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UNION OF DISTANCE MAGIC GRAPHS

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Abstract

A distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection ℓ from V to the set $\{1, \ldots, n\}$ such that the weight $w(x) = \sum_{y \in N_G(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element μ, called the *magic constant*. In this paper, we study unions of distance magic graphs as well as some properties of such graphs.

Keywords: distance magic labeling, magic constant, sigma labeling, graph labeling, union of graphs, lexicographic product, direct product, Kronecker product, Kotzig array.

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1. Definitions

All graphs $G = (V, E)$ are finite undirected simple graphs. For standard graph theoretic notation and definitions we refer to Diestel [10]. For a graph G, we use $V(G)$ for the vertex set and $E(G)$ for the edge set of G. The *open neighborhood* $N(x)$ (or more precisely $N_G(x)$, when needed) of a vertex x is the set of all vertices adjacent to x, and the *degree* $d(x)$ of x is $|N(x)|$, i.e., the size of the neighborhood of x. By $N[x]$ (or $N_G[x]$) we denote the *closed neighborhood* $N(x) \cup \{x\}$ of x. By C_n we denote a cycle on n vertices.

Different kinds of labelings have been an important part of graph theory for years. See a dynamic survey [14] which covers the field. The subject of our investigation is the distance magic labeling. A *distance magic labeling* of a graph G of order n is a bijection $\ell : V \to \{1, 2, \ldots, n\}$ such that there exists a positive

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integer μ such that the *weight* $w(v) = \sum_{u \in N(v)} \ell(u) = \mu$ for all $v \in V$, where $N(v)$ is the open neighborhood of v. The constant μ is called the *magic constant* of the labeling ℓ. Any graph which admits a distance magic labeling is called a *distance magic graph.* Closed distance magic graphs are a variation of distance magic graphs, where the sums are taken over the closed neighborhoods $N_G[x]$ instead of the open ones $N_G(x)$, see [3, 4].

The concept of distance magic labeling has been motivated by the equalized incomplete tournaments (see [11, 12]). Finding an r-regular distance magic labeling is equivalent to finding equalized incomplete tournament $\mathrm{EIT}(n, r)$ [12]. In an *equalized incomplete tournament* $\mathrm{EIT}(n, r)$ of n teams with r rounds, every team plays exactly r other teams and the total strength of the opponents that team i plays is k. Thus, it is easy to notice that finding an $\mathrm{EIT}(n, r)$ is the same as finding a distance magic labeling of some r-regular graph on n vertices.

From the point of view of this application it is interesting to find disconnected r-regular distance magic graphs (tournaments which could be played simultaneously in different locations). Therefore in the paper we show examples of distance magic graphs G such that the union of t disjoint copies of G, denoted tG, is distance magic as well.

We recall four graph products (see [16]). All four, the *Cartesian product* $G \square H$, *lexicographic product* $G \circ H$, *direct product* $G \times H$ and the *strong product* $G \boxtimes H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \square H$ if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G,
- $G \times H$ if g is adjacent to g' in G and h is adjacent to h' in H,
- $G \boxtimes H$ if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G, or g is adjacent to g' in G and h is adjacent to h' in H,
- $G \circ H$ if either g is adjacent to g' in G or $g = g'$ and h is adjacent to h' in H.

The graph $G \circ H$ is also called the *composition* and denoted by $G[H]$ (see [17]). The product $G \times H$ is also known as *Kronecker product*, *tensor product*, *categorical product* and *graph conjunction.* The direct product is commutative, associative, and it has several applications, for instance it may be used as a model for concurrency in multiprocessor systems [19]. Some other applications can be found in [18].

Some product related graphs, which are distance magic or closed distance magic can be found in [1–5, 9, 21, 22].

Theorem 1.1 [21]**.** *Let $r \geq 1$, $n \geq 3$, G be an r-regular graph and C_n be the cycle of length n. Then the graph $G \circ C_n$ admits a distance magic labeling if and only if $n = 4$.*

Theorem 1.2 [2]**.** *Let G be an arbitrary regular graph. Then $G \times C_4$ is distance magic.*

Theorem 1.3 [22]**.** *The Cartesian product $C_n \square C_m$ is distance magic if and only if $n \equiv m \equiv 2 \pmod 4$ and $n = m$.*

Theorem 1.4 [2]**.** *A graph $C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$, or $m \equiv n \equiv 0 \pmod 4$.*

Theorem 1.5 [3]**.** *A graph $C_m \boxtimes C_n$ is distance magic if and only if at least one of the following conditions holds:*

1. *$m \equiv 3 \pmod 6$ and $n \equiv 3 \pmod 6$.*
2. *$\{m, n\} = \{3, x\}$ and x is an odd number.*

Let $K(n; r)$ denote the complete r-partite graph $K(n, n, \ldots, n)$.

Theorem 1.6 [8]**.** *The Cartesian product $K(n; r)\square C_4$ is distance magic if and only if $n > 2$, $r > 1$ and n is even.*

The *d-dimensional hypercube* is denoted $\mathcal{Q}_d$ where the vertices are binary d-tuples and two vertices are adjacent if and only if the d-tuples differ precisely in one position.

Theorem 1.7 [15]**.** *A hypercube $\mathcal{Q}_d$ has a distance magic labeling if and only if $d \equiv 2 \pmod 4$.*

The *circulant graph* $C_n(s_1, s_2, \ldots, s_k)$ is the graph on the vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$ with edges (x_i, x_{i+s_j}) for $i = 0, \ldots, n-1$, $j = 1, \ldots, k$ where $i + s_j$ is taken modulo n.

Theorem 1.8 [7]**.** *Let $p \geq 2$ and $n = p^2 - 1$ when p is odd and $n = 2(p^2 - 1)$ when p is even. Then $C_n(1, p)$ is a distance magic graph.*

Theorem 1.9 [6]**.** *If $p > 1$ is odd, then $C_{2p(p+1)}(1, 2, \ldots, p)$ is a distance magic graph.*

By tG we denote t disjoint copies of a graph G. Here are some examples of disconnected distance magic graphs.

Theorem 1.10 [13, 20]**.** *Let nr be odd, t be even, $r > 1$ and $t \geq 2$. Then $tK(n; r)$ is distance magic if and only if $r \equiv 3 \pmod 4$.*

Theorem 1.11 [20]**.** *Let $m \geq 1$, $n \geq 2$ and $p \geq 3$. Then $mC_p \circ \overline{K_n}$ has a distance magic labeling if and only if n is even or mnp is odd or n is odd and $p \equiv 0 \pmod 4$.*

Theorem 1.12 [9]**.** *Let m and n be two positive even integers such that $m \leq n$. The graph $G = tK_{m,n}$ is distance magic if and only if the following conditions hold:*

- $m + n \equiv 0 \pmod 4$, *and*
- $1 = 2(2tn+1)^2 - (2tm+2tn+1)^2$ *or* $m \geq (\sqrt{2}-1)n + \frac{\sqrt{2}-1}{2t}$.

Theorem 1.13 [3]**.** *Given $n \geq 2$ and $t \geq 1$, the union tK_n is closed distance magic if and only if $n(t+1) \equiv 0 \pmod 2$.*

We say that an r-regular graph G has a *p-partition* if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ (that is, $V(G) = V_1 \cup V_2 \cup \cdots \cup V_p$ where $V_i \cap V_j = \emptyset$ for $i \neq j$) such that for every $x \in V(G)$

$$|N(x) \cap V_1| = |N(x) \cap V_2| = \cdots = |N(x) \cap V_p|.$$

Analogously we say that an r-regular graph G has a *closed p-partition* if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ such that for every $x \in V(G)$

$$|N[x] \cap V_1| = |N[x] \cap V_2| = \cdots = |N[x] \cap V_p|.$$

We show that if a distance magic graph H has a 2-partition, then tH is distance magic for every positive integer t. Moreover, for an r-regular graph G the products $G \circ H$ and $G \times H$ are distance magic as well, and thus we generalize Theorems 1.1 and 1.2.

2. Distance Magic Graphs

Lemma 2.1. *Let G be an r-regular graph of order n with a 2-partition (closed 2-partition). If G is a distance magic (closed distance magic) graph, then tG is a distance magic (closed distance magic) graph for any positive integer t.*

Proof. Let ℓ be a distance magic (closed distance magic) labeling of G with the magic constant μ. In each copy $G^1, G^2, \ldots, G^t$ of G we apply the partition defined above such that $V_1^j \cup V_2^j$ is the partition of the j-th copy G^j of G. Define

$$\ell'(x) = \begin{cases} \ell(x) + (j-1)n, & \text{if} \quad x \in V_1^j, \\ \ell(x) + (t-j)n, & \text{if} \quad x \in V_2^j. \end{cases}$$

Obviously, ℓ' is a distance magic (closed distance magic) labeling of the graph tG with the magic constant $\mu' = \mu + (t-1)nr/2$ (closed magic constant $\mu' = \mu + (t-1)n(r+1)/2$). ■

We will now use Kotzig arrays as a tool. A *Kotzig array* was defined in [23] to be a $j \times k$ matrix, each row being a permutation of $\{0, 1, \ldots, k-1\}$ and each column having a constant sum.

Lemma 2.2 [23]. *A Kotzig array of size $j \times k$ exists whenever $j > 1$ and $j(k-1)$ is even.*

The following lemma shows that even if an r-regular distance magic graph G has no 2-partition, the union tG can be distance magic.

Lemma 2.3. *Let $p \geq 2$ and G be an r-regular graph of order n having a p-partition (closed p-partition). If G is a distance magic (closed distance magic) graph, then for $t \geq 0$ where $p(t-1)$ is even the graph tG is also distance magic (closed distance magic).*

Proof. Let ℓ be a distance magic (closed distance magic) labeling of G with the magic constant μ. In each copy $G^1, G^2, \ldots, G^t$ of G we apply the partition defined above such that $V_1^j \cup V_2^j \cup \cdots \cup V_p^j$ is the partition of j-th copy G^j of G.

Let $A = (a_{i,j})$ be a Kotzig array of size $p \times t$. Define

$$\ell'(x) = \ell(x) + na_{a_{i,j}},\ x \in V_i^j.$$

Obviously, ℓ' is the distance magic (closed distance magic) labeling of the graph tG with a magic constant $\mu' = \mu + (t-1)nr/2$ (closed magic constant $\mu' = \mu + (t-1)n(r+1)/2$). ■

We will now present some examples of graphs that have the desired 2-partition.

Observation 1. *If*

1. $G = C_n \Box C_m$ *for* $n = m$ *and* $n \equiv m \equiv 2 \pmod 4$,
2. $G = C_n \times C_m$ *for* $n = 4$ *or* $m = 4$, *or* $m \equiv n \equiv 0 \pmod 4$,
3. $G = K(n; r) \Box C_4$ *for* $n > 2$, $r > 1$ *and* n *even,*
4. $G = \mathcal{Q}_d$ *for* $d \equiv 2 \pmod 4$,
5. $G = C_{p^2-1}(1, p)$ *for* p *odd,*
6. $G = C_{2(p^2-1)}(1, p)$ *for* p *even,*
7. $G = C_{2p(p+1)}(1, 2, \ldots, p)$ *for* p *odd,*

then G has a 2-partition.

Proof. 1. Let $V(C_m \Box C_n) = \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, where $N(v_{i,j}) = \{v_{i-1,j}, v_{i+1,j}, v_{i,j-1}, v_{i,j+1}\}$ and the addition in the first suffix is taken modulo m and in the second suffix modulo n. Let $V_1 = \{v_{i,j} : i = 0, 1, \ldots, m-1, j = 0, 2, \ldots, n-2\}$, $V_2 = \{v_{i,j} : i = 0, 1, \ldots, m-1, j = 1, 3, \ldots, n-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = 2$.

2. Let $V(C_m \times C_n) = \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, where $N(v_{i,j}) = \{v_{i-1,j-1}, v_{i-1,j+1}, v_{i+1,j-1}, v_{i+1,j+1}\}$ and the addition in the first suffix is taken modulo m and in the second suffix modulo n. Let $V_1 = \{v_{i,j} : i \equiv 0, 1 \pmod 4, j = 0, 1, \ldots, n-1\}$, $V_2 = \{v_{i,j} : i \equiv 2, 3 \pmod 4, j = 0, 1, \ldots, n-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = 2$.

3. Let $V(K(n;r)) = \{v_i^j : i = 1, \ldots, n, j = 1, \ldots, r\}$, $C_4 = xuywx$, and $H = K(n;r) \Box C_4$. Let $V_1 = \{(v_i^j, x), (v_i^j, u), (v_{n/2+i}^j, y), (v_{n/2+i}^j, w)$, where $i = 1, 2, \ldots, n/2, j = 1, 2, \ldots, r\}$, $V_2 = \{(v_{n/2+i}^j, x), (v_{n/2+i}^j, u), (v_i^j, y), (v_i^j, w)$, where $i = 1, 2, \ldots, n/2, j = 1, 2, \ldots, r\}$. Obviously for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = n(r-1)/2 + 1$.

4. Let us define the set of vertices of $\mathcal{Q}_n$ as the set of binary strings of length n, that is, $V = \{(a_1, a_2, \ldots, a_n)\}; a_i \in \{0, 1\}$. Two vertices are adjacent if and only if the corresponding strings differ in exactly one position. Then $V_1 = \{(a_1, \ldots, a_n)$, where $a_1 + \cdots + a_{n/2}$ is even$\}$, $V_2 = \{(a_1, \ldots, a_n)$, where $a_1 + \cdots + a_{n/2}$ is odd$\}$. Notice that each vertex has $n/2$ neighbours in V_1 and $n/2$ in V_2.

5. Let $V(G) = \{x_0, x_1, \ldots, x_{p^2-2}\}$, where $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and the addition in the suffix is taken modulo n. Let $V_1 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 0, 2, \ldots, p-1\}$, $V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, p\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = 2$.

6. Let $V(G) = \{x_0, x_1, \ldots, x_{2p^2-3}\}$, where $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and the addition in the suffix is taken modulo n. Let $V_1 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 0, 2, \ldots, 2p\}$, $V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, 2p-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = 2$.

7. Let $V(G) = \{x_0, x_1, \ldots, x_{2p(p+1)-1}\}$, where $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and the addition in the suffix is taken modulo n. Let $V_1 = \{x_{i+j(p+1)} : i = 0, 1, \ldots, p, j = 0, 2, \ldots, 2p-2\}$, $V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, 2p-1\}$. Notice that for any $v \in G$ we obtain $|N(v) \cap V_1| = |N(v) \cap V_2| = p$. ∎

Below we show some interesting properties of distance magic unions of graphs.

Theorem 2.4. *If G is an r-regular graph of order t and H is p-regular such that tH is distance magic, then the product $G \circ H$ is distance magic.*

Proof. Let ℓ be a distance magic labeling of the graph $tH = H_1 \cup H_2 \cup \cdots \cup H_t$ with a magic constant μ. For any $u \in V(H)$ let u_j be the corresponding vertex belonging to $V(H_j)$, $j = 1, 2, \ldots, t$. Let $V(G) = \{1, 2, \ldots, t\}$. Notice that for any $i = 1, 2, \ldots, t$ we have $\sum_{v \in V(H_i)} \ell(v) = \frac{|H|\mu}{p}$.

Define the labeling ℓ' of $G \circ H$ as $\ell'(j, u) = \ell(u_j)$ for $u \in V(H)$, $u_j \in V(H_j)$, $j = 1, 2, \ldots, t$. Obviously, ℓ' is a bijection. Moreover, for any $(g, h) \in V(G \circ H)$

we obtain

$$\begin{aligned} w(g,h) &= \sum_{(j,u)\in N_{G\circ H}((g,h))} \ell'(j,u) = \sum_{j\in N_G(g)} \sum_{u\in V(H)} \ell'(j,u) + \sum_{u\in N_H(h)} \ell'(g,u) \\ &= r \sum_{u_j\in V(H_j)} \ell(u_j) + \sum_{u_g\in N_{H_g}(h_g)} \ell(u_g) = r\frac{|H|\mu}{p} + \mu = \frac{(r|H|+p)\mu}{p}. \end{aligned}$$

■

Using the same technique we can prove an analogous theorem for closed distance magic labeling.

Theorem 2.5. *If G is an r-regular graph of order t and H is p-regular such that tH is closed distance magic, then the product $G \circ H$ is closed distance magic.*

Notice that the assumption that H is a regular graph is not necessary, as shown in the observation below.

Observation 2. *Let G be an r-regular graph of order t. If m and n are two positive even integers such $m + n \equiv 0 \pmod 4$ and either $2(2tn+1)^2 - (2tm + 2tn + 1)^2 = 1$ or $m \geq (\sqrt{2}-1)n + \frac{\sqrt{2}-1}{2t}$, then the product $G \circ K_{m,n}$ is distance magic.*

Proof. The graph $tK_{m,n}$ is distance magic by Theorem 1.12. Let ℓ be a distance magic labeling of the graph $tK_{m,n} = K^1_{m,n} \cup K^2_{m,n} \cup \cdots \cup K^t_{m,n}$ with the magic constant μ. For any $u \in V(K_{m,n})$ let u_j be the corresponding vertex belonging to $V(K^j_{m,n})$, $j = 1, 2, \ldots, t$. Let $V(G) = \{1, 2, \ldots, t\}$. We have $\sum_{v\in V(K^i_{m,n})} \ell(v) = 2\mu$ for any $i = 1, 2, \ldots, t$. Define the labeling ℓ' of $G \circ H$ as $\ell'(j,u) = \ell(u_j)$ for $u \in V(K_{m,n})$, $u_j \in V(K^j_{m,n})$, $j = 1, 2, \ldots, t$. As in the proof of Theorem 2.4 we have

$$\begin{aligned} w(g,h) &= \sum_{(j,u)\in N_{G\circ K_{m,n}}((g,h))} \ell'(j,u) \\ &= r \sum_{u_j\in V(K^j_{m,n})} \ell(u_j) + \sum_{u_g\in N_{K^g_{m,n}}(h_g)} \ell(u_g) = (2r+1)\mu, \end{aligned}$$

for any $(g,h) \in V(G \circ H)$. ■

Theorem 2.6. *If G is an r-regular graph of order t and H is such that tH is distance magic, then the product $G \times H$ is distance magic.*

Proof. Let ℓ be a distance magic labeling of the graph $tH = H_1 \cup H_2 \cup \cdots \cup H_t$ with the magic constant μ. For any $u \in V(H)$ let u_j be the corresponding vertex belonging to $V(H_j)$, $j = 1, 2, \dots, t$. Let $V(G) = \{1, 2, \dots, t\}$. Set the labeling ℓ' of $G \times H$ as $\ell'(j, u) = \ell(u_j)$ for $u \in V(H)$, $u_j \in V(H_j)$, $j = 1, 2, \dots, t$. Therefore

$$\begin{aligned} w(g,h) &= \sum_{(j,u)\in N_G(g)\times N_H(h)} \ell'(j,u) = \sum_{j\in N_G(g)} \sum_{u\in N_H(h)} \ell'(j,u) \\ &= \sum_{j\in N_G(g)} \sum_{u_j\in N_{H_j}(h_j)} \ell(u_j) = \sum_{j\in N_G(g)} \mu = r\mu, \end{aligned}$$

for any $(g, h) \in V(G \times H)$. ■

Now we present a theorem, which is a corollary of Lemma 2.1 and Theorems 2.4 and 2.6.

Theorem 2.7. *If G is an r-regular graph and H is a p-regular distance magic graph with a 2-partition, then the products $G \circ H$ and $G \times H$ are both distance magic.*

Notice that even if G and H are both regular distance magic graphs with 2-partitions, then the product $G \Box H$ is not necessarily distance magic (for instance $G = H = C_4$).

Below are presented some families of disconnected distance magic graphs.

Theorem 2.8. *If*

1. $H = C_n \Box C_m$ *for* $n = m$ *and* $m \equiv n \equiv 2 \pmod 4$,
2. $H = C_n \times C_m$ *for* $n = 4$ *or* $m = 4$, *or* $m \equiv n \equiv 0 \pmod 4$,
3. $H = K(n; r) \Box C_4$ *for* $n > 2$, $r > 1$ *and* n *even*,
4. $H = \mathcal{Q}_d$ *for* $d \equiv 2 \pmod 4$,
5. $H = C_{p^2-1}(1, p)$ *for* p *odd*,
6. $H = C_{2(p^2-1)}(1, p)$ *for* p *even*,
7. $H = C_{2p(p+1)}(1, 2, \dots, p)$ *for* p *odd*,

then tH is distance magic. Moreover, if G is an r-regular graph, then the products $G \circ H$ and $G \times H$ are distance magic as well.

Proof. We obtain that tH is distance magic by Lemma 2.1, Observation 1 and Theorems 1.3, 1.4, 1.6, 1.7, 1.8 and 1.9, respectively. By Theorem 2.7 we obtain now that $G \circ H$ and $G \times H$ are distance magic. ■

We conclude this section with an observation that can be obtained easily by applying Theorems 1.10, 1.11, 2.4 and 2.6.

Observation 3. *If G is an r-regular graph of order t and*

1. $H = K(n; p)$ *for* n *odd*, $t \geq 2$ *even and* $p \equiv 3 \pmod 4$,

2. *$H = C_p \circ \overline{K_n}$ for $t \geq 1$, $n \geq 3$ and $p \geq 3$, tnp odd or n odd and $p \equiv 0 \pmod 4$,*

then the products $G \circ H$ and $G \times H$ are distance magic.

3. Closed Distance Magic Graphs

We start with the following observations about closed distance magic graphs:

Observation 4 [4]**.** *Let u and v be vertices of a closed distance magic graph. Then $|N(u) \cup N(v)| = 0$ or $|N(u) \cup N(v)| > 2$.*

Observation 5 [3]**.** *If G is an r-regular graph on n vertices having a closed distance magic labeling with a magic constant μ', then $\mu' = \frac{(r+1)(n+1)}{2}$.*

We will present now two examples of graphs that have a closed 3-partition.

Observation 6. *If*

1. *$G = C_3$, or*
2. *$G = C_n \boxtimes C_m$ for $n = 3$ and m odd, or $m \equiv n \equiv 3 \pmod 6$,*

then G has the closed 3-partition.

Proof. 1. Let $V(C_3) = \{v_0, v_1, v_2\}$. Let $V_i = \{v_i\}$ for $i = 0, 1, 2$.
2. Let $V(C_m \boxtimes C_n) = \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, where $N(v_{i,j}) = \{v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}, v_{i,j-1}, v_{i,j+1}, v_{i+1,j-1}, v_{i+1,j}, v_{i+1,j+1}\}$ and the addition in the first suffix is taken modulo m and in the second suffix modulo n. Let $V_p = \{v_{i,j} : i + j \equiv p \pmod 3\}$. Notice that for any $v \in G$ we obtain $|N[v] \cap V_1|$ $= |N[v] \cap V_2| = |N[v] \cap V_3| = \frac{mn}{3}$. ■

Theorem 3.1. *If*

1. *$G = C_3$, or*
2. *$G = C_n \boxtimes C_m$ for $n = 3$ and m odd, or $m, n \equiv 3 \pmod 6$,*

then tG is closed distance magic if and only if t is odd.

Proof. Notice that if $G = C_3$ then it is closed distance magic. Note that $G = C_n \times C_m$ for $n = 3$ and m odd, or $m, n \equiv 3 \pmod 6$, is closed distance magic by Theorem 1.13. Since G has a closed 3-partition, then the graph tG is closed distance magic by Lemma 2.3 for odd t. Observe that G is an r-regular graph with r even. Suppose now that t is even. Then $|V(tG)|$ is even as well and $\frac{(r+1)(|V(tG)|+1)}{2}$ is not an integer. Therefore the graph G is not closed distance magic by Observation 5. ■

By Lemma 4 it is now obvious that tC_n is closed distance magic if and only if t is odd and $n = 3$. Moreover, by Theorem 2.4 we obtain immediately the following observation.

Observation 7. *When G is an r-regular graph with r odd and*

1. $H = C_3$, *or*
2. $H = C_n \boxtimes C_m$ *for* $n = 3$ *and* m *odd, or* $m \equiv n \equiv 3 \pmod 6$,

then the product $G \circ H$ is closed distance magic.

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