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THE DICHROMATIC NUMBER OF INFINITE FAMILIES OF CIRCULANT TOURNAMENTS

Nahid Javier and Bernardo Llano

Departamento de Matemáticas
Universidad Autónoma Metropolitana Iztapalapa
San Rafael Atlixco 186, Colonia Vicentina
09340, México, D.F., Mexico

e-mail: {nahid,llano}@xanum.uam.mx

Abstract

The *dichromatic number* $dc(D)$ of a digraph D is defined to be the minimum number of colors such that the vertices of D can be colored in such a way that every chromatic class induces an acyclic subdigraph in D. The *cyclic* circulant tournament is denoted by $T = \overrightarrow{C}_{2n+1}(1, 2, \ldots, n)$, where $V(T) = \mathbb{Z}_{2n+1}$ and for every jump $j \in \{1, 2, \ldots, n\}$ there exist the arcs $(a, a+j)$ for every $a \in \mathbb{Z}_{2n+1}$. Consider the circulant tournament $\overrightarrow{C}_{2n+1}\langle k\rangle$ obtained from the cyclic tournament by reversing one of its jumps, that is, $\overrightarrow{C}_{2n+1}\langle k\rangle$ has the same arc set as $\overrightarrow{C}_{2n+1}(1, 2, \ldots, n)$ except for $j = k$ in which case, the arcs are $(a, a-k)$ for every $a \in \mathbb{Z}_{2n+1}$. In this paper, we prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \in \{2, 3, 4\}$ for every $k \in \{1, 2, \ldots, n\}$. Moreover, we classify which circulant tournaments $\overrightarrow{C}_{2n+1}\langle k\rangle$ are vertex-critical r-dichromatic for every $k \in \{1, 2, \ldots, n\}$ and $r \in \{2, 3, 4\}$. Some previous results by Neumann-Lara are generalized.

Keywords: tournament, dichromatic number, vertex-critical r-dichromatic tournament.

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1. Introduction

A *tournament* T is an orientation of the complete graph. If T contains no directed cycles, T is called *transitive* and denoted by TT_k, where $k \in \mathbb{N}$ is its order.

The definition of the dichromatic number of a digraph and the first important results were introduced by Neumann-Lara in 1982 [9]. Independently, Jacob and Meyniel defined the same notion in 1983, see [6]. In 1977, Erdős visited Mexico and began to work on the dichromatic number of a graph (see the definition below) with Neumann-Lara. The results of this collaboration were summarized in a survey by Erdős in 1979 (see [3] for details).

Other results concerning this topic can be found in a paper by Erdős, Gimbel and Kratsch [4]. According to this paper, the *dichromatic number of a digraph* D, denoted by $dc(D)$, is " the minimum number of parts the vertex set of D must be partitioned into, so that each part induces an acyclic digraph." Equivalently, the dichromatic number of D is the minimum number of colors such that the vertices of D can be colored in such a way that every chromatic class induces an acyclic subdigraph in D. The main result of paper [4] for tournaments is the following: every tournament with n vertices can be colored with $O(n/\log n)$ and there exists tournaments (for example, random tournaments) having dichromatic number $\Omega(n/\log n)$ (see Theorem 5 of the aforementioned paper). There are more interesting asymptotic results in [5] by Harutyunyan. In particular, Theorem 2.3.8 states that if T is a tournament of order n, then $dc(T) \leq \frac{n}{\log n}(1+o(1))$.

The *dichromatic number of a graph* G was defined by Erdős and Neumann-Lara in [3] as

$$dc(G) = \max\left\{dc(\overrightarrow{G}) : \overrightarrow{G} \text{ is an orientation of } G\right\}.$$

Determining the dichromatic number of a general (di)graph is a very hard problem. Exact values of this parameter are only known for some special classes of digraphs, particularly in a few cases of circulant tournaments (see [1, 7, 9, 12, 10, 11] and [13]). In this paper, we prove that

(i) $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } 4 \leq n \leq 7, \\ 4 & \text{if } n \geq 8, \end{cases}$

(Section 3, Corollary to Theorem 11),

(ii) $dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8, \end{cases}$

(Section 3, Corollary to Theorem 17),

(iii) if $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$ for $n \geq 7$ (Section 4, Theorem 22),

(iv) if $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$ (Section 5, Theorem 26), and

(v) $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$ for $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$, where $n \geq 3$ (Section 5, Theorem 29).

Our results generalize some theorems obtained by Neumann-Lara. At the end of Section 5, we characterize the vertex-critical r-dichromatic circulant tournaments $\overrightarrow{C}_{2n+1}\langle k\rangle$ for every $k \in \{1, 2, \ldots, n\}$ and $r \in \{2, 3, 4\}$, see Theorem 32, the main theorem of this paper.

2. Preliminaries

Let $\mathbb{Z}_m$ be the cyclic group of integers modulo m, where $m \in \mathbb{N}$ and J is a nonempty subset of $\mathbb{Z}_m \setminus \{0\}$ such that $w \in J$ if and only if $-w \notin J$ for every $w \in \mathbb{Z}_m$. The *circulant digraph* $\overrightarrow{C}_m(J)$ is defined by $V(\overrightarrow{C}_m(J)) = \mathbb{Z}_m$ and

$$A(\overrightarrow{C}_m(J)) = \{(i, j) : i, j \ \in \mathbb{Z}_m \ \text{ and } j - i \in J\}.$$

Notice that $\overrightarrow{C}_{2n+1}(J)$ is a circulant (or rotational) tournament if and only if $|J| = n$. We recall that circulant tournaments are regular and their automorphism group is vertex-transitive. We define

$$\begin{aligned}\overrightarrow{C}_{2n+1}(1, 2, \ldots, n) &:= \overrightarrow{C}_{2n+1}\langle\emptyset\rangle \quad \text{and}\\ \overrightarrow{C}_{2n+1}(1, \ldots, k-1, -k, k+1, \ldots, n) &= \\ \overrightarrow{C}_{2n+1}(1, \ldots, k-1, k+1, \ldots, n, 2n+1-k) &:= \overrightarrow{C}_{2n+1}\langle k\rangle.\end{aligned}$$

Observe that the circulant $\overrightarrow{C}_m(1) = \overrightarrow{C}_m$ is the directed cycle of length m. If $V(\overrightarrow{C}_m) = \{a_1, a_2, \ldots, a_m\}$, we denote $\overrightarrow{C}_m = (a_1, a_2, \ldots, a_m, a_1)$. The tournament $\overrightarrow{C}_{2n+1}\langle\emptyset\rangle$ is called the *cyclic tournament*. It is straightforward to check that there is only one cyclic tournament on n vertices up to isomorphism for every $n \in \mathbb{N}$. The isomorphism between digraphs G and H is denoted by $G \cong H$. A digraph D is called *r-dichromatic* if $dc(D) = r$. It is *vertex-critical* r-dichromatic if $dc(D) = r$ and $dc(D - v) < r$ for every $v \in V(D)$. For general terminology, see [2]. In what follows, we will need the following results of [11] and [13].

Theorem 1 ([13], Theorem 1)**.** *If T_{2n+1} is a regular tournament on $2n+1$ vertices, then $dc(T_{2n+1}) = 2$ if and only if $T_{2n+1} \cong \overrightarrow{C}_{2n+1}\langle\emptyset\rangle$.*

Theorem 2 ([13], Theorem 2)**.** *$\overrightarrow{C}_{2n+1}\langle n\rangle$ is a vertex-critical 3-dichromatic circulant tournament for $n \geq 3$.*

Theorem 3 ([11])**.** *$\overrightarrow{C}_{6m+1}\langle 2m\rangle$ is a vertex-critical 4-dichromatic circulant tournament for $m \geq 2$.*

Let T be a tournament and $k, l \in \mathbb{N}$. We recall that a transitive subtournament TT_k of T is *maximal* if there does not exist a transitive subtournament TT_l of T $(k < l)$ such that TT_k is a subtournament of TT_l.

Remark 4 ([8]). Up to isomorphism

(i) there exists a unique circulant tournament of order 5, that is, $\overrightarrow{C}_5(1,2) = \overrightarrow{C}_5\langle\emptyset\rangle$,

(ii) there exist two circulant tournaments of order 7 which are

$$\overrightarrow{C}_7(1,2,3) = \overrightarrow{C}_7\langle\emptyset\rangle \cong \overrightarrow{C}_7\langle 1\rangle \cong \overrightarrow{C}_7\langle 2\rangle \text{ and}$$
$$\overrightarrow{C}_7(1,2,4) = \overrightarrow{C}_7\langle 3\rangle,$$

(iii) there exist three circulant tournaments of order 9 which are

$$\overrightarrow{C}_9(1,2,3,4) = \overrightarrow{C}_9\langle\emptyset\rangle,$$
$$\overrightarrow{C}_9(1,2,3,5) = \overrightarrow{C}_9\langle 4\rangle \cong \overrightarrow{C}_9\langle 1\rangle \cong \overrightarrow{C}_9\langle 3\rangle \quad \text{and}$$
$$\overrightarrow{C}_9(1,3,4,7) = \overrightarrow{C}_9\langle 2\rangle,$$

(where $\overrightarrow{C}_9\langle 2\rangle \cong \overrightarrow{C}_3[\overrightarrow{C}_3]$ is the composition of $\overrightarrow{C}_3$ and $\overrightarrow{C}_3$).

In the following three sections we determine the exact value of $dc(\overrightarrow{C}_{2n+1}\langle k\rangle)$ for every $n, k \in \mathbb{N}$. We subdivide the calculations into five cases (the cases are illustrated in Figure 1):

(i) $k = 1$,

(ii) $k = 2$,

(iii) $3 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$,

(iv) $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq \left\lfloor \frac{2}{3}n \right\rfloor$ and

(v) $\left\lfloor \frac{2}{3}n \right\rfloor + 1 \leq k \leq n$.

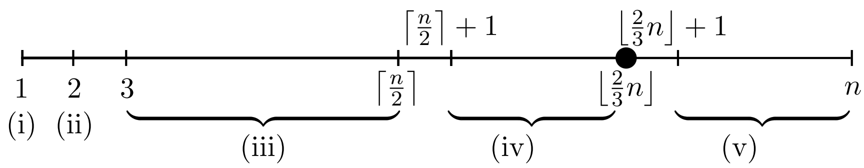

Figure 1

3. The Dichromatic Number of $\overrightarrow{C}_{2n+1}\langle 1\rangle$ and $\overrightarrow{C}_{2n+1}\langle 2\rangle$

To begin with, let us observe the following facts.

Remark 5.

(i) For every $j \in \mathbb{Z}_{2n+1}$, the set of vertices $\{j-2, j-1, j\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1\rangle$ and $\overrightarrow{C}_{2n+1}\langle 2\rangle$, respectively,

(ii) $\overrightarrow{C}_{11}\langle 1\rangle \cong \overrightarrow{C}_{11}\langle 4\rangle$,

(iii) $\overrightarrow{C}_{13}\langle 1\rangle \cong \overrightarrow{C}_{13}\langle 3\rangle$.

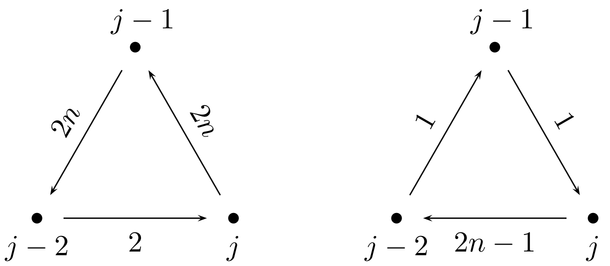

Figure 2. $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1\rangle$ and $\overrightarrow{C}_{2n+1}\langle 2\rangle$, respectively.

Proposition 6. $dc(\overrightarrow{C}_{11}\langle 1\rangle) = dc(\overrightarrow{C}_{13}\langle 1\rangle) = dc(\overrightarrow{C}_{15}\langle 1\rangle) = 3$.

Proof. Consider $\overrightarrow{C}_{11}\langle 1\rangle$. Since $\overrightarrow{C}_{11}\langle 1\rangle \ncong \overrightarrow{C}_{11}\langle\emptyset\rangle$, by Theorem 1, $dc(\overrightarrow{C}_{11}\langle 1\rangle) \geq 3$. The partition of the vertices of $\overrightarrow{C}_{11}\langle 1\rangle$ given by $P_1 = \{0, 3, 2, 5\}$, $P_2 = \{7, 10, 9, 1\}$ and $P_3 = \{4, 6, 8\}$ implies that $dc(\overrightarrow{C}_{11}\langle 1\rangle) = 3$. Observe that P_1 and P_2 induce a TT_4, and P_3 induces a TT_3 in $\overrightarrow{C}_{11}\langle 1\rangle$, respectively. By Remark 5(ii), we have that $dc(\overrightarrow{C}_{11}\langle 4\rangle) = 3$. Notice that the transitive subtournaments induced by P_1, P_2 and P_3 are maximal. If $\langle P_1\rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is the vertex 10, but $(2, 5, 10, 2) \cong \overrightarrow{C}_3$. Then P_1 induces a maximal transitive subtournament. The arguments are similar for P_2 and P_3. Analogously we can prove the others cases $n = 6$ and $n = 7$. ■

The following lemmas will be useful tools in order to prove Theorem 11. Let a and b nonnegative integers such that $0 \leq a < b \leq n$. We define the interval $[a, b] = \{a, a + 1, \ldots, b\}$, $X_0 = [0, n]$, $X_3 = [0, 2n]$ and

$$Y_0 = \{j \in X_0 : j \equiv 1 \bmod 3\}.$$

Lemma 7. *The tournament $\overrightarrow{C}_{2n+1}\langle 1\rangle$ contains a maximal transitive subtournament of order $n + 1 - \left\lfloor \frac{n}{3} \right\rfloor$ if $n \equiv 0 \bmod 3$ and of order $n + 2 - \left\lceil \frac{n}{3} \right\rceil$ vertices if $n \equiv 1 \bmod 3$.*

Proof. Consider $\overrightarrow{C}_{2n+1}\langle 1\rangle = \overrightarrow{C}_{2n+1}(2, 3, 4, 5, 6, \ldots, n, 2n)$. Recall that circulant tournaments are vertex-transitive, so it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that

$$N^+(0) = \{2, 3, 4, 5, 6, \ldots, n, 2n\}.$$

We have two cases.

Case 1. $n \equiv 0 \bmod 3$. Let $n = 3k$ where $k \in \mathbb{N}$. Notice that the subset of vertices $j \equiv 0, 2 \bmod 3$ belonging to the set X_0 induces a transitive subtournament

$$H_0 = \langle X_0 \setminus Y_0\rangle,$$

by Remark 5(i). Observe that $|Y_0| = \lfloor \frac{n}{3} \rfloor$ and $|H_0| = n + 1 - \lfloor \frac{n}{3} \rfloor = 2k + 1$. It remains to prove that H_0 is maximal. If H_0 was not maximal, then the only vertex we can add is $2n$ by Remark 5(i). Observe that in this case the set $\{n, 2n, n-4\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1\rangle$, which implies that $H_0 \cup \{2n\}$ cannot induce a maximal transitive subtournament.

Case 2. $n \equiv 1 \bmod 3$. This case is analogous to Case 1. The maximal transitive subtournament is

$$H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0\rangle.$$

■

Lemma 8. *Let* $\overrightarrow{C}_{2n+1}\langle 1\rangle$ *be such that* $n \equiv 0, 1 \bmod 3$. *Then the subtournaments induced by*

$$X_3 \setminus (X_0 \setminus Y_0) \quad \textit{if} \quad n \equiv 0 \bmod 3 \quad \textit{and}$$

$$X_3 \setminus ((X_0 \cup \{2n\}) \setminus Y_0) \quad \textit{if} \quad n \equiv 1 \bmod 3$$

contain a maximal transitive subtournament of $n - \lfloor \frac{n}{3} \rfloor$ *and* $n - \lfloor \frac{n}{3} \rfloor - 1$ *vertices, respectively.*

Proof. Suppose that $n \equiv 0 \bmod 3$. By Lemma 7, $\overrightarrow{C}_{2n+1}\langle 1\rangle$ contains the transitive subtournament H_0. Consider $X_1 = [n+1, 2n]$ and define

$$Y_1 = \{j \in X_1 : j \equiv 2 \bmod 3\} \text{ and } J_0 = \langle X_1 \setminus Y_1\rangle.$$

By Remark 5(i), J_0 is transitive. Notice that J_0 has order $n - \lfloor \frac{n}{3} \rfloor$. We prove that J_0 is maximal in the same way as in the proof of Lemma 7. For a contradiction, if J_0 was not maximal, then the only vertex we can add is vertex 1 by Remark 5(i). Observe that in this case, the set $\{1, n+1, n+3\}$ induces a $\overrightarrow{C}_3$ in $\overrightarrow{C}_{2n+1}\langle 1\rangle$, which implies that $J_0 \cup \{1\}$ cannot induce a maximal transitive subtournament.

When $n \equiv 1 \bmod 3$, the arguments are similar. The maximal transitive subtournament J_1 is given by

$$J_1 = \langle X_1 \setminus (Y_2 \cup \{2n\})\rangle,$$

where $Y_2 = \{j \in X_1 : j \equiv 0 \bmod 3\}$. ■

Lemma 9. *A maximal transitive subtournament contained in* $\overrightarrow{C}_{2n+1}\langle 1\rangle$ *has* $n + 1 - \lceil \frac{n}{3} \rceil$ *vertices if* $n \equiv 2 \bmod 3$.

Proof. It is similar to the proof of Lemma 7. The maximal transitive subtournament is

$$H_2 = \langle X_0 \setminus Y_0\rangle.$$

■

Lemma 10. *Let* $\overrightarrow{C}_{2n+1}\langle 1\rangle$ *be such that* $n \equiv 2 \bmod 3$*. Then the subtournament induced by*

$$X_3 \setminus (X_0 \setminus Y_0)$$

contains a maximal transitive subtournament of order $n - \left\lfloor \frac{n}{3} \right\rfloor$*.*

Proof. It is similar to the proof of Lemma 8. In this case, every vertex $j \equiv 0, 1 \bmod 3$ in X_1 induces a transitive subtournament

$$J_2 = \langle X_1 \setminus Y_3 \rangle,$$

where $Y_3 = \{j \in X_1 : j \equiv 2 \bmod 3\}$. ■

Theorem 11. *Let* $n \in \mathbb{N}$*. Then* $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 4$ *for every* $n \geq 8$*.*

Proof. By Theorem 1, we have that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) \geq 3$. In the first place, we prove that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) \geq 4$. For a contradiction, suppose that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 3$. Thus, $\overrightarrow{C}_{2n+1}\langle 1\rangle$ has a partition of its vertices inducing three transitive subtournaments. Suppose that $n \equiv 0 \bmod 3$ (it is similar when $n \equiv 1, 2 \bmod 3$). By Lemmas 7 and 8, two maximal disjoint transitive subtournaments in $\overrightarrow{C}_{2n+1}\langle 1\rangle$ are H_0 and J_0. Hence, the remaining vertex set $X_3 \setminus \{V(H_0) \cup V(J_0)\}$,

$$\{1, 4, 7, \ldots, n+2, n+5, \ldots\},$$

induces the third transitive subtournament. Observe that the vertex set $\{1, 7, n+2\}$ induces a $\overrightarrow{C}_3$, this is a contradiction. Therefore, $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) \geq 4$. We show that $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 4$. By Lemmas 7 and 8, we have that the two maximal transitive subtournaments H_0 and J_0 have cardinality $n + 1 - \left\lfloor \frac{n}{3} \right\rfloor$ and $n - \left\lfloor \frac{n}{3} \right\rfloor$, respectively. Define a third subtournament

$$K_0 = \langle \{1\} \cup Y_1 \rangle.$$

Notice that $|K_0| = \left\lfloor \frac{n}{3} \right\rfloor + 1$ and K_0 is transitive by the definition of Y_1. We will prove that K_0 is a maximal transitive subtournament in $\overrightarrow{C}_{2n+1}\langle 1\rangle \setminus \{H_0 \cup J_0\}$. If K_0 was not a maximal transitive subtournament, then we can add at least one vertex of $Y_0 \setminus \{1\}$. Notice that if $i \in Y_0 \setminus \{1\}$, we have that $(i, i+n-2, i+n+1, i) \cong \overrightarrow{C}_3$. Therefore, K_0 is a maximal transitive subtournament. Finally, the subtournament $L_0 = \langle Y_0 \setminus \{1\} \rangle$ is transitive by the definition of Y_0 and maximal. Thus, $dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = 4$. The proof is completely analogous for the cases when $n \equiv 1, 2 \bmod 3$. The partition into transitive subtournaments is

$$H_1 = \langle (X_0 \cup \{2n\}) \setminus Y_0 \rangle,\ J_1 = \langle X_1 \setminus Y_2 \cup \{2n\} \rangle,\ K_1 = \langle Y_2 \cup \{1\} \rangle,\ L_1 = \langle Y_0 \setminus \{1\} \rangle,$$

for $n \equiv 1 \bmod 3$. For $n \equiv 2 \bmod 3$, we have that

$$H_2 = \langle X_0 \setminus Y_0 \rangle,\ J_2 = \langle X_1 \setminus Y_3 \rangle,\ K_2 = \langle (Y_3 \cup \{1\}) \rangle,\ L_2 = \langle Y_0 \setminus \{1\} \rangle.$$ ■

From Proposition 6, Theorems 1, 2 and 11 and Remark 4(iii), we obtain the following consequence.

Corollary 12.

$$dc(\overrightarrow{C}_{2n+1}\langle 1\rangle) = \begin{cases} 2 & \textit{if } n = 3, \\ 3 & \textit{if } 4 \le n \le 7, \\ 4 & \textit{if } n \ge 8. \end{cases}$$

Theorem 13. *Let $r \in \{2,3,4\}$. Then $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical r-dichromatic circulant tournament if and only if $n \in \{1,4\}$.*

Proof. If $r = 2$, clearly $\overrightarrow{C}_3\langle 1\rangle$ is a vertex-critical 2-dichromatic.

If $r = 3$, we need to check for which values of $4 \le n \le 7$, the circulant tournament $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical 3-dichromatic. Notice that $\overrightarrow{C}_9\langle 1\rangle \cong \overrightarrow{C}_9\langle 4\rangle$ by Remark 4(iii). It is a vertex-critical 3-dichromatic by Theorem 2. For $n = 5$, the circulant tournament $\overrightarrow{C}_{11}\langle 1\rangle$ is not a vertex-critical 3-dichromatic by Proposition 6. Using analogous arguments, $\overrightarrow{C}_{13}\langle 1\rangle$ and $\overrightarrow{C}_{15}\langle 1\rangle$ are not vertex-critical.

If $r = 4$, the circulant tournament $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is 4-dichromatic for every $n \ge 8$ by Theorem 11. It was partitioned into four maximal transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0, 1, 2$. Notice that $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical 4-dichromatic if the cardinality of L_i is equal to one for $i = 0, 1, 2$. Since $|L_i| = |Y_0| - 1 = \lfloor \frac{n}{3} \rfloor - 1$, we have that $|L_i| = 1$ if and only if $\lfloor \frac{n}{3} \rfloor = 2$. It occurs when $n = 6, 7$ or 8. By Theorem 11, it is only possible for $n \ge 8$. Observe that $|L_2| \ge 2$ for $T = \overrightarrow{C}_{2n+1}\langle 1\rangle$ if $n \ge 8$. Since this partition is maximal, T is not vertex-critical.

Therefore, $\overrightarrow{C}_{2n+1}\langle 1\rangle$ is a vertex-critical r-dichromatic circulant tournament if and only if n is 1 or 4. ■

Let us recall that

Remark 14. $\overrightarrow{C}_9\langle 2\rangle = \overrightarrow{C}_3[\overrightarrow{C}_3]$ is 3-dichromatic, a particular case of Theorem 8 from [9]. Notice that it is not vertex-critical.

Remark 15 ([10], Theorem 2.6)**.** $\overrightarrow{C}_{11}\langle 2\rangle$ is vertex-critical 4-dichromatic.

Proposition 16. *If $n = 6$ and 7, then $dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = 3$.*

Proof. Observe that $\overrightarrow{C}_{15}\langle 2\rangle \ncong \overrightarrow{C}_{15}\langle \emptyset\rangle$. Then by Theorem 1, $dc(\overrightarrow{C}_{15}\langle 2\rangle) \ge 3$. Consider the following partition of $V(\overrightarrow{C}_{15}\langle 2\rangle)$:

$$P_1 = \{0,1,3,4,6,7\}, P_2 = \{5,8,9,11,12\} \text{ and } P_3 = \{2,10,13,14\}.$$

We have that $\langle P_1\rangle \cong TT_6$, $\langle P_2\rangle \cong TT_5$ and $\langle P_3\rangle \cong TT_4$. Therefore, $dc(\overrightarrow{C}_{15}\langle 2\rangle) = 3$. Note that the transitive subtournaments induced by P_1, P_2 and P_3 are maximal. If $\langle P_1\rangle$ was not a maximal transitive subtournament, then the only vertex that we can add is vertex 13. We cannot add the vertex 5 by Remark 5(i). But $(4, 7, 13, 4) \cong \overrightarrow{C}_3$. Then P_1 induces a maximal transitive subtournament. The same conclusion is valid for P_2 and P_3. Observe that $\overrightarrow{C}_{15}\langle 2\rangle$ is not vertex-critical. The proof is analogous for $n = 6$. ■

Theorem 17. *Let $n \in \mathbb{N}$. Then $dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = 4$ for every $n \geq 8$.*

Proof. It is analogous to the proof of Theorem 11. Therefore, Remark 5(i) is applied for $\overrightarrow{C}_{2n+1}\langle 2\rangle$. The corresponding partitions are following.

(i) $n \equiv 0 \bmod 3$, we define $Y_4 = \{j \in X_0 : j \equiv 2 \bmod 3\}$,

$$H_0 = \langle X_0 \setminus Y_4\rangle,\ J_0 = \langle X_1 \setminus Y_2\rangle,\ K_0 = \langle Y_2 \cup \{2\}\rangle,\ L_0 = \langle Y_4 \setminus \{2\}\rangle.$$

(ii) $n \equiv 1 \bmod 3$, we define $Y_5 = \{j \in X_1 : j \equiv 1 \bmod 3\}$,

$$H_1 = \langle X_0 \setminus Y_4\rangle,\ J_1 = \langle X_1 \setminus Y_5\rangle,\ K_1 = \langle Y_5 \cup \{2\}\rangle,\ L_1 = \langle Y_4 \setminus \{2\}\rangle.$$

(iii) $n \equiv 2 \bmod 3$, we define $Y_6 = \{j \in X_1 : j \equiv 1 \bmod 3\}$,

$$H_2 = \langle X_0 \setminus Y_4\rangle,\ J_2 = \langle (X_1 \cup \{n\}) \setminus Y_6\rangle,\ K_2 = \langle Y_6\rangle,\ L_2 = \langle Y_4 \setminus \{n\}\rangle.$$ ■

The next corollary is an immediate consequence of Remarks 4(ii)–(iii), 14, 15, Proposition 16 and Theorems 1, 2 and 17.

Corollary 18.

$$dc(\overrightarrow{C}_{2n+1}\langle 2\rangle) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, 6, 7, \\ 4 & \text{if } n = 5 \text{ and } n \geq 8. \end{cases}$$

Theorem 19. *Let $r \in \{2, 3, 4\}$. Then $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is a vertex-critical r-dichromatic circulant tournament if and only if $n = 5$.*

Proof. If $r = 2$, by Theorem 1, $\overrightarrow{C}_7\langle 2\rangle \cong \overrightarrow{C}_7\langle\emptyset\rangle$ is 2-dichromatic, but it is not vertex-critical.

Let $r = 3$. For $n = 4$, we have that $\overrightarrow{C}_9\langle 2\rangle$ is not vertex-critical by Remark 14. For $n = 6$ and 7 by Proposition 16, $\overrightarrow{C}_{13}\langle 2\rangle$ and $\overrightarrow{C}_{15}\langle 2\rangle$ are not vertex-critical.

If $r = 4$, then by Remark 15, $\overrightarrow{C}_{11}\langle 2\rangle$ is vertex-critical. By Theorem 17, $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is 4-dichromatic for every $n \geq 8$. It was partitioned into four maximal transitive subtournaments, where $|L_i| = \min\{|H_i|, |J_i|, |K_i|, |L_i|\}$ for $i = 0, 1, 2$. Notice that $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is vertex-critical 4-dichromatic if the cardinality of L_i is equal to one for $i = 0, 1, 2$. Since $|L_i| = |Y_4| - 1 = \lfloor \frac{n}{3} \rfloor - 1$, we have that $|L_i| = 1$ if and only if $\lfloor \frac{n}{3} \rfloor = 2$. It occurs when $n = 6, 7$ or 8. By Theorem 17 it is only possible for $n \geq 8$. Observe that $|L_2| \geq 2$ for $T = \overrightarrow{C}_{2n+1}\langle 2\rangle$ if $n \geq 8$. Since this partition is maximal, T is not vertex-critical. Therefore, $\overrightarrow{C}_{2n+1}\langle 2\rangle$ is vertex-critical if and only if $n = 5$. ■

4. The Dichromatic Number of $\overrightarrow{C}_{2n+1}\langle k\rangle$ for $3 \leq k \leq \lceil \frac{n}{2} \rceil$

We prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$, for $3 \leq k \leq \lceil \frac{n}{2} \rceil$ and $n \geq 7$.

Lemma 20. *If $3 \leq k \leq \lceil \frac{n}{2} \rceil$, then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains a maximal transitive subtournament H.*

Proof. Let n and k be nonnegative integers and consider the interval $[0, n]$. Applying the Euclidean division algorithm to $n+1$ and $2k-1$, there exist unique α, $r \in \mathbb{N}$ such that

$$n + 1 = \alpha(2k - 1) + r \quad \text{where} \quad 0 \leq r < 2k - 1.$$

Consider the partition of the interval $[0, 2k - 2] = [0, k - 1] \cup [k, 2k - 2]$ and define

$$n + 1 = \begin{cases} \alpha(2k - 1) + s_1 & \text{if } s_1 \in [0, k - 1], \\ \alpha(2k - 1) + s_2 & \text{if } s_2 \in [k, 2k - 2]. \end{cases}$$

Observe that since $3 \leq k \leq \lceil \frac{n}{2} \rceil$, we have that $s_1 \in [1, k - 1]$.

Let

$$W = \bigcup_{i=0}^{\alpha-1} [i(2k - 1), i(2k - 1) + (k - 1)].$$

We define the subtournament H of $\overrightarrow{C}_{2n+1}\langle k\rangle$ in the following way.

(i) If $s_1 \in [1, k - 1]$, then $H = \langle W \cup [\alpha(2k - 1), n]\rangle$. Moreover, if $k = \frac{n+1}{2}$ and n is odd, then $H = \langle W \cup \{n, 2n + 1 - k\}\rangle$.

(ii) $H = \langle W \cup [\alpha(2k - 1), \alpha(2k - 1) + k - 1]\rangle$ for every $s_2 \in [k, 2k - 2]$.

Note that H is a transitive subtournament by construction, since its vertex set does not contain induced $\overrightarrow{C}_3$'s. We prove that H is maximal by contradiction. Since $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex-transitive, without loss of generality, choose the vertex 0. Observe that $N^+(0) = \{1, 2, \ldots, k - 1, k + 1, \ldots, n, 2n + 1 - k\}$ and

$$N^+(0) \setminus V(H) = (X_0 \cup \{2n + 1 - k\}) \setminus (V(H) \cup \{k\}).$$

For every vertex $x \in N^+(0) \setminus V(H)$ there exist $h_1, h_2 \in V(H)$ such that the vertex set $\{h_1, h_2, x\}$ induces a $\overrightarrow{C}_3$ (for instance, $x = k+1$, $h_1 = 1$ and $h_2 = k-1$), a contradiction. Therefore, H is maximal. ■

Lemma 21. *If* $3 \leq k \leq \lceil \frac{n}{2} \rceil$, *then* $X_3 \setminus V(H)$ *contains a maximal transitive subtournament* J, *where* H *is the subtournament defined in Lemma* 20.

Proof. The construction of J is similarly obtained as in the proof of Lemma 20 for H, but we have two ways of defining J.

Case 1. $\alpha = 1$.

(i) If $s_1 \in [1, k-1]$, then $J = [k, 2k-2] \cup [3k-2, 3k+s_1-2]$. Notice that if $k = \frac{n+1}{2}$ with n is odd if and only if $s_1 = 1$. Then $J = [k, 2k-2] \cup \{3k-2\}$ by the construction of H.

(ii) If $s_2 \in [k, 2k-2]$, then $J = [k, 2k-2] \cup [3k-1, 4k-2]$.

Case 2. $\alpha > 1$. Let

$$U = \bigcup_{i=0}^{\alpha-1} [(n+1) + i(2k-1), (n+1) + i(2k-1) + (k-1)].$$

(i) If $s_1 \in [1, k-1]$, then $J = \langle U \cup [(n+1) + \alpha(2k-1), 2n] \rangle$.

(ii) $J = \langle U \cup [(n+1)+\alpha(2k-1), (n+1)+\alpha(2k-1)+k-1] \rangle$ for every $s_2 \in [k, 2k-2]$.

Notice that H is a maximal transitive subtournament in $\overrightarrow{C}_{2n+1}\langle k \rangle$ by Lemma 20. We claim that J is maximal in $V(\overrightarrow{C}_{2n+1}\langle k \rangle \setminus V(H)$. If J was not maximal, we could add at least one vertex of $V(\overrightarrow{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J))$.

For Case 1, consider

$$\{n+k+1, 3k-3\} \subseteq V(\overrightarrow{C}_{2n+1}\langle k \rangle) \setminus (V(H) \cup V(J))).$$

We have that $(k, n+k, n+k+1, k) \cong \overrightarrow{C}_3$ or $(2k-2, 3k-3, 3k-2, 2k-2) \cong \overrightarrow{C}_3$. Therefore, J is maximal.

For Case 2, consider

$$k \in V(\overrightarrow{C}_{2n+1}\langle k \rangle \setminus (V(H) \cup V(J)).$$

We have that $(k, n+k, n+2k+1, k) \cong \overrightarrow{C}_3$. Hence, J is maximal. ■

Theorem 22. *If* $3 \leq k \leq \lceil \frac{n}{2} \rceil$, *then* $dc(\overrightarrow{C}_{2n+1}\langle k \rangle) = 4$ *for* $n \geq 7$.

Proof. By Theorem 1, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 3$. We prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. For a contradiction, suppose that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$. Thus, $\overrightarrow{C}_{2n+1}\langle k\rangle$ has a partition of its vertices consisting of three transitive subtournaments. By Lemmas 20 and 21, two maximal transitive disjoint subtournaments in $\overrightarrow{C}_{2n+1}\langle k\rangle$ are H and J. Hence, the remaining vertex set $X_3 \setminus (V(H) \cup V(J))$ induces the third transitive subtournament.

We consider three cases.

Case 1. $J = \langle [k, 2k-2] \cup [3k-2, 3k+s_1-2]\rangle$ obtained by Case 1(i) of Lemma 21. Therefore, $K = \langle [3k+s_1-1, 2n]\rangle$. Moreover, $|J| = k+s_1$ and $|H| = n-k+2$. Since $k \leq \lceil \frac{n}{2} \rceil$, we have that $|K| = 2n+1-(|H|+|J|) = 2k-3 > k$. In this case, K is induced by at least $k+1$ consecutive vertices. Therefore, K cannot be a transitive subtournament by the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. The following cases are necessary because the structure of K and L changes with different values of s_1.

(i) If $s_1 = 1, 2$ or 3, then

$$K = \langle [n+1, 3k-3] \cup [4k-3, 2n]\rangle \text{ and } L = \langle [3k+s_1-1, 4k-4]\rangle.$$

(ii) If $s_1 \in [4, k-5]$, then

$$K = \langle [n+1, 3k-3] \cup [4k-3, 5k-4]\rangle \text{ and}$$
$$L = \langle [3k+s_1-1, 4k-4] \cup [5k-3, 2n]\rangle.$$

(iii) If $s_1 = k-4$, then

$$K = \langle [n+1 = 3k-4, 3k-3] \cup [4k-3, 5k-4]\rangle \text{ and}$$
$$L = \langle [4k-5, 4k-4] \cup [5k-3, 2n]\rangle.$$

(iv) If $s_1 = k-3$, then

$$K = \langle [n+1 = 3k-5, 3k-3] \cup [4k-3, 5k-4]\rangle \text{ and}$$
$$L = \langle \{4k-4\} \cup [5k-3, 2n]\rangle.$$

(v) If $s_1 = k-1$ or $k-2$, then

$$K = \langle [n+k+1, n+2k]\rangle \text{ and } L = \langle [n+2k+1, 2n]\rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the subtournaments K and L are transitive. Observe that if $s_1 \in [1, k-3]$, $4k-4 \notin V(K)$ and $(4k-4, 4k-3, 3k-3, 4k-4) \cong \overrightarrow{C}_3$. Therefore, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$. If $s_1 = k-1$ or $k-2$, then $n+2k+1 \notin V(K)$ and $(n+k+1, n+2k, n+2k+1, n+k+1) \cong \overrightarrow{C}_3$. Thus, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle = 4$.

Case 2. $J = [k, 2k-2] \cup [3k-1, 4k-2]$ obtained by Case 1(ii) of Lemma 21. We have that $X_3 \setminus (V(H) \cup V(J)) = [4k-1, 2n]$, but $2n - 4k + 2 > k$ implies that the subtournament induced by $X_3 \setminus (V(H) \cup V(J))$ has at least $k+1$ consecutive vertices and a $\overrightarrow{C}_3$ is induced by $X_3 \setminus (V(H) \cup V(J))$. Therefore, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle \geq 4$. The following cases show the partition of $\overrightarrow{C}_{2n+1}\langle k\rangle$ into transitive subtournaments.

(i) If $s_2 = k$, then

$$K = \langle [4k-1, 5k-2] \rangle \text{ and } L = \langle [5k-1, 2n] \rangle.$$

(ii) If $s_2 = k+1$, then

$$K = \langle [4k-1, 5k-2] \cup \{6k-2 = 2n\} \rangle \text{ and } L = \langle [5k-1, 6k-3] \rangle.$$

(iii) If $s_2 = k+2$, then

$$K = \langle [4k-1, 5k-2] \cup [6k-2, 6k] \rangle \text{ and } L = \langle [5k-1, 6k-3] \rangle.$$

(iv) If $s_2 \in [k+3, 2k-2]$, then

(a) if $2n \leq 7k-3$, we have that

$$K = \langle [4k-1, 5k-2] \cup [6k-2, 2n] \rangle \text{ and}$$
$$L = \langle [5k-1, 6k-3] \rangle,$$

(b) if $2n > 7k-3$, then

$$K = \langle [4k-1, 5k-2] \cup [6k-2, 7k-3] \rangle \text{ and}$$
$$L = \langle [5k-1, 6k-3] \cup [7k-2, 2n] \rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the subtournaments K and L are transitive. Observe that $5k-1 \notin V(K)$ and $(4k-1, 5k-2, 5k-1, 4k-1) \cong \overrightarrow{C}_3$. Therefore, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$.

Case 3. J is obtained by Case 2(i) and (ii) of Lemma 21. If $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$, then $X_3 \setminus (V(H) \cup V(J))$ induces a transitive subtournament, but the vertex set $\{k, 3k-1, n+k+1\} \subseteq X_3 \setminus (V(H) \cup V(J))$ induces a $\overrightarrow{C}_3$. Hence, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. The partition of $\overrightarrow{C}_{2n+1}\langle k\rangle$ into transitive subtournaments is H, J, $K = \langle X_1 \cup \{k\} \setminus V(J) \rangle$ and

$$L = \langle X_3 \setminus (V(H) \cup V(J) \cup V(K)) \rangle.$$

By construction and the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the subtournaments K and L are maximal transitive. Observe that $k+1 \notin V(K)$. Then the vertex set $\{k, k+1, n+k+1\}$ induces a $\overrightarrow{C}_3$. Therefore, K is maximal in $\overrightarrow{C}_{2n+1}\langle k\rangle \setminus (H \cup J)$.

This proves that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$. ∎

The following example illustrates Theorem 22, Case 2(ii). The tournament $\overrightarrow{C}_{29}\langle 5\rangle$ has the following partition into four transitive subtournaments

$H = \langle [0,4] \cup [9,13]\rangle$, $K = \langle [5,8] \cup [14,18]\rangle$, $J = \langle [19,23] \cup \{28\}\rangle$ and $L = \langle [24,27]\rangle$.

Observe that $H \cong TT_{10}$, $J \cong TT_9$, $K \cong TT_6$ and $L \cong TT_4$.

Theorem 23. *If* $3 \leq k \leq \lceil \frac{n}{2} \rceil$, *then* $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is a vertex-critical* 4*-dichromatic circulant tournament if and only if*

(i) $n = 7$ *and* $k \in \{3, 4\}$,
(ii) $n = 9$ *and* $k = 4$,
(iii) $n = 10$ *and* $k = 5$,
(iv) $n = 13$ *and* $k = 6$.

Proof. By Theorem 22, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is 4-dichromatic, where H, J, K and L are maximal transitive subtournaments. Note that by the partition of the vertices of $\overrightarrow{C}_{2n+1}\langle k\rangle$, the cases that need to be considered are when $\alpha = 1$, because it is when the order of L can be one. In this case, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is a vertex-critical 4-dichromatic. We have two cases when $\alpha = 1$.

Case 1. $s_1 \in [1, k-1]$.

(i) If $s_1 \in \{1, 2, 3\}$, then by Theorem 22 Case 1(i), we have that $|L| = k-3, k-4, k-5$, respectively. The tournament is vertex-critical if and only if $|L| = 1$ if and only if $k = 4$ and $n = 7$, $k = 5$ and $n = 10$, $k = 6$ and $n = 13$, respectively
(ii) If $s_1 = k-3$, then by the proof of Theorem 22 Case 1(iv), it is vertex-critical if and only if $|L| = 1$ if and only if $2n = 5k-4$ and $n = 3k-5$ if and only if $k = 6$ and $n = 13$.
(iii) If $s_1 = k-2$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = k-2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
(iv) If $s_1 = k-1$, then by the proof of Theorem 22 Case 1(v), we have that $|L| = n-2k$. It is vertex-critical if and only if $|L| = 1$ if and only if $n = 2k+1$ and $n = 3k-3$ if and only if $k = 4$ and $n = 9$.

Case 2. $s_2 \in [k, 2k-2]$.

(i) If $s_2 = k$, then by the proof of Theorem 22 Case 2(i), we have that $|L| = k-2$. It is vertex-critical if and only if $|L| = 1$ if and only if $k = 3$ and $n = 7$.
(ii) If $s_2 \in [k+1, 2k-2]$, then by the proof of Theorem 22 Case 2(ii)–(iv)(a), we have that $|L| = k-2$, but it is not necessarily vertex-critical if $|L| = 1$, because the last vertices remain in K. When L is obtained by the proof of Theorem 22 Case 2(iv)(b), $|L|$ never is one. In any case, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is not a vertex-critical 4-dichromatic circulant tournament. ∎

5. THE DICHROMATIC NUMBER OF $\overrightarrow{C}_{2n+1}\langle k\rangle$ FOR $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n$.

In this part we prove that the tournaments $\overrightarrow{C}_{2n+1}\langle k\rangle$ are 4-dichromatic if $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$ for $n \geq 8$.

Lemma 24. *If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains a maximal transitive subtournament of order k.*

Proof. Since $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex-transitive, it is enough to consider a maximal transitive subtournament containing vertex 0. Observe that $N^+(0) = \{1, 2, \ldots, k-1, k+1, \ldots, n, 2n+1-k\}$. We define $H = \langle [0, k-1] \rangle$. It is transitive by the definition of $\overrightarrow{C}_{2n+1}\langle k\rangle$. If H was not maximal, then we could add one vertex of $N^+(0) \setminus [0, k-1]$. Let $j \in [k+1, n]$. Without loss of generality, choose $j = k+1$. Thus, the set of vertices $\{1, t, k+1\}$ with $t \in [2, k-1]$ induces a $\overrightarrow{C}_3$. The same occurs for the vertex $2n+1-k$. Observe that $(3, k-1, 2n+1-k, 3) \cong \overrightarrow{C}_3$, a contradiction. Therefore, H is maximal. ■

Lemma 25. *If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains three maximal transitive subtournaments of k vertices.*

Proof. By Lemma 24, $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains a maximal transitive subtournament H. Notice that $|N^+(0)| - |H| < k$. Consider the following subtournaments

$$J = \langle [k, 2k-1] \rangle \quad \text{and } K = \langle [2k, 3k-1] \rangle.$$

Observe that J and K are isomorphic to H. Let $\varphi_1 : H \to J$ such that $\varphi_1(j) = j + k$ with $0 \leq j \leq k-1$, (φ_1 is bijective and it is clear that H is isomorphic to J). Analogously, $\varphi_2 : H \to K$ is an isomorphism between H and K. As in Lemma 24, we can prove that J and K are maximal transitive subtournaments. Then $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains three maximal transitive subtournaments on k vertices. ■

Theorem 26. *If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$.*

Proof. First we prove that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. By Lemma 25, we have that $\overrightarrow{C}_{2n+1}\langle k\rangle$ contains three maximal transitive subtournaments of k vertices. Then $|\overrightarrow{C}_{2n+1}\langle k\rangle| - 3k > 0$. Thus, $V(\overrightarrow{C}_{2n+1}\langle k\rangle)$ cannot be partitioned into three transitive subtournaments. Then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 4$. We verify that $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 4$. By Lemma 25, we have that H, J and K are maximal transitive subtournaments of order k. The fourth transitive subtournament is $L = \langle [3k, 2n] \rangle$. Therefore, $\overrightarrow{C}_{2n+1}\langle k\rangle)$ is 4-dichromatic. ■

Theorem 27. *If* $\lceil \frac{n}{2} \rceil + 1 \leq k \leq \lfloor \frac{2}{3}n \rfloor$, *then* $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is a vertex-critical* 4-*dichromatic circulant tournament if and only if* $n \equiv 0 \bmod 3$.

Proof. By Theorem 26, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is 4-dichromatic. Observe that the order of H, J and K is k and $|L| = 2n - 3k + 1$. Notice that $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex critical 4-dichromatic if the cardinality of L is equal to one, and it occurs if and only if $k = \frac{2}{3}n$ when $n \equiv 0 \bmod 3$. By Theorem 3, $\overrightarrow{C}_{2n+1}\langle \frac{2}{3}n\rangle$ with $n \equiv 0 \bmod 3$ is a vertex-critical circulant tournament 4-dichromatic. ■

Corollary 28 ([11])**.** $\overrightarrow{C}_{6m+1}\langle 2m\rangle$ *is a vertex-critical* 4-*dichromatic circulant tournament for* $m \geq 2$.

Theorem 29. *Let* $n \geq 3$. *Then* $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$ *for* $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$.

Proof. Let $n \geq 3$. By Theorem 1, $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) \geq 3$. Take the following partition of the vertices of $\overrightarrow{C}_{2n+1}\langle k\rangle$:

$$H = [0, k-1], J = [k, 2k-1] \text{ and } K = [2k, 2n].$$

Observe that H induces a TT_k because $N^+(i) = \{i+1, i+2, \ldots, k+1\}$ for $k \leq i \leq 2k-1$, also J and K induce a TT_k and a $TT_{2n-2k+1}$, respectively. Then $dc(\overrightarrow{C}_{2n+1}\langle k\rangle) = 3$. ■

Theorem 30. *If* $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$, $n \geq 3$. *Then* $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is a vertex-critical* 3-*dichromatic circulant tournament if and only if* $n = k$.

Proof. By Theorem 29, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is 3-dichromatic and its partition into three maximal transitive subtournaments was

$$|H| = |J| = k \text{ and } |K| = 2n - 2k + 1.$$

Since $k = \lfloor \frac{2}{3}n \rfloor + 1, \ldots, n$, we have that $k \geq 2n - 2k + 1$. Hence, $\overrightarrow{C}_{2n+1}\langle k\rangle$ is vertex-critical if and only if $2n - 2k + 1 = 1$, if and only if $n = k$. ■

Corollary 31 ([13], Theorem 2)**.** $\overrightarrow{C}_{2n+1}\langle n\rangle$ *is a vertex-critical* 3-*dichromatic circulant tournament for* $n \geq 3$.

By Theorems 13, 19, 23, 27 and 30, we have the following.

Theorem 32. *Let* $r \in \{2, 3, 4\}$, $\overrightarrow{C}_{2n+1}\langle k\rangle$ *is vertex-critical* r-*dichromatic if and only if*

(i) $r = 2$, $n = 1$ *and* $k = 1$;

(ii) $r = 3$,

  (a) $n = 4$ *and* $k = 1$,

(b) $n \geq 3$ *and* $k = n$;

(iii) $r = 4$,

(a) $n = 5$ *and* $k = 2$,

(b) $n = 7$ *and* $k \in \{3, 4\}$,

(c) $n = 9$ *and* $k = 4$,

(d) $n = 10$ *and* $k = 5$,

(e) $n = 13$ *and* $k = 6$,

(f) $n = 3m$ *and* $k = 2m$ $(m \geq 2)$.

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