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SIGNED TOTAL ROMAN DOMINATION IN DIGRAPHS

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Abstract

Let D be a finite and simple digraph with vertex set $V(D)$. A signed total Roman dominating function (STRDF) on a digraph D is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v, and (ii) every vertex u for which $f(u) = -1$ has an inner neighbor v for which $f(v) = 2$. The weight of an STRDF f is $w(f) = \sum_{v \in V(D)} f(v)$. The signed total Roman domination number $\gamma_{stR}(D)$ of D is the minimum weight of an STRDF on D. In this paper we initiate the study of the signed total Roman domination number of digraphs, and we present different bounds on $\gamma_{stR}(D)$. In addition, we determine the signed total Roman domination number of some classes of digraphs. Some of our results are extensions of known properties of the signed total Roman domination number $\gamma_{stR}(G)$ of graphs G.

Keywords: digraph, signed total Roman dominating function, signed total Roman domination number.

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1. Introduction

In this paper we continue the study of Roman dominating functions in graphs and digraphs. Let G be a finite and simple graph with vertex set $V(G)$, and let $N(v) = N_G(v)$ be the neighborhood of the vertex v. A *signed total Roman dominating function* (STRDF) on a graph G is defined in [8] as a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{x \in N_G(v)} f(x) \geq 1$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRDF f on a graph G is $w(f) = \sum_{v \in V(G)} f(v)$. The *signed total*

Roman domination number $\gamma_{stR}(G)$ of G is the minimum weight of an STRDF on G. Following [8], we initiate the study of signed total Roman dominating functions on digraphs D.

Let D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. We write $d_D^+(v) = d^+(v)$ for the out-degree of a vertex v and $d_D^-(v) = d^-(v)$ for its in-degree. The minimum and maximum in-degree are $\delta^- = \delta^-(D)$ and $\Delta^- = \Delta^-(D)$ and the minimum and maximum out-degree are $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$. The sets $N_D^+(v) = N^+(v) = \{x \mid (v,x) \in A(D)\}$ and $N_D^-(v) = N^-(v) = \{x \mid (x,v) \in A(D)\}$ are called the *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise, $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x,y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y, and we also say that x *dominates* y or y *is dominated by* x. The *underlying graph* of a digraph D is the graph obtained by replacing each arc (u,v) or symmetric pairs $(u,v),(v,u)$ of arcs by the edge uv. A digraph D is *connected* if its underlying graph is connected. For a real-valued function $f : V(D) \to \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. Consult [2] and [3] for notation and terminology which are not defined here.

A set $S \subseteq V(D)$ is a *total dominating set* of D if for all $v \in V(D)$, there exists a vertex $u \in S$ such that v is dominated by u. The minimum cardinality of a total dominating set in D is the *total domination number* $\gamma_t(D)$.

A *signed total dominating function* on a graph G is defined in [9] as a function $f : V(G) \to \{-1,1\}$ such that $\sum_{x \in N(v)} f(x) \geq 1$ for every $v \in V(G)$. The minimum cardinality of a signed total dominating function is the *signed total domination number* $\gamma_{st}(G)$. This parameter is studied by several authors, see, for example [4, 5]. Analogously, a *signed total dominating function* on a digraph D is defined in [6] as a function $f : V(D) \to \{-1,1\}$ such that $\sum_{x \in N^-(v)} f(x) \geq 1$ for every $v \in V(D)$.

A *signed total Roman dominating function* (abbreviated STRDF) on D is defined as a function $f : V(D) \to \{-1,1,2\}$ such that $f(N^-(v)) = \sum_{x \in N^-(v)} f(x) \geq 1$ for every $v \in V(D)$ and every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The *weight* of an STRDF f on a digraph D is $w(f) = \sum_{v \in V(D)} f(v)$. The *signed total Roman domination number* $\gamma_{stR}(D)$ of D is the minimum weight of an STRDF on D. A $\gamma_{stR}(D)$-*function* is a signed total Roman dominating function on D of weight $\gamma_{stR}(D)$. For an STRDF f on D, let $V_i = V_i(f) = \{v \in V(D) \mid f(v) = i\}$ for $i \in \{-1,1,2\}$. An STRDF $f : V(D) \to \{-1,1,2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(D)$.

A signed total Roman dominating function on a digraph combines the properties of both a Roman dominating function (see [7]) and a signed total dominating function. The signed total Roman domination number exists when $\delta^- \geq 1$. Thus we assume throughout this paper that $\delta^-(D) \geq 1$. We present different sharp lower and upper bounds on $\gamma_{stR}(D)$. In addition, we determine the signed total Roman domination number of some classes of digraphs. Some of our results imply known properties of the signed total Roman domination number $\gamma_{stR}(G)$ of graphs G, given in [8].

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1. *If $D(G)$ is the associated digraph of a graph G, then $\gamma_{stR}(D(G)) = \gamma_{stR}(G)$.*

Let K_n and K_n^* be the complete graph and complete digraph of order n, respectively. In [8], the author determines the signed total Roman domination number of complete graphs.

Proposition 2 ([8])**.** *If $n \geq 3$, then $\gamma_{stR}(K_n) = 3$.*

Using Observation 1 and Proposition 2, we obtain the signed total Roman domination number of complete digraphs.

Corollary 3. *If $n \geq 3$, then $\gamma_{stR}(K_n^*) = 3$.*

Let $K_{p,p}$ be the complete bipartite graph of order $2p$ with equal size of partite sets, and let $K^*_{p,p}$ be its associated digraph.

Proposition 4 ([8])**.** *For $p \geq 1$, $\gamma_{stR}(K_{p,p}) = 2$, unless $p = 3$ in which case $\gamma_{stR}(K_{3,3}) = 4$.*

Using Observation 1 and Proposition 4, we obtain the signed total Roman domination number of complete bipartite digraphs $K^*_{p,p}$.

Corollary 5. *For $p \geq 1$, $\gamma_{stR}(K^*_{p,p}) = 2$, unless $p = 3$ in which case $\gamma_{stR}(K^*_{3,3}) = 4$.*

Proposition 6 ([8])**.** *Let C_n be a cycle of order $n \geq 3$. Then $\gamma_{stR}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stR}(C_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stR}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.*

The next result follows from Observation 1 and Proposition 6.

Corollary 7. *Let C_n^* be the associated digraph of the cycle C_n of order $n \geq 3$. Then $\gamma_{stR}(C_n^*) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stR}(C_n^*) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stR}(C_n^*) = (n+6)/2$ when $n \equiv 2 \pmod 4$.*

2. Preliminary Results

In this section we present basic properties of the signed total Roman dominating functions and the signed total Roman domination numbers of digraphs.

Proposition 8. *If $f = (V_{-1}, V_1, V_2)$ is an* STRDF *on a digraph D of order n and minimum in-degree $\delta^-(D) \geq 1$, then*

(a) $|V_{-1}| + |V_1| + |V_2| = n$,

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$,

(c) $V_1 \cup V_2$ *is a total dominating set of D.*

Proof. Since (a) and (b) are immediate, we only prove (c). By the definition, every vertex of V_{-1} has an in-neighbor in V_2. Thus V_2 dominates V_{-1}. Suppose that $D[V_1 \cup V_2]$ contains a vertex v without an in-neighbor in $V_1 \cup V_2$. As $\delta^-(D) \geq 1$, the vertex v has an in-neighbor in V_{-1} and all its in-neighbors are in V_{-1}. This leads to the contradiction $f(N^-(v)) \leq -1$. Consequently, $V_1 \cup V_2$ is a total dominating set of D. ■

Proposition 9. *Assume that $f = (V_{-1}, V_1, V_2)$ is an* STRDF *on a digraph D of order n with $\delta^-(D) \geq 1$. If $\Delta^+(D) = \Delta^+$ and $\delta^+(D) = \delta^+$, then*

(i) $(2\Delta^+ - 1)|V_2| + (\Delta^+ - 1)|V_1| \geq (\delta^+ + 1)|V_{-1}|$,

(ii) $(2\Delta^+ + \delta^+)|V_2| + (\Delta^+ + \delta^+)|V_1| \geq (\delta^+ + 1)n$,

(iii) $(\Delta^+ + \delta^+)\omega(f) \geq (\delta^+ + 2 - \Delta^+)n + (\delta^+ - \Delta^+)|V_2|$,

(iv) $\omega(f) \geq (\delta^+ + 2 - 2\Delta^+)n/(2\Delta^+ + \delta^+) + |V_2|$.

Proof. (i) It follows from Proposition 8 (a) that

$$\begin{aligned}
|V_{-1}| + |V_1| + |V_2| &= n \leq \sum_{v \in V(D)} \sum_{x \in N^-(v)} f(x) = \sum_{v \in V(D)} d^+(v) f(v) \\
&= \sum_{v \in V_2} 2d^+(v) + \sum_{v \in V_1} d^+(v) - \sum_{v \in V_{-1}} d^+(v) \\
&\leq 2\Delta^+|V_2| + \Delta^+|V_1| - \delta^+|V_{-1}|.
\end{aligned}$$

This inequality chain yields to the desired bound in (i).

(ii) Proposition 8 (a) implies that $|V_{-1}| = n - |V_1| - |V_2|$. Using this identiy and part (i) of Proposition 9, we arrive at (ii).

(iii) According to Proposition 8 and part (ii) of Proposition 9, we obtain part (iii) of Proposition 9 as follows

$$\begin{aligned}
(\Delta^+ + \delta^+)\omega(f) &= (\Delta^+ + \delta^+)(2(|V_1| + |V_2|) - n + |V_2|) \\
&\geq 2(\Delta^+ + \delta^+)|V_2| + 2(\delta^+ + 1)n - 2(2\Delta^+ + \delta^+)|V_2| \\
&+ (\Delta^+ + \delta^+)(|V_2| - n) = (\delta^+ + 2 - \Delta^+)n + (\delta^+ - \Delta^+)|V_2|.
\end{aligned}$$

(iv) The inequality chain in the proof of part (i) and Proposition 8 (a) show that

$$\begin{aligned} n &\le 2\Delta^+|V_1 \cup V_2| - \delta^+|V_{-1}| = 2\Delta^+|V_1 \cup V_2| - \delta^+(n - |V_1 \cup V_2|) \\ &= (2\Delta^+ + \delta^+)|V_1 \cup V_2| - \delta^+ n \end{aligned}$$

and thus

$$|V_1 \cup V_2| \ge \frac{n(\delta^+ + 1)}{2\Delta^+ + \delta^+}.$$

Using this inequality and Proposition 8, we obtain

$$\begin{aligned} \omega(f) &= 2|V_1 \cup V_2| - n + |V_2| \ge \frac{2n(\delta^+ + 1)}{2\Delta^+ + \delta^+} - n + |V_2| \\ &= \frac{n(\delta^+ + 2 - 2\Delta^+)}{2\Delta^+ + \delta^+} + |V_2|. \end{aligned}$$

This is the bound in part (iv), and the proof is complete. ■

3. Bounds on the Signed Total Roman Domination Number

We start with a simple but sharp upper bound on the signed total Roman domination number of a digraph.

Proposition 10. *If D is a digraph of order n with minimum in-degree $\delta^- \ge 1$, then $\gamma_{stR}(D) \le n$.*

Proof. Define the function $f : V(D) \to \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(D)$. Since $\delta^- \ge 1$, the function f is an STRDF on D of weight n and thus $\gamma_{stR}(D) \le n$. ■

If $\delta^- \ge 3$, then we can improve the bound in Proposition 10.

Theorem 11. *If D is a digraph of order n with minimum in-degree $\delta^- \ge 3$, then*

$$\gamma_{stR}(D) \le n + 1 - 2\left\lceil \frac{\delta^- - 2}{2} \right\rceil.$$

Proof. Define $t = \left\lceil \frac{\delta^- - 2}{2} \right\rceil$. Since

$$n \cdot \Delta^+(D) \ge \sum_{x \in V(D)} d^+(x) = \sum_{x \in V(D)} d^-(x) \ge n \cdot \delta^-(D),$$

we observe that $\Delta^+(D) \ge \delta^- \ge t$. Let now $v \in V(D)$ be a vertex of maximum out-degree, and let $A = \{u_1, u_2, \ldots, u_t\}$ be a set of t out-neighbors of v. Define the function $f : V(D) \to \{-1, 1, 2\}$ by $f(v) = 2$, $f(u_i) = -1$ for $1 \le i \le t$ and $f(x) = 1$ for $x \in V(D) \setminus (A \cup \{v\})$. Then

$$f(N^-(w)) \ge -t + (\delta^- - t) = \delta^- - 2t = \delta^- - 2\left\lceil \frac{\delta^- - 2}{2} \right\rceil \ge 1$$

for each vertex $w \in V(D)$. Therefore f is an STRDF on D of weight $-t + 2 + (n - t - 1) = n + 1 - 2t$ and thus $\gamma_{stR}(D) \le n + 1 - 2t$. ■

Corollary 3 shows that Theorem 11 is sharp for even $n \ge 4$.

Corollary 12. *If D is a digraph of order n with minimum in-degree $\delta^- \ge 3$, then $\gamma_{stR}(D) \le n - 1$.*

Theorem 13 ([1])**.** *If D is a conneted digraph of order n with $\delta^-(D) \ge 1$, then $\gamma_t(D) = n$ if and only if D is an oriented cycle.*

Theorem 14. *Let D be a conneted digraph of order n with $\delta^-(D) \ge 1$. Then $\gamma_{stR}(D) \ge 2\gamma_t(D) - n$ with equality if and only if D is an oriented cycle.*

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stR}(D)$-function. If $V_2 = \emptyset$, then $V(D) = V_1$ and thus $\gamma_{stR}(D) = \omega(f) = n$. Since $\gamma_t(D) \le n$, we obtain $\gamma_{stR}(D) = n \ge 2\gamma_t(D) - n$. Applying Theorem 13, we see that $\gamma_{stR}(D) = 2\gamma_t(D) - n$ if and only if D is an oriented cycle in this case.

Now we assume that $|V_2| \ge 1$. Using Proposition 8, we deduce that

$$\begin{aligned} \gamma_{stR}(D) &= |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \\ &> 2|V_1 \cup V_2| - n \ge 2\gamma_t(D) - n. \end{aligned}$$

■

Corollary 15. *Let D be digraph of order n with $\delta^-(D) \ge 1$. Then $\gamma_{stR}(D) \ge 2\gamma_t(D) - n$ with equality if and only if the components of D are oriented cycles.*

A digraph D is *out-regular* or *r-out-regular* if $\delta^+(D) = \Delta^+(D) = r$. As an application of Proposition 9 (iii), we obtain a lower bound on the signed total Roman domination number for r-out-regular digraphs.

Corollary 16. *If D is an r-out-regular digraph of order n with $r \ge 1$, then $\gamma_{stR}(D) \ge n/r$.*

Using Corollary 16 and Observation 1, we obtain the next known result.

Corollary 17 ([8])**.** *If G is an r-regular graph of order n with $r \ge 1$, then $\gamma_{stR}(G) \ge n/r$.*

A digraph D is *r-regular* if $\Delta^+(D) = \Delta^-(D) = \delta^+(D) = \delta^-(D) = r$.

Example 18. If H is a 1-regular digraph of order n, then it follows from Corollary 16 that $\gamma_{stR}(H) \geq n$ and so $\gamma_{stR}(H) = n$, according to Proposition 10.

Example 18 demonstrates that Proposition 10 and Corollary 16 are both sharp. Proposition 4 implies that $\gamma_{stR}(K^*_{p,p}) = 2$ when $p \neq 3$. If $n \equiv 0 \pmod 4$, then it follows from Corollary 7 that $\gamma_{stR}(C^*_n) = n/2$. These are further examples which show that Corollary 16 is sharp.

If D is a 1-regular digraph of order n, then we have seen that $\gamma_{stR}(D) = n$. According to Corollary 7, we have $\gamma_{stR}(C^*_3) = 3$ and $\gamma_{stR}(C^*_6) = 6$. Thus Corollary 12 is not valid in general for $\delta^-(D) \leq 2$.

If D is not out-regular, then the next lower bound on the signed total Roman domination number is valid.

Corollary 19. *Let D be a digraph of order n, minimum in-degree $\delta^- \geq 1$, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then*

$$\gamma_{stR}(D) \geq \left(\frac{2\delta^+ + 3 - 2\Delta^+}{2\Delta^+ + \delta^+}\right) n.$$

Proof. Multiplying both sides of the inequality in Proposition 9 (iv) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 9 (iii), we obtain the desired lower bound. ■

Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Corollary 19 and Observation 1 lead to the next known corollary.

Corollary 20 ([8])**.** *Let G be a graph of order n, minimum degree $\delta \geq 1$ and maximum degree Δ. If $\delta < \Delta$, then*

$$\gamma_{stR}(G) \geq \left(\frac{2\delta + 3 - 2\Delta}{2\Delta + \delta}\right) n.$$

Example 11 in [8] demonstrate that Corollary 20 is sharp. This example together with Observation 1 show that Corollary 19 is sharp too.

Proposition 21. *If D is a digraph of order n with $\delta^-(D) \geq 1$, then*

$$\gamma_{stR}(D) \geq 1 + \Delta^-(D) - n.$$

Proof. Let $w \in V(D)$ be a vertex of maximum in-degree, and let f be a $\gamma_{stR}(D)$-function. Then the definitions imply

$$\begin{aligned}\gamma_{stR}(D) &= \sum_{x \in V(D)} f(x) = \sum_{x \in N^-(w)} f(x) + \sum_{x \in V(D) \setminus N^-(w)} f(x)\\ &\geq 1 + \sum_{x \in V(D) \setminus N^-(w)} f(x) \geq 1 - (n - \Delta^-(D)) = 1 + \Delta^-(D) - n,\end{aligned}$$

and the proof is complete. ■

Example 22. (1) Let $p \geq 2$ be an integer, and let $\{w\}$, $U = \{u_1, u_2, \ldots, u_{p-1}\}$ and $X = \{x_1, x_2, \ldots, x_{p+1}\}$ be the vertex set of the digraph H_{2p+1} such that w dominates $U \cup X$, u_1 dominates w, and $U \cup (X - \{x_1\})$ dominates x_1. Define the function $f : V(H_{2p+1}) \to \{-1, 1, 2\}$ by $f(w) = 2$, $f(u) = 1$ for $u \in U$ and $f(x) = -1$ for $x \in X$. Then f is an STRDF on H_{2p+1} of weight 0 and so $\gamma_{stR}(H_{2p+1}) \leq 0$. Since $n(H_{2p+1}) = 2p + 1$ and $\Delta^-(H_{2p+1}) = 2p$, Proposition 21 leads to

$$\gamma_{stR}(H_{2p+1}) \geq 1 + \Delta^-(H_{2p+1}) - n(H_{2p+1}) = 0$$

and thus $\gamma_{stR}(H_{2p+1}) = 0$.

(2) Let $p \geq 3$ be an integer, and let $U = \{u_1, u_2, \ldots, u_{p-2}\}$, $X = \{x_1, x_2, \ldots, x_{p+1}\}$ and $\{w\}$ be the vertex set of the digraph Q_{2p} such that w dominates $U \cup X$, u_1 dominates w and $U \cup (X - \{x_1\})$ dominates x_1. Define the function $g : V(Q_{2p}) \to \{-1, 1, 2\}$ by $g(w) = g(u_1) = 2$, $g(u) = 1$ for $u \in (U - \{u_1\})$ and $g(x) = -1$ for $x \in X$. Then g is an STRDF on Q_{2p} of weight 0 and so $\gamma_{stR}(Q_{2p}) \leq 0$. Since $n(Q_{2p}) = 2p$ and $\Delta^-(Q_{2p}) = 2p - 1$, Proposition 21 leads to

$$\gamma_{stR}(Q_{2p}) \geq 1 + \Delta^-(Q_{2p}) - n(Q_{2p}) = 0$$

and thus $\gamma_{stR}(Q_{2p}) = 0$.

The digraphs presented in Example 22 show that Proposition 21 is sharp for each $\Delta^- \geq 4$.

Proposition 23. *If D is a digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$, then*

$$\gamma_{stR}(D) \geq 4 + \delta^-(D) - n.$$

Proof. Let f be a $\gamma_{stR}(D)$-function. If $f(x) = 1$ for all $x \in V(D)$, then $\gamma_{stR}(D) = n \geq 4 + \delta^-(D) - n$. Now assume that there exists a vertex u with $f(u) = -1$. Then u has an in-neighbor w with $f(w) = 2$, and it follows that

$$\begin{aligned}\gamma_{stR}(D) &= \sum_{x \in V(D)} f(x) = f(w) + \sum_{x \in N^-(w)} f(x) + \sum_{x \in V(D) \setminus N^-[w]} f(x) \\ &\geq 2 + 1 + \sum_{x \in V(D) \setminus N^-[w]} f(x) \geq 3 - (n - d^-(w) - 1)) \geq 4 + \delta^-(D) - n\end{aligned}$$

and the proof of the desired lower bound is complete. ■

Corollary 3 shows that Proposition 23 is sharp.

Let F_n be the digraph of order $n \geq 3$ with the vertex set $\{u, w, x_1, x_2, \ldots, x_{n-2}\}$ such that w dominates $u, x_1, x_2, \ldots, x_{n-2}$ and u dominates w. Let $A = \{x_1, x_2, \ldots, x_{n-2}\}$. Now let $\mathcal{F}_n$ be the following family of digraphs. The digraph F_n belongs to $\mathcal{F}_n$. There is no arc from A to w. One arc from A to u is admissible.

If $A' \subseteq A$ is an abitrary subset, then it is admissible that u dominates A'. In addition, there are admissible arcs between vertices of A such that $d^-(y) \leq 4$ for $y \in A'$ and $d^-(y) \leq 2$ for $y \in A \setminus A'$.

Theorem 24. *Let D be a digraph of order $n \geq 3$ such that $\delta^-(D) \geq 1$. Then $\gamma_{stR}(D) \geq 5 - n$, with equality if and only if D is an element of the family $\mathcal{F}_n$.*

Proof. Since $\delta^-(D) \geq 1$, Proposition 23 implies $\gamma_{stR}(D) \geq 5 - n$ immediately.

Assume now that $\gamma_{stR}(D) = 5 - n$, and let f be a $\gamma_{stR}(D)$-function. This implies that D has exactly one vertex w with $f(w) = 2$, one vertex u with $f(u) = 1$ and $n - 2$ vertices $x_1, x_2, \ldots, x_{n-2}$ with $f(x_i) = -1$ for $1 \leq i \leq n - 2$. By the definition, w dominates x_i for $1 \leq i \leq n - 1$, w dominates u, and u dominates w. Let $A = \{x_1, x_2, \ldots, x_{n-2}\}$. If there is an arc, say (x_1, w), from A to w, then $f(N^-(w)) \leq 0$, a contradiction. Hence there is no arc from A to w. If there are at least two arcs from A to u, then we obtain the contradiction $f(N^-(u)) \leq 0$ and so there is at most one arc from A to u. Now assume that $A' \subseteq A$ such that u dominates A'. If there is a vertex $y \in A'$ with $d^-(y) \geq 5$, then $f(N^-(y) \leq 0$, a contradiction. Hence $d^-(y) \leq 4$ for $y \in A'$. If there is a vertex $y \in A \setminus A'$ with $d^-(y) \geq 3$, then $f(N^-(y) \leq 0$, a contradiction. Consequently, $d^-(y) \leq 2$ for $y \in A \setminus A'$. Altogether, we observe that D is a member of the family $\mathcal{F}_n$.

Conversely, if H is a member of the family $\mathcal{F}_n$, then define the function $g : V(H) \to \{-1, 1, 2\}$ by $f(w) = 2$, $f(u) = 1$ and $f(x_i) = -1$ for $1 \leq i \leq n - 2$. Then g is an STRDF on H of weight $5 - n$ and thus $\gamma_{stR}(H) = 5 - n$. ∎

Let $n = 2r + 1$ with an integer $r \geq 1$. We define the *circulant tournament* $CT(n)$ of order n with vertex set $\{u_0, u_1, \ldots, u_{n-1}\}$ as follows. For each $i \in \{0, 1, \ldots, n - 1\}$ the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.

Theorem 25. *Let $n = 2r + 1$ with an integer $r \geq 1$. Then $\gamma_{stR}(CT(3)) = 3$, $\gamma_{stR}(CT(7)) = 5$ and $\gamma_{stR}(CT(n)) = 4$ for $n \geq 5$ with $n \neq 7$.*

Proof. According to Example 18, $\gamma_{stR}(CT(3)) = 3$. Let now $r \geq 2$, and let f be a $\gamma_{stR}(CT(n))$-function. If $f(x) = 1$ for each $x \in V(CT(n))$, then $\omega(f) = n \geq 5$. If $f(x) = -1$ for a vertex x, then there exists a vertex, say u_r, such that $f(u_r) = 2$. Consider the sets $N^-(u_0) = \{u_{r+1}, u_{r+2}, \ldots, u_{2r}\}$ and $N^-(u_r) = \{u_0, u_1, \ldots, u_{r-1}\}$. As f is an STRDF on $CT(n)$, we deduce that

$$\omega(f) = f(N^-(u_0)) + f(N^-(u_r)) + f(u_r) \geq 1 + 1 + 2 = 4.$$

Consequently, $\gamma_{stR}(CT(n)) \geq 4$. In the special case $r = 3$, we observe that $f(x) = -1$ for at most two vertices and $f(y) = 2$ for at least two vertices. Therefore $\gamma_{stR}(CT(7)) \geq 5$. In addition, define the function $g : V(CT(7)) \to \{-1, 1, 2\}$ by $g(u_3) = g(u_6) = 2$, $g(u_1) = g(u_2) = g(u_5) = 1$ and $g(u_0) = g(u_4) = -1$.

Clearly, g is an STRDF on $CT(7)$ of weight 5 and thus $\gamma_{stR}(CT(7)) \le 5$ and so $\gamma_{stR}(CT(7)) = 5$. For the proof of $\gamma_{stR}(CT(n)) \le 4$ for $n \ge 5$ with $n \ne 7$, we distinguish two cases.

First let $r = 2p$ for an integer $p \ge 1$. If $p = 1$, then define the function $g : V(CT(5)) \to \{-1, 1, 2\}$ by $g(u_0) = g(u_1) = g(u_3) = 2$ and $g(u_2) = g(u_4) = -1$. Obviously, g is an STRDF on $CT(5)$ of weight 4 and thus $\gamma_{stR}(CT(5)) \le 4$. If $p \ge 2$, then define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ by $g(u_0) = g(u_1) = g(u_{2p+1}) = 2$, $g(u_2) = g(u_3) = \cdots, g(u_p) = 1$, $g(u_{p+1}) = g(u_{p+2}) = \cdots = g(u_{2p}) = -1$, $g(u_{2p+2}) = g(u_{2p+3}) = \cdots = g(u_{3p}) = 1$ and $g(u_{3p+1}) = g(u_{3p+2}) = \cdots = g(u_{4p}) = -1$. Then it is straightforward to verify that g is an STRDF on $CT(n)$ of weight 4 and thus $\gamma_{stR}(CT(n)) \le 4$.

Now let $r = 2p + 1$ for an integer $p \ge 2$.

If $p = 2$, then define the function $h : V(CT(11)) \to \{-1, 1, 2\}$ by $h(u_0) = h(u_1) = h(u_2) = h(u_6) = h(u_7) = 2$ and $h(x) = -1$ otherwise. Obviously, h is an STRDF on $CT(11)$ of weight 4 and thus $\gamma_{stR}(CT(11)) \le 4$. If $p \ge 3$, then define the function $h : V(CT(n)) \to \{-1, 1, 2\}$ by $h(u_0) = h(u_1) = h(u_2) = h(u_{2p+2}) = h(u_{2p+3}) = 2$, $h(u_3) = h(u_4) = \cdots = h(u_p) = 1$, $h(u_{p+1}) = h(u_{p+2}) = \cdots = h(u_{2p+1}) = -1$, $h(u_{2p+4}) = h(u_{2p+5}) = \cdots = h(u_{3p+1}) = 1$ and $h(u_{3p+2}) = h(u_{3p+3}) = \cdots = h(u_{4p+2}) = -1$. Then it is easy to see that h is an STRDF on $CT(n)$ of weight 4 and thus $\gamma_{stR}(CT(n)) \le 4$. ■

We call a set $S \subseteq V(D)$ a 2-*packing* of the digraph D if $N^-[u] \cap N^-[v] = \emptyset$ for any two distinct vertices of $u, v \in S$. The maximum cardinality of a 2-packing is the 2-*packing number* of D, denoted by $\rho(D)$.

Theorem 26. *If D is a digraph of order n with $\delta^-(D) \ge 1$, then*

$$\gamma_{stR}(D) \ge \rho(D)(\delta^-(D) + 1) - n.$$

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(D)}\}$ be a 2-packing of D, and let f be a $\gamma_{stR}(D)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(D)} N^-(v_i)$, then, since $\{v_1, v_2, \ldots, v_{\rho(D)}\}$ is a 2-packing of D, we have

$$|A| = \sum_{i=1}^{\rho(D)} d^-(v_i) \ge \rho(D) \cdot \delta^-(D).$$

It follows that

$$\begin{aligned} \gamma_{stR}(D) &= \sum_{x \in V(D)} f(x) = \sum_{i=1}^{\rho(D)} f(N^-(v_i)) + \sum_{x \in V(D) \setminus A} f(x) \\ &\ge \rho(D) + \sum_{x \in V(D) \setminus A} f(x) \ge \rho(D) - n + |A| \\ &\ge \rho(D) - n + \rho(D) \cdot \delta^-(D) = \rho(D)(\delta^-(D) + 1) - n. \end{aligned}$$ ■

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v the arc (u,v) belongs to $\overline{D}$ if and only if (u,v) does not belong to D. Finally, we present a so called Nordhaus-Gaddum type inequality for the signed total Roman domination number of regular digraphs.

Theorem 27. *If D is an r-regular digraph of order n such that $r \geq 1$ and $n-r-1 \geq 1$, then*

$$\gamma_{stR}(D) + \gamma_{stR}(\overline{D}) \geq \frac{4n}{n-1}.$$

If n is even, then $\gamma_{stR}(D) + \gamma_{stR}(\overline{D}) \geq 4(n-1)/(n-2)$.

Proof. Since D is r-regular, the complement $\overline{D}$ is $(n-r-1)$-regular. Therefore it follows from Corollary 16 that

$$\gamma_{stR}(D) + \gamma_{stR}(\overline{D}) \geq n\left(\frac{1}{r} + \frac{1}{n-r-1}\right).$$

The conditions $r \geq 1$ and $n-r-1 \geq 1$ imply that $1 \leq r \leq n-2$. As the function $g(x) = 1/x + 1/(n-x-1)$ has its minimum for $x = (n-1)/2$ when $1 \leq x \leq n-2$, we obtain

$$\gamma_{stR}(D) + \gamma_{stR}(\overline{D}) \geq n\left(\frac{1}{r} + \frac{1}{n-r-1}\right) \geq n\left(\frac{2}{n-1} + \frac{2}{n-1}\right) = \frac{4n}{n-1},$$

and this is the desired bound. If n is even, then the function g has its minimum for $r = x = (n-2)/2$ or $r = x = n/2$, since r is an integer. Hence this case leads to

$$\gamma_{stR}(D) + \gamma_{stR}(\overline{D}) \geq n\left(\frac{1}{r} + \frac{1}{n-r-1}\right) \geq n\left(\frac{2}{n} + \frac{2}{n-2}\right) = \frac{4(n-1)}{n-2},$$

and the proof is complete. ■

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