Discussiones Mathematicae
Graph Theory 37 (2017) 251–259
doi:10.7151/dmgt.1928

A CONSTRUCTIVE EXTENSION OF THE CHARACTERIZATION ON POTENTIALLY $K_{s,t}$-BIGRAPHIC PAIRS [1]

Ji-Yun Guo and Jian-Hua Yin [2]

Department of Mathematics
College of Information Science and Technology
Hainan University, Haikou 570228, P.R. China

e-mail: yinjh@hainu.edu.cn

Abstract

Let $K_{s,t}$ be the complete bipartite graph with partite sets of size s and t. Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $c_1 \geq c_2 \geq \cdots \geq c_n$. We say that $L = (L_1; L_2)$ is potentially $K_{s,t}$ (resp. $A_{s,t}$)-bigraphic if there is a simple bipartite graph G with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $a_i \leq d_G(x_i) \leq b_i$ for $1 \leq i \leq m$, $c_i \leq d_G(y_i) \leq d_i$ for $1 \leq i \leq n$ and G contains $K_{s,t}$ as a subgraph (resp. the induced subgraph of $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ in G is a $K_{s,t}$). In this paper, we give a characterization of L that is potentially $A_{s,t}$-bigraphic. As a corollary, we also obtain a characterization of L that is potentially $K_{s,t}$-bigraphic if $b_1 \geq b_2 \geq \cdots \geq b_m$ and $d_1 \geq d_2 \geq \cdots \geq d_n$. This is a constructive extension of the characterization on potentially $K_{s,t}$-bigraphic pairs due to Yin and Huang (Discrete Math. **312** (2012) 1241–1243).

Keywords: degree sequence, bigraphic pair, potentially $K_{s,t}$-bigraphic pair.

2010 Mathematics Subject Classification: 05C07.

1. Introduction

Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be two nonincreasing sequences of nonnegative integers. The pair $S = (A; B)$ is said to be *bigraphic* if there exists a

[1]Supported by National Natural Science Foundation of China (No. 11561017) and Natural Science Foundation of Hainan Province (Nos. 20151004 and 2016CXTD004).

[2]Corresponding author.

simple bipartite graph G with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $d_G(x_i) = a_i$ for $1 \le i \le m$ and $d_G(y_i) = b_i$ for $1 \le i \le n$. In this case, G is referred to as a *realization* of S. The following well-known theorem due to Gale [2] and Ryser [4] independently gave a characterization of S that is bigraphic.

Theorem 1 [2, 4]**.** *$S = (A; B)$ is bigraphic if and only if $\sum_{i=1}^m a_i = \sum_{i=1}^n b_i$ and*

$$\sum_{i=1}^{k} a_i \le \sum_{i=1}^{n} \min\{k, b_i\} \text{ for all } k \text{ with } 1 \le k \le m.$$

The pair $S = (A; B)$ is said to be *potentially $K_{s,t}$-bigraphic* if there is a realization of S containing $K_{s,t}$ as a subgraph. Yin and Huang [6] presented a characterization of S that is potentially $K_{s,t}$-bigraphic.

Theorem 2 [6]**.** *$S = (A; B)$ is potentially $K_{s,t}$-bigraphic if and only if $a_s \ge t$, $b_t \ge s$, $\sum_{i=1}^m a_i = \sum_{i=1}^n b_i$ and*

$$\sum_{i=1}^{p} a_i + \sum_{i=s+1}^{s+q} a_i \le \sum_{i=1}^{t} \min\{p+q, b_i - s + p\} + \sum_{i=t+1}^{n} \min\{p+q, b_i\}$$

for all p and q with $0 \le p \le s$ and $0 \le q \le m - s$.

Let $L_1 = ([a_1, b_1], \ldots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \ldots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \ge a_2 \ge \cdots \ge a_m$ and $c_1 \ge c_2 \ge \cdots \ge c_n$. We say that $L = (L_1; L_2)$ is *bigraphic* if there exists a simple bipartite graph G with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $a_i \le d_G(x_i) \le b_i$ for $1 \le i \le m$ and $c_i \le d_G(y_i) \le d_i$ for $1 \le i \le n$. In this case, G is referred to as a *realization* of L. Garg *et al.* [3] obtained a characterization of L that is bigraphic.

Theorem 3 [3]**.** *$L = (L_1; L_2)$ is bigraphic if and only if*

$$\sum_{i=1}^{k} a_i \le \sum_{j=1}^{n} \min\{k, d_j\} \text{ for all } k \text{ with } 1 \le k \le m$$

and

$$\sum_{i=1}^{k} c_i \le \sum_{j=1}^{m} \min\{k, b_j\} \text{ for all } k \text{ with } 1 \le k \le n.$$

Theorem 3 reduces to Theorem 1 when $a_i = b_i$ for $1 \le i \le m$ and $c_i = d_i$ for $1 \le i \le n$. We say that $L = (L_1; L_2)$ is *potentially $K_{s,t}$ (resp. $A_{s,t}$)-bigraphic* if there is a simple bipartite graph G with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $a_i \le d_G(x_i) \le b_i$ for $1 \le i \le m$, $c_i \le d_G(y_i) \le d_i$

for $1 \le i \le n$ and G contains $K_{s,t}$ as a subgraph (resp. the induced subgraph of $\{x_1, \dots, x_s, y_1, \dots, y_t\}$ in G is a $K_{s,t}$).

The purpose of this paper is to investigate a characterization of L that is potentially $K_{s,t}$-bigraphic. We first give a characterization of L that is potentially $A_{s,t}$-bigraphic as follows.

Theorem 4. *Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \ge a_2 \ge \cdots \ge a_m$ and $c_1 \ge c_2 \ge \cdots \ge c_n$. If $a_s \ge t$ and $c_t \ge s$, then $L = (L_1; L_2)$ is potentially $A_{s,t}$-bigraphic if and only if*

$$(1) \qquad \sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i \le \sum_{i=1}^{t} \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_i\}$$

for all p_1 and q_1 with $0 \le p_1 \le s$ and $0 \le q_1 \le m - s$ and

$$(2) \qquad \sum_{i=1}^{p_2} c_i + \sum_{i=t+1}^{t+q_2} c_i \le \sum_{i=1}^{s} \min\{p_2 + q_2, b_i - t + p_2\} + \sum_{i=s+1}^{m} \min\{p_2 + q_2, b_i\}$$

for all p_2 and q_2 with $0 \le p_2 \le t$ and $0 \le q_2 \le n - t$.

If $s = t = 0$, then $p_1 = p_2 = 0$ and Theorem 4 reduces to Theorem 3. If we further assume that $b_1 \ge b_2 \ge \cdots \ge b_m$ and $d_1 \ge d_2 \ge \cdots \ge d_n$, then we can prove the following theorem.

Theorem 5. *Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \ge a_2 \ge \cdots \ge a_m$ and $c_1 \ge c_2 \ge \cdots \ge c_n$. If $b_1 \ge b_2 \ge \cdots \ge b_m$ and $d_1 \ge d_2 \ge \cdots \ge d_n$, then $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic if and only if it is potentially $A_{s,t}$-bigraphic.*

Combining Theorem 4 with Theorem 5, we have the following corollary.

Corollary 6. *Let $L_1 = ([a_1, b_1], \dots, [a_m, b_m])$ and $L_2 = ([c_1, d_1], \dots, [c_n, d_n])$ be two sequences of intervals consisting of nonnegative integers with $a_1 \ge a_2 \ge \cdots \ge a_m$ and $c_1 \ge c_2 \ge \cdots \ge c_n$. If $a_s \ge t$, $c_t \ge s$, $b_1 \ge b_2 \ge \cdots \ge b_m$ and $d_1 \ge d_2 \ge \cdots \ge d_n$, then $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic if and only if* (1) *holds for all p_1 and q_1 with $0 \le p_1 \le s$ and $0 \le q_1 \le m - s$ and* (2) *holds for all p_2 and q_2 with $0 \le p_2 \le t$ and $0 \le q_2 \le n - t$.*

Corollary 6 reduces to Theorem 2 when $a_i = b_i$ for $1 \le i \le m$ and $c_i = d_i$ for $1 \le i \le n$.

2. Proofs of Theorems 4 and 5

The proof technique of Theorem 4 was developed earlier by Tripathi, Venugopalan and West [5].

***Proof of Theorem* 4.** For the necessity, we suppose that G is a realization of $L = (L_1; L_2)$ with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \le d_G(x_i) \le b_i$ for $1 \le i \le m$, $c_i \le d_G(y_i) \le d_i$ for $1 \le i \le n$ and the induced subgraph of $\{x_1, \dots, x_s, y_1, \dots, y_t\}$ in G is a $K_{s,t}$. For p_1 and q_1 with $0 \le p_1 \le s$ and $0 \le q_1 \le m - s$, it is easy to see that $\sum_{i=1}^{t} \min\{p_1 + q_1, d_G(y_i) - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_G(y_i)\}$ is the maximum contribution to $\sum_{i=1}^{p_1} d_G(x_i) + \sum_{i=s+1}^{s+q_1} d_G(x_i)$ from edges incident to $y_1, \dots, y_n$. Thus,

$$\begin{aligned}\sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i &\le \sum_{i=1}^{p_1} d_G(x_i) + \sum_{i=s+1}^{s+q_1} d_G(x_i)\\ &\le \sum_{i=1}^{t} \min\{p_1 + q_1, d_G(y_i) - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_G(y_i)\}\\ &\le \sum_{i=1}^{t} \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_i\},\end{aligned}$$

that is, (1) holds for p_1 and q_1. Similarly, we can prove that (2) holds for p_2 and q_2 with $0 \le p_2 \le t$ and $0 \le q_2 \le n - t$.

For the sufficiency, we assume that (1) holds for p_1 and q_1 with $0 \le p_1 \le s$ and $0 \le q_1 \le m - s$ and (2) holds for p_2 and q_2 with $0 \le p_2 \le t$ and $0 \le q_2 \le n - t$. A *subrealization* of $L = (L_1; L_2)$ is a bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $d_G(x_i) \le b_i$ for $1 \le i \le m$ and $d_G(y_i) \le d_i$ for $1 \le i \le n$. If $a_i \le d_G(x_i) \le b_i$ for $1 \le i \le m$ and $c_i \le d_G(y_i) \le d_i$ for $1 \le i \le n$, then G is a realization of L. We will construct a realization of L through successive subrealizations. The initial subrealization is $K_{s,t} \cup \overline{K}_{m-s} \cup \overline{K}_{n-t}$, where $\overline{K}_r$ is the complement of K_r, $K_{s,t}$ has partite sets $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$, $V(\overline{K}_{m-s}) = \{x_{s+1}, \dots, x_m\}$ and $V(\overline{K}_{n-t}) = \{y_{t+1}, \dots, y_n\}$.

In each successive subrealization, let p_1 be the largest index such that $d(x_i) = a_i$ for $1 \le i < p_1$ and $d(x_{p_1}) < a_{p_1}$ and q_1 be the largest index such that $d(x_i) = a_i$ for $s + 1 \le i < s + q_1$ and $d(x_{s+q_1}) < a_{s+q_1}$. While $p_1 \le s$ or $q_1 \le m - s$, we can obtain a new subrealization containing the initial subrealization and having smaller deficiency $(a_{p_1} - d(x_{p_1})) + (a_{s+q_1} - d(x_{s+q_1}))$ at x_{p_1} and x_{s+q_1} while not changing the degree of any vertex x_i with $i \in \{1, \dots, p_1 - 1, s + 1, \dots, s + q_1 - 1\}$.

Let $X_1 = \{x_{p_1+1}, \dots, x_s\}$ and $X_2 = \{x_{s+q_1+1}, \dots, x_m\}$. We maintain the condition that $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, there is no edge between $\{y_1, \dots, y_t\}$ and X_2 and there is no edge between $\{y_{t+1}, \dots, y_n\}$ and $X_1 \cup X_2$, which certainly hold initially. For convenience, we write $v_i \leftrightarrow v_j$ for "v_i is adjacent to v_j" and $v_i \not\leftrightarrow v_j$ for "v_i is not adjacent to v_j".

Case 0. Suppose $x_{p_1} \not\leftrightarrow y_k$ for some $k > t$ such that $d(y_k) < d_k$. Add $x_{p_1}y_k$.

Case 1. Suppose $x_{s+q_1} \not\leftrightarrow y_k$ for some k such that $d(y_k) < d_k$. Add $x_{s+q_1}y_k$.

Case 2. Suppose $d(y_k) \neq \min\{p_1 + q_1, d_k\}$ for some k with $k \geq t + 1$. In a subrealization, $d(y_k) \leq d_k$. Since there is no edge between $\{y_{t+1}, \ldots, y_n\}$ and $X_1 \cup X_2$, $d(y_k) \leq p_1 + q_1$. Hence, $d(y_k) < \min\{p_1 + q_1, d_k\}$. Case 0 and Case 1 apply, unless $x_{p_1} \leftrightarrow y_k$ and $x_{s+q_1} \leftrightarrow y_k$. Since $d(y_k) < p_1 + q_1$, there exists $i \in \{1, \ldots, p_1 - 1, s + 1, \ldots, s + q_1 - 1\}$ such that $x_i \not\leftrightarrow y_k$. If $i \in \{1, \ldots, p_1 - 1\}$, by $p_1 \leq s$ and $d(x_i) = a_i \geq a_{p_1} > d(x_{p_1})$, there exists $u \in N(x_i) \setminus N(x_{p_1})$, then replace ux_i by $\{x_iy_k, ux_{p_1}\}$. If $i \in \{s + 1, \ldots, s + q_1 - 1\}$, by $d(x_i) > d_{s+q_1}$, there exists $u \in N(x_i) \setminus N(x_{s+q_1})$, then replace ux_i by $\{x_iy_k, ux_{s+q_1}\}$.

Case 3. Suppose $d(y_k)-s+p_1 \neq \min\{p_1+q_1, d_k-s+p_1\}$ for some k with $k \leq t$. In a subrealization, $d(y_k) - s + p_1 \leq d_k - s + p_1$. Since there is no edge between $\{y_1, \ldots, y_t\}$ and X_2, $d(y_k) - s + p_1 \leq p_1 + q_1$. Hence $d(y_k) - s + p_1 < \min\{p_1 + q_1, d_k - s + p_1\}$. Case 1 applies unless $x_{s+q_1} \leftrightarrow y_k$. Since $d(y_k) - s + p_1 < p_1 + q_1$ and $x_i \leftrightarrow y_k$ for $1 \leq i \leq p_1$, there exists $i \in \{s + 1, \ldots, s + q_1 - 1\}$ such that $x_i \not\leftrightarrow y_k$. By $d(x_i) > d(x_{s+q_1})$, there exists $u \in N(x_i) \setminus N(x_{s+q_1})$, then replace ux_i by $\{x_iy_k, ux_{s+q_1}\}$.

If none of Cases 0–3 applies, then $d(y_k) = \min\{p_1 + q_1, d_k\}$ for $k \geq t + 1$ and $d(y_k) - s + p_1 = \min\{p_1 + q_1, d_k - s + p_1\}$ for $k \leq t$. Since $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$ form a $K_{s,t}$, there is no edge between $\{y_1, \ldots, y_t\}$ and X_2 and there is no edge between $\{y_{t+1}, \ldots, y_n\}$ and $X_1 \cup X_2$, we have that

$$\sum_{i=1}^{p_1} d(x_i) + \sum_{i=1}^{q_1} d(x_{s+i}) = \sum_{i=1}^{t} \min\{p_1 + q_1, d_i - s + p_1\} + \sum_{i=t+1}^{n} \min\{p_1 + q_1, d_i\}.$$

By (1) and the observation that $d(x_i) = a_i$ for $1 \leq i \leq p_1 - 1$ and $d(x_{s+i}) = a_{s+i}$ for $1 \leq i \leq q_1 - 1$, we get that $\sum_{i=1}^{p_1} a_i + \sum_{i=s+1}^{s+q_1} a_i = \sum_{i=1}^{p_1} d(x_i) + \sum_{i=1}^{q_1} d(x_{s+i})$, which implies that $d(x_{p_1}) = a_{p_1}$ and $d(x_{s+q_1}) = a_{s+q_1}$. Now we have shown that while $p_1 \leq s$ or $q_1 \leq m - s$, we obtain a new subrealization containing the initial subrealization and having $d(x_{p_1}) = a_{p_1}$ and $d(x_{s+q_1}) = a_{s+q_1}$ while not changing the degree of any vertex x_i with $i \in \{1, \ldots, p_1 - 1, s + 1, \ldots, s + q_1 - 1\}$. Increase p_1 by 1 and q_1 by 1, and repeat the process from Case 0 to Case 3. Thus when $p_1 = s$ and $q_1 = m - s$, a subrealization G' containing the initial subrealization can be obtained so that $d(x_i) = a_i$ for $1 \leq i \leq m$ and $d(y_i) \leq d_i$ for $1 \leq i \leq n$.

We now regard G' as a new initial subrealization. In the following, for each successive subrealization, we define p_2 to be the largest index such that $d(y_i) \geq c_i$ for $1 \leq i < p_2$ and $d(y_{p_2}) < c_{p_2}$, and q_2 to be the largest index such that $d(y_i) \geq c_i$ for $t+1 \leq i < t+q_2$ and $d(y_{t+q_2}) < c_{t+q_2}$. While $p_2 \leq t$ or $q_2 \leq n-t$, we can obtain a new subrealization having smaller deficiency $(c_{p_2}-d(y_{p_2}))+(c_{t+q_2}-d(y_{t+q_2}))$ at y_{p_2} and y_{t+q_2} while maintaining the conditions that $\{x_1,\ldots,x_s\}$ and $\{y_1,\ldots,y_t\}$ form a $K_{s,t}$, $d(y_i)\ge c_i$ for $i\in\{1,\ldots,p_2-1,t+1,\ldots,t+q_2-1\}$ and $a_i\le d(x_i)\le b_i$ for $1\le i\le m$. The process can only stop when the subrealization is a realization of L.

Case 4. Suppose, for some $j>s$, $x_j\leftrightarrow y_k$ for some $p_2+1\le k\le t$ and $x_j\not\leftrightarrow y_\ell$ for some $\ell\le p_2$. If $\ell=p_2$, then replace y_kx_j by $y_{p_2}x_j$. If $\ell<p_2$, then replace $\{y_kx_j,y_\ell v\}$ by $\{y_\ell x_j,y_{p_2}v\}$, where $v\in N(y_\ell)\setminus N(y_{p_2})$.

Case 5. Suppose, for some $j\in\{1,\ldots,m\}$, $x_j\leftrightarrow y_k$ for some $k>t+q_2$ and $x_j\not\leftrightarrow y_\ell$ for some $1+t\le\ell\le t+q_2$. If $\ell=t+q_2$, then replace x_jy_k by $x_jy_{t+q_2}$. If $t+1\le\ell<t+q_2$, then replace $\{x_jy_k,y_\ell v\}$ by $\{vy_{t+q_2},y_\ell x_j\}$, where $v\in N(y_\ell)\setminus N(y_{t+q_2})$.

Case 6. Suppose $d(x_j)<b_j$ for some $j>s$ and $x_j\not\leftrightarrow y_\ell$ for some $\ell\le p_2$. If $\ell=p_2$, then add $x_jy_{p_2}$. If $\ell<p_2$, then replace vy_ℓ by $\{vy_{p_2},y_\ell x_j\}$, where $v\in N(y_\ell)\setminus N(y_{p_2})$.

Case 7. Suppose $d(x_j)<b_j$ for some $j\in\{1,\ldots,m\}$ and $x_j\not\leftrightarrow y_\ell$ for some $t+1\le\ell\le t+q_2$. If $\ell=t+q_2$, then add $x_jy_{t+q_2}$. If $t+1\le\ell<t+q_2$, then replace vy_ℓ by $\{vy_{t+q_2},y_\ell x_j\}$, where $v\in N(y_\ell)\setminus N(y_{t+q_2})$.

Case 8. Suppose, for some $j>s$, $x_j\leftrightarrow y_k$ for some $p_2+1\le k\le t$ and $x_j\not\leftrightarrow y_\ell$ for some $t+1\le\ell\le t+q_2$. If $\ell=t+q_2$, then replace x_jy_k by $x_jy_{t+q_2}$. If $t+1\le\ell<t+q_2$, then replace $\{x_jy_k,y_\ell v\}$ by $\{vy_{t+q_2},y_\ell x_j\}$, where $v\in N(y_\ell)\setminus N(y_{t+q_2})$.

Case 9. Suppose, for some $j>s$, $x_j\leftrightarrow y_k$ for some $k>t+q_2$ and $x_j\not\leftrightarrow y_\ell$ for some $\ell\le p_2$. If $\ell=p_2$, then replace x_jy_k by $x_jy_{p_2}$. If $\ell<p_2$, then replace $\{x_jy_k,vy_\ell\}$ by $\{vy_{p_2},x_jy_\ell\}$, where $v\in N(y_\ell)\setminus N(y_{p_2})$.

Case 10. Suppose $d(y_i)>c_i$ for some $i\in\{1,\ldots,p_2-1,t+1,\ldots,t+q_2-1\}$. If $i\in\{1,\ldots,p_2-1\}$, then replace vy_i by vy_{p_2}, where $v\in N(y_i)\setminus N(y_{p_2})$. If $i\in\{t+1,\ldots,t+q_2-1\}$, then replace vy_i by vy_{t+q_2}, where $v\in N(y_i)\setminus N(y_{t+q_2})$.

If none of Cases 4–9 applies, we can prove the following claim.

Claim. Assume that none of Cases 4–9 applies. Then

(i) For each $x_j\in\{x_1,\ldots,x_s\}$, $\min\{p_2+q_2,d(x_j)-t+p_2\}=\min\{p_2+q_2,b_j-t+p_2\}$ and $\min\{p_2+q_2,b_j-t+p_2\}$ is the maximum contribution to $\sum_{i=1}^{p_2}d(y_i)+\sum_{i=1}^{q_2}d(y_{t+i})$ from edges incident to x_j.

(ii) For each $x_j\in\{x_{s+1},\ldots,x_m\}$, $\min\{p_2+q_2,d(x_j)\}=\min\{p_2+q_2,b_j\}$ and $\min\{p_2+q_2,b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2}d(y_i)+\sum_{i=1}^{q_2}d(y_{t+i})$ from edges incident to x_j.

Proof. If $x_j\in\{x_1,\ldots,x_s\}$, we consider the following two cases de- pending on whether x_j is adjacent to all the vertices in $\{y_{t+1},\ldots,y_{t+q_2}\}$ or not.

Suppose $x_j \leftrightarrow y_k$ for all $t+1 \leq k \leq t+q_2$. Since $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$ form a $K_{s,t}$, x_j is adjacent to every vertex in $\{y_1, \ldots, y_t\}$. Thus, $p_2 + q_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j. By $b_j - t + p_2 \geq d(x_j) - t + p_2 \geq p_2 + q_2$, we have that $\min\{p_2+q_2, d(x_j)-t+p_2\} = \min\{p_2 + q_2, b_j - t + p_2\} = p_2 + q_2$ and $\min\{p_2 + q_2, b_j - t + p_2\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j.

Suppose $x_j \not\leftrightarrow y_k$ for some $t + 1 \leq k \leq t + q_2$. Since Case 5 and Case 7 cannot apply, we have that $x_j \not\leftrightarrow y_\ell$ for all $\ell > t + q_2$ and $d(x_j) = b_j$. This implies that $b_j - t + p_2 = d(x_j) - t + p_2 < p_2 + q_2$, $\min\{p_2 + q_2, d(x_j) - t + p_2\} = \min\{p_2 + q_2, b_j - t + p_2\} = b_j - t + p_2$ and $b_j - t + p_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j.

If $x_j \in \{x_{s+1}, \ldots, x_m\}$, we consider the following two cases depending on whether x_j is adjacent to all the vertices in $\{y_1, \ldots, y_{p_2}\}$ or not.

Suppose $x_j \leftrightarrow y_k$ for all $k \leq p_2$. If $x_j \leftrightarrow y_\ell$ for all $t + 1 \leq \ell \leq t + q_2$, then $p_2 + q_2$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j. By $b_j \geq d(x_j) \geq p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = p_2 + q_2$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j. Assume that $x_j \not\leftrightarrow y_\ell$ for some $t+1 \leq \ell \leq t+q_2$. Since Case 5, Case 7 and Case 8 cannot apply, we have that $x_j \not\leftrightarrow y_k$ for all $k > t+q_2$, $d(x_j) = b_j$ and $x_j \not\leftrightarrow y_k$ for all $p_2+1 \leq k \leq t$. Thus, b_j is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j. By $b_j = d(x_j) < p_2 + q_2$, we have that $\min\{p_2 + q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = b_j$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j.

Suppose $x_j \not\leftrightarrow y_k$ for some $k \leq p_2$. Since Case 4, Case 6 and Case 9 cannot apply, we have that $x_j \not\leftrightarrow y_\ell$ for all $p_2 + 1 \leq \ell \leq t$, $x_j \not\leftrightarrow y_\ell$ for all $\ell > t + q_2$ and $d(x_j) = b_j$. By $b_j = d(x_j) < p_2+q_2$, we have that $\min\{p_2+q_2, d(x_j)\} = \min\{p_2 + q_2, b_j\} = b_j$ and $\min\{p_2 + q_2, b_j\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to x_j. The claim is proved. $\square$

We now continue to proceed with the proof of theorem. By the previous claim, we have that $\sum_{i=1}^{s} \min\{p_2+q_2, d(x_i)-t+p_2\} + \sum_{i=s+1}^{m} \min\{p_2+q_2, d(x_i)\} = \sum_{i=1}^{s} \min\{p_2+q_2, b_i-t+p_2\} + \sum_{i=s+1}^{m} \min\{p_2+q_2, b_i\}$, and $\sum_{i=1}^{s} \min\{p_2+q_2, b_i - t + p_2\} + \sum_{i=s+1}^{m} \min\{p_2 + q_2, b_i\}$ is the maximum contribution to $\sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i})$ from edges incident to $x_1, \ldots, x_m$. If Case 10 cannot apply, then $d(y_i) = c_i$ for all $i \in \{1, \ldots, p_2 - 1, t + 1, \ldots, t + q_2 - 1\}$. Thus, we obtain that

$$\begin{aligned}\sum_{i=1}^{p_2-1} c_i + \sum_{i=1}^{q_2-1} c_{t+i} + d(y_{p_2}) + d(y_{t+q_2}) &= \sum_{i=1}^{p_2} d(y_i) + \sum_{i=1}^{q_2} d(y_{t+i}) \\ = \sum_{j=1}^{s} \min\{p_2 + q_2, b_j - t + p_2\} + \sum_{j=s+1}^{m} \min\{p_2 + q_2, b_j\}.\end{aligned}$$

By (2), we further have that $c_{p_2} + c_{t+q_2} \le d(y_{p_2}) + d(y_{t+q_2})$, which implies that $d(y_{p_2}) = c_{p_2}$ and $d(y_{t+q_2}) = c_{t+q_2}$. Now we have shown that while $p_2 \le t$ or $q_2 \le n-t$, we obtain a new subrealization having $d(y_{p_2}) = c_{p_2}$ and $d(y_{t+q_2}) = c_{t+q_2}$ while maintaining the conditions that $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ form a $K_{s,t}$, $d(y_i) \ge c_i$ for $i \in \{1, \dots, p_2 - 1, t+1, \dots, t+q_2-1\}$ and $a_i \le d(x_i) \le b_i$ for $1 \le i \le m$. Increase p_2 by 1 and q_2 by 1, and repeat the process from Case 4 to Case 10. When $p_2 = t$ and $q_2 = n-t$, we finally get a realization of L containing $K_{s,t}$ and satisfying $a_i \le d(x_i) \le b_i$ for $1 \le i \le m$ and $c_i \le d(y_i) \le d_i$ for $1 \le i \le n$, where $V(K_{s,t}) = \{x_1, \dots, x_s, y_1, \dots, y_t\}$. In other words, L is potentially $A_{s,t}$-bigraphic. The proof of Theorem 4 is completed. ∎

This constructive proof can be implemented as an algorithm to construct a realization of L containing $K_{s,t}$. The following lemma due to Ferrara *et al.* [1] will be useful as we proceed with the proof of Theorem 5.

Lemma 7 [1]**.** *Let S be a bigraphic pair with realization $G = (X \cup Y, E)$ having partite sets X and Y. Let $H = (X' \cup Y', E')$ be a subgraph of G such that X' and Y' are contained in X and Y, respectively. Then there exists a realization $G_1 = (X \cup Y, E_1)$ of S containing H as a subgraph such that X' and Y' lie on the vertices of highest degree in X and Y, respectively.*

***Proof of Theorem* 5.** We only need to show that if $L = (L_1; L_2)$ is potentially $K_{s,t}$-bigraphic, then it is potentially $A_{s,t}$-bigraphic. Let G be a simple bipartite graph with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $a_i \le d_G(x_i) \le b_i$ for $1 \le i \le m$, $c_i \le d_G(y_i) \le d_i$ for $1 \le i \le n$ and G contains $K_{s,t} = (X' \cup Y', E')$ as a subgraph. Denote $d_{1i} = d_G(x_i)$ for $1 \le i \le m$ and $d_{2i} = d_G(y_i)$ for $1 \le i \le n$. Let $A = (d_{11}, \dots, d_{1m})$ and $B = (d_{21}, \dots, d_{2n})$. By Lemma 7, $(A; B)$ has a realization $G_1 = (X \cup Y, E_1)$ satisfying $d_{G_1}(x_i) = d_{1i}$ for $1 \le i \le m$, $d_{G_1}(y_i) = d_{2i}$ for $1 \le i \le n$ and G_1 contains $K_{s,t}$ so that X' and Y' lie on the vertices of highest degree in X and Y, respectively. Let $D = \{x_1, \dots, x_s\} \setminus X'$, $D' = \{x_{s+1}, \dots, x_m\} \cap X'$, $C = \{y_1, \dots, y_t\} \setminus Y'$ and $C' = \{y_{t+1}, \dots, y_n\} \cap Y'$. Then, it is easy to see that

$$\begin{aligned}
&\max\{a_i | x_i \in D\} \le \max\{d_{1i} | x_i \in D\} \le d_{1j} \le \min\{b_i | x_i \in D\} \text{ for each } x_j \in D',\\
&\max\{a_i | x_i \in D'\} \le d_{1j} \le \min\{d_{1i} | x_i \in D'\} \le \min\{b_i | x_i \in D'\} \text{ for each } x_j \in D,\\
&\max\{c_i | y_i \in C\} \le \max\{d_{2i} | y_i \in C\} \le d_{2j} \le \min\{d_i | y_i \in C\} \text{ for each } y_j \in C',\\
&\max\{c_i | y_i \in C'\} \le d_{2j} \le \min\{d_{2i} | y_i \in C'\} \le \min\{d_i | y_i \in C'\} \text{ for each } y_j \in C.
\end{aligned}$$

Thus, we can see that $(L_1; L_2)$ is potentially $A_{s,t}$-bigraphic by exchanging D with D' and C with C'. ∎

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable comments and suggestions.

References

[1] M.J. Ferrara, M.S. Jacobson, J.R. Schmitt and M. Siggers, *Potentially H-bigraphic sequences*, Discuss. Math. Graph Theory **29** (2009) 583–596.
doi:10.7151/dmgt.1466

[2] D. Gale, *A theorem on flows in networks*, Pacific J. Math. **7** (1957) 1073–1082.
doi:10.2140/pjm.1957.7.1073

[3] A. Garg, A. Goel and A. Tripathi, *Constructive extensions of two results on graphic sequences*, Discrete Appl. Math. **159** (2011) 2170–2174.
doi:10.1016/j.dam.2011.06.017

[4] H.J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. **9** (1957) 371–377.
doi:10.4153/CJM-1957-044-3

[5] A. Tripathi, S. Venugopalan and D.B. West, *A short constructive proof of the Erdös-Gallai characterization of graphic lists*, Discrete Math. **310** (2010) 843–844.
doi:10.1016/j.disc.2009.09.023

[6] J.H. Yin and X.F. Huang, *A Gale-Ryser type characterization of potentially $K_{s,t}$-bigraphic pairs*, Discrete Math. **312** (2012) 1241–1243.
doi:10.1016/j.disc.2011.12.016

Received 6 July 2015
Revised 13 April 2016
Accepted 13 April 2016