollaries.
Corollary 6. Let $r, s \geq 3$ be positive integers and $s \mid r$. Then every $r$-regular bipartite graph can be decomposed into any double-star of size $s$.

Proof. Let $r=s k$ and $S_{k_{1}, k_{2}}$ be a double-star of size $s$. Since $G$ is 1-factorable, $G$ can be decomposed into spanning subgraphs $G_{1}, \ldots, G_{k}$, each of them is $s$ regular. Now, Theorem 5 implies that each $G_{i}$ can be decomposed into $S_{k_{1}, k_{2}}$, and this completes the proof.

Corollary 7. Let $r, s, k$ and $t$ be positive integers such that $r=s k+t$ and $r, s, t \geq 3$. Moreover, suppose that $S_{1}$ and $S_{2}$ are two double-stars of size $s$ and $t$, respectively. Then every r-regular bipartite graph $G$ is $\left\{S_{1}, S_{2}\right\}$-decomposable.

Proof. Similarly to the proof of the previous corollary, $G$ can be decomposed into $G_{1}, \ldots, G_{k+1}$, where $G_{1}, \ldots, G_{k}$ are $s$-regular and $G_{k+1}$ is $t$-regular. Now,

Theorem 5 implies that $G_{1}, \ldots, G_{k}$ and $G_{k+1}$ can be decomposed into $S_{1}$ and $S_{2}$, respectively. This completes the proof.

Another generalization of Theorem 5 is as follows.
Theorem 8. Let $r \geq 3$ be an integer and $G=(A, B)$ be a bipartite graph such that for every $v \in V(G), r \mid d(v)$. Then every double-star of size $r$ decomposes $G$.

Proof. Let $v \in A$ and $d(v)=r k$, for some positive integer $k$, and $S$ be a doublestar of size $r$. Suppose that $N(v)=\left\{u_{1}, \ldots, u_{r k}\right\}$. Let $G^{\prime}$ be the graph obtained from $G$ by removing $v$ and adding $v_{1}, \ldots, v_{k}$ to $A$. For $i=1, \ldots, k$, join $v_{i}$ to every vertex of the set $\left\{u_{(i-1) r+1}, \ldots, u_{i r}\right\}$. It is not hard to see that if $G^{\prime}$ is $S$-decomposable, then $G$ is $S$-decomposable, too.

By repeating this procedure for all vertices of $G$, one can obtain an $r$-regular bipartite graph, say $H$. Now, Theorem 5 implies that $H$ is $S$-decomposable and hence $G$ is $S$-decomposable.

Now, we generalize Theorem 3. We prove the following.
Theorem 9. Let $k$ be an integer and $G$ be a graph in which every vertex has degree $2 k+1$ or $2 k+2$. Also, suppose that $k_{1}$ and $k_{2}$ are two positive integers such that $k_{1}+k_{2}=k$. If $G$ contains a 2 -factor, then $G$ is $\left\{S_{k_{1}, k_{2}}, S_{k_{1}-1, k_{2}}\right\}$ decomposable.

Proof. We will use the following structure given in [3]. Let $G$ be a $2 k$-regular graph. Then Petersen Theorem [1, Theorem 3.1] implies that $G$ is 2 -factorable. Let $F$ be a 2 -factor of $G$ with cycles $C_{1}, \ldots, C_{l}$. Note that $G \backslash F$ has an Eulerian orientation. Also, orient $C_{i}$ clockwise, for $i=1, \ldots, l$, to obtain an Eulerian orientation of $G$. We define $G_{C_{i}}$ as the subgraph of $G$ with the edge set $E=$ $\left\{(u, v): u \in V\left(C_{i}\right)\right\}$. Clearly, $\left\{G_{C_{1}}, \ldots, G_{C_{l}}\right\}$ partitions $E(G)$. So, if we show that each $G_{C_{i}}$ is $H$-decomposable, for some $H$, then $G$ is $H$-decomposable too.

As mentioned, it suffices to decompose $G_{C_{i}}$, for $i=1, \ldots, l$, into $S_{k_{1}, k_{2}}$ and $S_{k_{1}-1, k_{2}}$. Add a new vertex $z$ adjacent to all vertices of degree $2 k+1$ and call the resulting graph $H$. Now, we use the method given in the proof of Theorem 3. Note that all vertices in $H$ have even degree. So, $H$ has an Eulerian orientation in which cycles of $F$ are directed. Let $C: v_{0}, e_{1}, v_{1}, \ldots, v_{t-1}, e_{t}, v_{t}=v_{0}$ be a cycle in $F$. Then $d^{+}\left(v_{i}\right)=k_{1}+k_{2}+1$, for $i=1, \ldots, t$. We use $e_{1}, \ldots, e_{t}$ as the central edges of double-stars.

Start with an edge $e_{i}$ in which $\left(u_{i}, z\right)$ is a directed edge (if such an edge exists). With no loss of generality assume that $i=1$. Choose $k_{1}$ edges directed out from $v_{1}$ such that $\left(v_{1}, z\right)$ is chosen. Note that since $e_{2}$ is going to be a central edge, it cannot be one of these $k_{1}$ edges. Call the set of end vertices of these edges $Y_{1}$. Now, we start constructing double-stars. In the first step, use $e_{2}$ as the
central edge. Choose the remaining $k_{2}$ edges directed out from $v_{1}$ and call them $X_{2}$. We choose $k_{1}$ edges directed out from $v_{2}$ such that:
(1) no edge $\left(v_{2}, u\right)$ is chosen in which $u \in X_{2}$;
(2) include all the edges $\left(v_{2}, u\right)$ in which $u \in Y_{1}$.

Note that since $e_{3}$ is going to be a central edge, it cannot be one of these $k_{1}$ edges. Call the set of these vertices $Y_{2}$. Our first double-star consists of central edge $e_{2}$ together with $k_{2}$ edges from $v_{1}$ to $X_{2}$, and $k_{1}$ or $k_{1}-1$ edges from $v_{2}$ to $Y_{2}$ (excluding $v_{2} z$ if this is an edge). We repeat this procedure for $i=3, \ldots, t, 1$. Note that this can be done because in the $i$-th step, all edges directed out from $v_{i+1}$ to $Y_{1}$ are chosen in $Y_{i}$ and hence $X_{i+1} \cap Y_{1}=\emptyset$. Also, it is clear that if a double-star contains $z$, then $z$ is adjacent to a vertex of degree $k_{1}$. Hence, $G$ is $\left\{S_{k_{1}-1, k_{2}}, S_{k_{1}, k_{2}}\right\}$-decomposable.

We have an immediate corollary for $(2 k+1)$-regular graphs.
Corollary 10. Let $k$ be a positive integer and $G$ be a $(2 k+1)$-regular graph containing a-factor. Then $G$ is $\left\{S_{k_{1}, k_{2}}, S_{k_{1}-1, k_{2}}\right\}$-decomposable, for any doublestar $S_{k_{1}, k_{2}}$ of size $k$.

## Acknowledgement

We thank the reviewer for his/her thorough review and highly appreciate comments and suggestions, which significantly contributed to improving the quality of the publication.

## References

[1] J. Akiyama and M. Kano, Factors and Factorizations of Graphs (London, Springer, 2011).
doi:10.1007/978-3-642-21919-1
[2] A. Bondy and U.S.R. Murty, Graph Theory (Graduate Texts in Mathematics, Springer, 2008).
[3] S.I. El-Zanati, M. Ermete, J. Hasty, M.J. Plantholt and S. Tipnis, On decomposing regular graphs into isomorphic double-stars, Discuss. Math. Graph Theory 35 (2015) 73-79. doi:10.7151/dmgt. 1779
[4] M. Jacobson, M. Truszczyński and Zs. Tuza, Decompositions of regular bipartite graphs, Discrete Math. 89 (1991) 17-27.
doi:10.1016/0012-365X(91)90396-J
[5] F. Jaeger, C. Payan and M. Kouider, Partition of odd regular graphs into bistars, Discrete Math. 46 (1983) 93-94.
doi:10.1016/0012-365X(83)90275-3
[6] A. Kötzig, Problem 1, in: Problem session, Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing, Congr. Numer. XXIV (1979) 913-915.
[7] G. Ringel, Problem 25, in: Theory of Graphs and its Applications, Proc. Symposium Smolenice 1963 (Prague, 1964), 162.

Received 4 August 2015
Revised 9 May 2016
Accepted 9 May 2016


[^0]:    ${ }^{1}$ Corresponding author.

