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DOMINATION GAME: EXTREMAL FAMILIES FOR THE 3/5-CONJECTURE FOR FORESTS

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Abstract

In the *domination game* on a graph G, the players Dominator and Staller alternately select vertices of G. Each vertex chosen must strictly increase the number of vertices dominated. This process eventually produces a dominating set of G; Dominator aims to minimize the size of this set, while Staller aims to maximize it. The size of the dominating set produced under optimal play is the game domination number of G, denoted by $\gamma_g(G)$. Kinnersley, West and Zamani [SIAM J. Discrete Math. **27** (2013) 2090–2107] posted their 3/5-Conjecture that $\gamma_g(G) \le \frac{3}{5}n$ for every isolate-free forest on n vertices. Brešar, Klavžar, Košmrlj and Rall [Discrete Appl. Math. **161** (2013) 1308–1316] presented a construction that yields an infinite family of trees that attain the conjectured 3/5-bound. In this paper, we provide a much larger, but simpler, construction of extremal trees. We conjecture that if G is an isolate-free forest on n vertices satisfying $\gamma_g(G) = \frac{3}{5}n$, then every component of G belongs to our construction.

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1. Introduction

The domination game in graphs was first introduced by Brešar, Klavžar, and Rall [4] and extensively studied afterwards in [1–3, 5–9, 11, 12] and elsewhere. A *neighbor* of a vertex v is a vertex adjacent to v. We say that a vertex *dominates* itself and its neighbors; a *dominating set* in a graph G is a set of vertices of G that dominates all vertices in the graph. An *isolate-free* graph is a graph having no vertices of degree 0.

In this paper, we study a game in which two agents collaboratively build a dominating set. The game played on a graph G consists of two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. The game ends when the set of vertices chosen becomes a dominating set in G. Dominator wishes to end the game with a minimum number of vertices chosen, and Staller wishes to end the game with as many vertices chosen as possible.

The *game domination number* (resp. *Staller-start game domination number*), $\gamma_g(G)$ (resp. $\gamma_g'(G)$), of G is the size of the dominating set produced under optimal play when Dominator (resp. Staller) starts the game. Kinnersley, West, and Zamani posted the following 3/5-Conjecture in [11] on the game domination number.

$\frac{3}{5}$-Conjecture ([11]). *If G is an isolate-free forest on n vertices, then $\gamma_g(G) \le \frac{3}{5}n$.*

We remark that there are two "3/5-Conjectures": one for isolate-free forests, and one for general isolate-free graphs. It is not known whether the 3/5-Conjecture for isolate-free forests implies the 3/5-Conjecture for general isolate-free graphs. In this paper, we focus on the 3/5-Conjecture for isolate-free forests stated above. In [11], the authors showed that the 3/5-Conjecture holds when G is an isolate-free forest of caterpillars. When G is only required to be an n-vertex isolate-free forest, they showed that $\gamma_g(G) \le 7n/11$. Recently, Brešar, Klavžar, Košmrlj, and Rall [3] verified the 3/5-Conjecture for all trees on at most 20 vertices, and listed those meeting the conjectured bound with equality (when $n = 20$, there are only ten such trees). In addition, Bujtás [6] has proved the 3/5-Conjecture for isolate-free forests in which no two leaves are at distance 4 apart.

Adopting the terminology in [11], a *partially-dominated graph* is a graph in which we suppose that some vertices have already been dominated, and need not be dominated again to complete the game. A vertex v is *saturated* if v and all of its neighbors have already been dominated. Once a vertex is saturated, it plays no role in the remainder of the game and can be deleted from the partially-dominated graph. Further, an edge joining two dominated vertices plays no role in the game, and can be deleted. Therefore in what follows, we may assume a partially-dominated graph contains no saturated vertex and contains no edge joining two dominated vertices. The resulting partially-dominated graph is called a *residual graph* in [11]. We will also say that the original graph G, before any moves have been made in the game, is a residual graph.

Given a graph G and a subset S of vertices of G, we denote by $G|S$ the residual graph in which the vertices of S in G are already dominated. We use $\gamma_g(G|S)$ (resp. $\gamma_g'(G|S)$) to denote the number of turns remaining in the game on $G|S$ under optimal play when Dominator (resp. Staller) has the next turn.

The *game domination number* (resp. *Staller-start game domination number*), $\gamma_g(G)$ (resp. $\gamma_g'(G)$), of G is the size of the dominating set produced under optimal play when Dominator (resp. Staller) starts the game.

The *Staller-pass game* is the domination game in which, on each turn, Staller may pass her move. Let $\hat{\gamma}_g(G)$ (resp. $\hat{\gamma}_g'(G)$), be the size of the dominating set produced under optimal play when Dominator (resp. Staller) starts the Staller-pass game. The turns when Staller passes do not count as moves. The following results from [11] often prove useful.

Lemma 1 (Continuation Principle — [11], Lemma 2.1)**.** *Let G be a graph and let $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma_g'(G|A) \le \gamma_g'(G|B)$.*

Theorem 2 ([11], Theorem 4.6)**.** *If F is a forest and $S \subseteq V(F)$, then $\gamma_g(F|S) \le \gamma_g'(F|S)$.*

Lemma 3 ([11], Corollary 4.7)**.** *If F is a forest and $S \subseteq V(F)$, then $\hat{\gamma}_g(F|S) = \gamma_g(F|S)$ and $\hat{\gamma}_g'(F|S) = \gamma_g'(F|S)$.*

1.1. Terminology and Notation

For notation and graph theory terminology that are not defined herein, we refer the reader to [10]. Let G be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$, and let v be a vertex in $V(G)$. We denote the *degree* of v in G by $d_G(v)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v)-path in G. A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. We use the standard notation $[k] = \{1, \ldots, k\}$.

2. The Family $\mathcal{T}$

In this section, we construct a family $\mathcal{T}$ of trees with game domination number three-fifths their order. For this purpose, we introduce the notion of a 2-wing.

Definition 1. A tree T is a 2-*wing* if T has maximum degree at most 4 with no vertex of degree 3, and with the vertices of degree 2 in T precisely the support vertices of T, except for one vertex of degree 2 in T. This exceptional vertex of degree 2 in T that is not a support vertex is called the *gluing vertex* of T. We define a *base vertex* in T as a vertex of degree 4 or the gluing vertex. We note that every vertex in a 2-wing is either a leaf or a support vertex (of degree 2) or a base vertex.

We remark that the smallest 2-wing is a path on five vertices, with its central vertex as the gluing vertex. A 2-wing with gluing vertex v is illustrated in Figure 1.

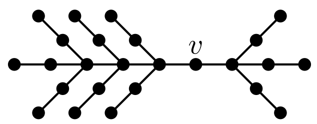

Figure 1. A 2-wing with gluing vertex v.

Definition 2. A tree T belongs to the family $\mathcal{T}$ if T is obtained from $k \geq 1$ vertex-disjoint 2-wings by adding $k-1$ edges between the gluing vertices. As in Definition 1, a *base vertex* in T is a vertex of degree 4 or a gluing vertex. We note that every vertex in T is either a leaf or a support vertex (of degree 2) or a base vertex.

3. Preliminary Results

In this section, we present some preliminary results that will be useful in proving our main result, namely Theorem 6. We start with the following fundamental and well-known property of a tree.

Observation 4. *If v, w, x are three distinct vertices of a tree T, then there is a unique vertex that is common to the (v,w)-path, to the (v,x)-path, and to the (w,x)-path.*

The following property of 2-wings will prove to be useful.

Proposition 5. *If T is a 2-wing of order n, then $n \equiv 0 \pmod 5$. Further, T has $\frac{2}{5}n$ leaves, $\frac{2}{5}n$ support vertices, and $\frac{1}{5}n$ base vertices.*

Proof. Let T be a 2-wing of order n with ℓ leaves. Then, T has $\ell+1$ vertices of degree 2, and therefore $(\ell-2)/2$ vertices of degree 4, implying that $\ell = \frac{2}{5}n$ and that $n \equiv 0 \pmod 5$. Thus, T has $\frac{2}{5}n$ support vertices, $\frac{1}{5}n - 1$ vertices of degree 4, and $\frac{1}{5}n$ base vertices. ■

4. Main Result

As remarked by Brešar, Klavžar, Košmrlj, and Rall in [3], "the domination game is very non-trivial even when played on trees." The author's in [3] present a construction that yields an infinite family of trees that attain the bound in the $\frac{3}{5}$-Conjecture. Their ingenious construction is relatively complicated, the details of which are nicely explained in [3]. In this paper, we provide a much larger, but simpler, construction of extremal trees by showing that every tree in our family $\mathcal{T}$ attains the conjectured $\frac{3}{5}$-bound. An example of a tree that belongs to our family $\mathcal{T}$ but does not belong to the family of trees constructed in [3] is shown in Figure 2. We remark that the family of trees constructed in [3] is a subfamily of trees in the family $\mathcal{T}$.

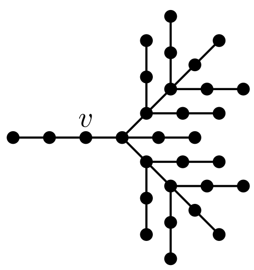

Figure 2. A tree $T \in \mathcal{T}$.

By the Continuation Principle, it is never in Dominator's best interests to play a leaf that belongs to a component of order at least 3, since in this case Dominator can always do at least as well by playing its neighbor (which results in the saturation of both the support vertex and its leaf-neighbor). Hence, to simplify the arguments in our proof of Theorem 6, we make the assumption that Dominator never plays a leaf that belongs to a component of order at least 3.

Theorem 6. *If $T \in \mathcal{T}$ has order n, then $\gamma_g(T) = \gamma_g'(T) = \frac{3}{5}n$.*

Proof. Let $T \in \mathcal{T}$ have order n. As an immediate consequence of Proposition 5, we note that $n \equiv 0 \,(\mathrm{mod}\, 5)$, and that T has $\frac{2}{5}n$ leaves, $\frac{2}{5}n$ support vertices, and $\frac{1}{5}n$ base vertices. By Theorem 2 and Lemma 3, $\hat{\gamma}_g(T) = \gamma_g(T) \le \gamma_g'(T)$. It suffices for us to therefore prove that $\frac{3}{5}n \le \hat{\gamma}_g(T)$ and that $\gamma_g'(T) \le \frac{3}{5}n$. First, we will prove $\gamma_g'(T) \le \frac{3}{5}n$.

Claim 6.A. $\gamma_g'(T) \le \frac{3}{5}n$.

Proof. Dominator's strategy is to guarantee that exactly one vertex is played from every leaf and its neighbor. Dominator can readily achieve his strategy by playing according to the following two rules.

Dominator's Rules

Rule 1. If Staller's last move was a vertex u that is a leaf in T and the (unique) base vertex w that is of distance 2 from u in T is still playable in the residual forest, then Dominator plays the vertex w.

Rule 2. Otherwise, Dominator plays a vertex that is not a leaf in T.

Dominator's strategy guarantees that exactly $\frac{2}{5}n$ vertices are played from the leaves and support vertices, since one vertex is played from every leaf and its neighbor. Since at most $\frac{1}{5}n$ additional (base) vertices can be played, we have that $\gamma_g'(T) \le \frac{3}{5}n$. □

We prove next that $\frac{3}{5}n \le \hat{\gamma}_g(T)$. We recall our prior assumption that Dominator never plays a leaf that belongs to a component of order at least 3, since this is never in his best interests.

Claim 6.B. $\frac{3}{5}n \le \hat{\gamma}_g(T)$.

Proof. In order to explain Staller's strategy, we introduce some further notation. Suppose that $T \in \mathcal{T}$ is obtained from $k \ge 1$ vertex-disjoint 2-wings, $T_1, \dots, T_k$, by adding $k-1$ edges between the gluing vertices. Let F denote the residual forest of T at each stage of the game. The components of $T_i[V(F)]$, $i \in [k]$, we call the *units* of F. We note that a unit of F is a subtree of both F and T. For a subtree C of F, we define $B(C)$ to be the set of vertices of C that are base vertices in the original tree T. Recall that every base vertex of T is a vertex of degree 4 in T_i or is the gluing vertex v_i of T_i (of degree 2 in T_i) for some $i \in [k]$. We note that for each unit U of F, the subtree $U[B(U)]$ is connected.

Staller's strategy is to guarantee that every base vertex is played. Staller can achieve her strategy by playing according to the following two rules.

Staller's Rules

Rule 1. If there is a unit $\hat{U}$ in the residual forest such that

$$\sum_{v \in B(\hat{U})} d_{\hat{U}}(v) = 4|B(\hat{U})| - 3,$$

then Staller plays a vertex of that unit in the following way.

(a) If there is a (base-) vertex $v \in B(\hat{U})$ of degree at most 2 in $\hat{U}$, then Staller plays that vertex.

(b) Otherwise, there are three distinct vertices $x, y, z \in B(\hat{U})$ of degree 3 in C. In this case, Staller plays the unique vertex (see Observation 4) that belongs to the (x, y)-path, to the (x, z)-path, and to the (y, z)-path. Since $\hat{U}[B(\hat{U})]$ is connected, we note that such a vertex belongs to $B(\hat{U})$.

Rule 2. If Staller cannot play according to Rule 1, then Staller passes.

Subclaim 6.B.1. *Let U be a unit of the residual forest with $B(U) \neq \emptyset$. If Staller plays according to her Rules* 1 *and* 2*, then all the vertices in the unit U that are not base vertices in the original tree T are undominated.*

Proof. Suppose, to the contrary, that there is a vertex $v \in V(U) \setminus B(U)$ that is dominated. Since v is not a base vertex in T, we note that v is either a leaf or a support vertex in T. Since v belongs to the residual forest, it is not saturated. If v is a leaf in T, then its neighbor is dominated as well and thus v is saturated, a contradiction. Hence we may assume that v is a support vertex in T. Let u be the leaf-neighbor of v in T, and let w be the neighbor of v different from u. We note that w is a base vertex in T. Since v is dominated, at least one of u, v, w is played. If w is played, then the unit U consists only of the support vertex v and its leaf-neighbor u, implying that $B(U) = \emptyset$, a contradiction. If v is played, then it is saturated, a contradiction. But according to the Continuation Principle and to Staller's Rules 1 and 2, neither Dominator nor Staller played the leaf u. □

Subclaim 6.B.2. *Let U be a unit of the residual forest F and let $v \in B(U)$. If v is not a gluing-vertex in T, then the following holds. The vertex v is undominated if and only if $d_U(v) = 4$.*

Proof. Since v is a base vertex but not a gluing-vertex in T, we have $d_T(v) = 4$. Since v belongs to F, it is not played yet. If v is dominated, a neighbor of v in T is played. That vertex is saturated and does not belong to U, which implies $d_U(v) \leq 3$. If v is undominated, all of its neighbors in T are non-saturated. Therefore, all edges incident with v in T exist in F and in U. □

We now prove by induction on the number of moves played, that

(i) for all units U of the residual forest F,

$$\sum_{v \in B(U)} d_U(v) \geq 4|B(U)| - 2 \tag{1}$$

holds before Dominator's first move and after each move of Staller, and

(ii) after each move of Dominator, there is at most one unit $\hat{U}$ of F violating Inequality (1), and in such a unit, if it exists, Equation (2) below holds.

$$\sum_{v \in B(\hat{U})} d_{\hat{U}}(v) = 4|B(\hat{U})| - 3. \tag{2}$$

Subclaim 6.B.3. *Condition* (i) *holds before Dominator's first move.*

Proof. Before Dominator's first move, the units of the residual forest are precisely the vertex-disjoint 2-wings of T. Let U be such a unit. All base vertices of U have degree 4 in U, except for the gluing vertex, which has degree 2 in U. Hence, Inequality (1) holds (in fact with equality) for U before Dominator's first move. □

Before we proceed with the induction, we prove the following.

Subclaim 6.B.4. *Let U be a unit of the residual forest F that satisfies Inequality* (1)*. If no vertex of U is played on the next move, then U is a unit of the residual forest F' after that move is played and Inequality* (1) *holds for U after that move.*

Proof. If U is a component of F, the move does not change anything to that unit and thus U is a unit of F' and Inequality (1) holds for that unit after the move.

If U is not a component in F, then U contains a gluing vertex. Since Inequality (1) holds for U before the move, Subclaim 6.B.1 and Subclaim 6.B.2 imply that all vertices of U different from the gluing vertex are undominated before the move and that the gluing vertex is of degree 2 in U. Since all vertices of U different from the gluing vertex do not have neighbors outside U, these vertices stay undominated after the move. This implies that each vertex of U is not saturated after the move and that every edge of U is incident to at least one undominated vertex. Hence, U is a unit in F' satisfying Inequality (1). □

By Subclaim 6.B.3, Condition (i) holds before Dominator's first move. This establishes the base case. For the inductive hypothesis, we first assume that it is Staller's move and Condition (ii) holds after Dominator's last move. We show then that Condition (i) holds after Staller's move. Let F and F' be the residual forests before and after Staller's move, respectively.

Subclaim 6.B.5. *Condition* (i) *holds after Staller's move.*

Proof. If Condition (i) holds after Dominator's last move, then according to Staller's Rules 1 and 2 she passes. Hence, Condition (i) holds after Staller's move. Otherwise, there is a unit $\hat{U}$ of F that violates Inequality (1). By the inductive hypothesis, we may assume that

$$\sum_{u \in B(\hat{U})} d_{\hat{U}}(u) = 4|B(\hat{U})| - 3,$$

and that all units of F different from $\hat{U}$ satisfy Inequality (1). Let v be the vertex Staller plays according to her Rule 1. By Staller's Rule 1, we note that $v \in B(\hat{U})$. Therefore, Subclaim 6.B.4 implies that all units of F different from $\hat{U}$ are units of F' satisfying Inequality (1).

Let C be a component of $\hat{U} - v$ and let $w \in V(C)$ be the neighbor of v in $\hat{U}$. If each vertex of C is non-saturated after Staller's move and every edge of C is incident to at most one dominated vertex after Staller's move, then C is a unit of F'. Otherwise, there are units of F' whose vertex sets partition the non-saturated vertices of C. If $B(C) = \emptyset$, then it is easy to see that C consists of exactly two vertices, one of them, namely w, is dominated after Staller's move and the other vertex is undominated after Staller's move. Therefore, C is a unit of F', which trivially satisfies Inequality (1). Hence we may assume that $B(C) \neq \emptyset$. Since $\hat{U}[B(\hat{U})]$ is connected, we have $w \in B(C)$. Subclaim 6.B.1 implies that all vertices in $V(C) \setminus B(C)$ are undominated before Staller's move and, since those vertices are not adjacent to v, they are undominated after Staller's move.

If every vertex of $B(C)$ has degree 4 in $\hat{U}$, then, using Subclaim 6.B.2, every vertex of $B(C)$ has degree 4 in $B(C)$ and is undominated after Staller's move, except for the vertex w, which has degree 3 in $B(C)$ and is dominated after Staller's move. Since no vertex of C is saturated after Staller's move and no edge of C is incident to two dominated vertices after Staller's move, C is a unit of F' with

$$\sum_{u \in B(C)} d_C(u) = 4|B(C)| - 1.$$

Hence we may assume that there is a vertex $w' \in B(C)$ with $d_{\hat{U}}(w') \leq 3$. Possibly, $w' = w$. If $d_{\hat{U}}(w') \leq 2$, then, by Staller's Rule 1, $d_{\hat{U}}(v) \leq 2$, which contradicts Condition (ii) before Staller's move. Hence we have $d_{\hat{U}}(w') = 3$.

Suppose there is a vertex $w'' \in B(C)$ different from w' with $d_{\hat{U}}(w'') = 3$. Condition (ii) implies that no vertex of $B(\hat{U})$ has degree at most 2 in $\hat{U}$. According to Staller's Rule 1, the vertex v belongs to the (w', w'')-path, which contradicts the fact that w' and w'' belong to the same component of $\hat{U} - v$. Therefore, w' is the only vertex in $B(C)$ with $d_{\hat{U}}(w') = 3$, implying that every vertex in $B(C)$ different from w' has degree 4 in $\hat{U}$.

Using Subclaim 6.B.2, every vertex of $B(C)$ different from w and w' is undominated after Staller plays her move v. By Subclaim 6.B.2, w' is dominated before and therefore after Staller's move. Clearly, w is dominated after Staller's move. If $w \neq w'$, then $d_C(w) = d_C(w') = 3$. Otherwise, if $w = w'$, then $d_C(w) = 2$. Since no vertex of C is saturated after Staller's move, all vertices of C belong to F'. If $ww' \notin E(C)$, then each edge of C is incident to an undominated vertex, which implies that C is a unit of F' satisfying Inequality (1). If $ww' \in E(C)$, then the two components of $C - ww'$ are units of F'. Let C^* be such a component and let $\{w^*\} = V(C^*) \cap \{w, w'\}$. Every vertex of $B(C^*)$ has degree 4 in C^*, except for the vertex w^*, which has degree 2 in C^*. Hence C^* satisfies Inequality (1). □

Next we assume that it is Dominator's move and Condition (i) holds before his move. We show then that Condition (ii) holds after Dominator's move. Let

F and F' be the residual forests before and after Dominator's move, respectively.

Subclaim 6.B.6. *Condition* (ii) *holds after each move of Dominator.*

Proof. Let v be the vertex played by Dominator and let U be the unit of F that contains v. The inductive hypothesis and Subclaim 6.B.4 imply that all units of F different from U are units of F' satisfying Inequality (1).

Let C be a component of $U - v$ and let $w \in V(C)$ be the neighbor of v in U. If each vertex of C is non-saturated after Dominator's move and every edge of C is incident to at most one undominated vertex after Dominator's move, then C is a unit of F'. Otherwise, there are units of F' whose vertex sets partition the non-saturated vertices of C. If $B(C) = \emptyset$, then it is easy to see that C consists of one or two vertices. If $n(C) = 1$, then its vertex, namely w, is saturated after Dominator's move and thus no vertex of C belongs to F'. If $n(C) = 2$, then w is dominated after Staller's move and the other vertex of C is undominated after Staller's move. Therefore, C is a unit of F', which trivially satisfies Inequality (1). Hence we may assume that $B(C) \neq \emptyset$. According to Dominator's rules, he never plays a leaf in T. In particular, v is not a leaf in T. Since $U[B(U)]$ is connected, we have $w \in B(C)$. Subclaim 6.B.1 implies that all vertices in $V(C) \setminus B(C)$ are undominated before Dominator's move and, since those vertices are not adjacent to v, they are undominated after Dominator's move.

By Condition (i), there is at most one component C' of $U - v$ with

$$\sum_{u \in B(C')} d_U(u) = 4|B(C')| - 2.$$

Furthermore, for each component C of $U - v$ different from C', there is at most one vertex in $B(C)$ that has degree less than 4 in U, in particular, that vertex has degree 3 in U. With the same argumentation as in the proof of Subclaim 6.B.5, we deduce that all units of F' that contain vertices of those components satisfy Inequality (1).

We consider C'. Let $w \in V(C')$ be the neighbor of v in U. Recall that $w \in B(C')$. Since $d_{C'}(w) = d_U(w) - 1$ and $d_{C'}(u) = d_U(u)$ for all $u \in B(C') \setminus \{w\}$, we have

$$\sum_{u \in B(C')} d_{C'}(u) = 4|B(C')| - 3.$$

Let $Q \subseteq B(C')$ be the set of vertices of C' that have degree less than 4 in C'. We note that $w \in Q$. Clearly, $1 \leq |Q| \leq 3$. Subclaim 6.B.1, Subclaim 6.B.2, and the fact that no vertex of $V(C') \setminus Q$ is adjacent to v in F imply that the vertices of Q are dominated after Dominator's move and the vertices of $V(C') \setminus Q$ are undominated after Dominator's move.

We first suppose that $|Q| = 1$. In this case the unique vertex of Q, namely w, has degree 1 in C'. This implies that w is adjacent to an undominted vertex and thus w is non-saturated after Dominator's move. Since no edge of C' is incident to two dominated vertices after Staller's move, C' is the only unit of F' satisfying Equality (2).

We next suppose that $|Q| = 2$. In this case one vertex a of Q has degree 2 in C' and the other vertex b of Q has degree 3 in C'. Since both vertices of Q have an undominated neighbor after Staller's move, they are non-saturated after Dominator's move. If $ab \notin E(C')$, then C' is the only unit of F' satisfying Equality (2). If $ab \in E(C')$, then the component of $C' - ab$ that contains b is a unit of F' satisfying Inequality (1) and the component of $C' - ab$ that contains a is the only unit of F' satisfying Equality (2).

We finally suppose that $|Q| = 3$. In this case all three vertices a, b, c of Q have degree 3 in C'. Since all three vertices of Q have an undominated neighbor after Staller's move, they are non-saturated after Dominator's move. By symmetry, we assume that $ac \notin E(C')$. If $ab \notin E(C')$ and $bc \notin E(C')$, then C' is the only unit of F' satisfying Equality (2). If $ab \in E(C')$ and $bc \notin E(C')$, then the component of $C' - ab$ that does not contain c is a unit of F' satisfying Inequality (1) and the component of $C' - ab$ that contains c is the only unit of F' satisfying Equality (2). If $ab \in E(C')$ and $bc \in E(C')$, then the two components of $C' - \{ab, bc\}$ that do not contain b are units of F' each satisfying Inequality (1) and the component of $C' - \{ab, bc\}$ that contains b is the only unit of F' satisfying Equality (2). □

By Subclaims 6.B.3, 6.B.5, and 6.B.6, we have shown that by playing according to her two rules, Staller can achieve that Condition (i) holds before Dominator's first move and after each of her moves, and that Condition (ii) holds after each move of Dominator.

Subclaim 6.B.7. *Upon completion of the game, every base vertex is played.*

Proof. Suppose, to the contrary, that there is a base vertex v, that is not played upon completion of the game. Let F be the residual forest immediately before v became saturated. If it was Dominator's move, Condition (i) implies that the vertex v had at least two undominated neighbors in F. Dominator cannot dominate both these vertices by one move without playing v. Hence it was Staller's move and the vertex v had exactly one undominated neighbor in F. In that case, the unit of F that contains v is the only unit that satisfies Equality (2). According to Rule 1, Staller plays vertex v, a contradiction. □

By Subclaim 6.B.7, Staller's strategy guarantee that upon completion of the game, all $\frac{1}{5}n$ base vertices were played. At least $\frac{2}{5}n$ additional vertices must have been played from the leaves and support vertices in order to dominate the $\frac{2}{5}n$ leaves, implying that $\hat{\gamma}_g(T) \geq \frac{3}{5}n$. This completes the proof of Claim 6.B. □

We now return to the proof of Theorem 6. By Theorem 2 and Lemma 3, and by Claim 6.A and Claim 6.B, we have that $\frac{3}{5}n \le \hat{\gamma}_g(T) = \gamma_g(T) \le \gamma'_g(T) \le \frac{3}{5}n$. Consequently, we must have equality throughout this inequality chain, implying that $\gamma_g(T) = \gamma'_g(T) = \frac{3}{5}n$. This completes the proof of Theorem 6. ■

We pose the following conjecture.

Conjecture 1. *If F is an isolate-free forest on n vertices satisfying $\gamma_g(F) = \frac{3}{5}n$, then every component of F belongs to the family $\mathcal{T}$.*

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