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STRUCTURAL PROPERTIES OF RECURSIVELY PARTITIONABLE GRAPHS WITH CONNECTIVITY 2

Olivier Baudon

Université Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France
CNRS, LaBRI, UMR 5800, F-33400 Talence, France

e-mail: olivier.baudon@labri.fr

Julien Bensmail

Department of Applied Mathematics and Computer Science
Technical University of Denmark
DK-2800 Lyngby, Denmark

e-mail: julien.bensmail.phd@gmail.com

Florent Foucaud

LIMOS – CNRS UMR 6158
Université Blaise Pascal
Clermont-Ferrand, France

e-mail: florent.foucaud@gmail.com

and

Monika Pilśniak

Department of Discrete Mathematics
AGH University of Science and Technology
A. Mickiewicza 30, 30-059 Kraków, Poland

e-mail: pilsniak@agh.edu.pl

Abstract

A connected graph G is said to be *arbitrarily partitionable* (AP for short) if for every partition $(n_1, \ldots, n_p)$ of $|V(G)|$ there exists a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G on n_i vertices. Some stronger versions of this property were introduced, namely the ones of being *online arbitrarily partitionable* and *recursively arbitrarily partitionable* (OL-AP and R-AP for short, respectively), in which the subgraphs induced by a partition of G must not only be connected but also fulfil additional conditions. In this paper, we point out some structural properties of OL-AP and R-AP graphs with connectivity 2. In particular, we show that deleting a cut pair of these graphs results in a graph with a bounded number of components, some of whom have a small number of vertices. We obtain these results by studying a simple class of 2-connected graphs called *balloons*.

Keywords: online arbitrarily partitionable graph, recursively arbitrarily partitionable graph, graph with connectivity 2, balloon graph.

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1. Introduction

Let G be a graph of order n. A sequence $\tau = (n_1, \ldots, n_p)$ of positive integers is said to be *admissible for* G if it performs a partition of n, that is if $\sum_{i=1}^{p} n_i = n$. When, for such an admissible sequence for G, there exists a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G on n_i vertices, then τ is called *realizable in* G. If every admissible sequence for G is also realizable in G, then G is said to be *arbitrarily partitionable* (AP for short).

The notion of AP graphs was introduced in [1] (and, independently, in [11]) to deal with the following problem. Suppose that we want to share a network of n computing resources among p users, where the ith user needs n_i resources, and that, for the sake of performance, we do not want the sharing to be performed arbitrarily but in such a way that the following two conditions are met:

- a resource must be allocated to exactly one user,
- the subnetwork attributed to a user must be connected[1].

We can use the previously introduced notions to deduce an optimal sharing of the resources. Indeed, let G be the graph modelling the network; then we can satisfy the users with this specific resource demand if the sequence $(n_1, \ldots, n_p)$ is realizable in G. Moreover, observe that, regarding this allocation problem, the networks which are of most interest are those which can be shared among an arbitrary number of users no matter how many resources they need. Clearly, these networks are the ones that have an AP graph topology.

It has to be known that the problem of deciding whether a sequence is realizable in a graph is computationally hard, even when restricted to trees [2], although some exceptions can be pointed out. As an illustration, let us mention the early result proved independently by Győri and Lovász, obtained long before the introduction of AP graphs, that states that k-connected graphs are always

[1]In the sense that two resources of a subnetwork must be able to communicate within it.

partitionable into k connected subgraphs no matter what their requested orders are [9, 12]. For a deeper overview of the background of AP graphs, the interested reader is referred to [1, 2, 13].

Observe that the definition of AP graphs is quite static and thus not representing the difficulties we can encounter while actually partitioning a network; notably, one could point out the following two issues.

1. In the definition, a graph is fully partitioned at once; from the network sharing point of view, this constraint is like waiting for every single resource of the network to be needed before eventually supplying the users. This is, of course, not satisfactory since we would like to satisfy them as soon as possible (immediately, ideally).
2. When a sequence is realized in a graph, the induced subgraphs must only meet the connectivity constraint. But according to our network analogy, it would be more convenient to make sure that the allocated subnetworks themselves have the property of being shareable at will. This can be quite useful if, for example, a user wants himself to share his subnetwork with several other users.

Motivated by these deficiencies, the following augmented definitions have been introduced.

Definition 1.1 [10]**.** A graph G is said to be *online arbitrarily partitionable* (OL-AP for short) if and only if one of the following two conditions holds.

- G is isomorphic to K_1.
- G is connected and for every $\lambda \in \{1, \dots, n-1\}$, there exists a partition (S, T) of $V(G)$ such that $G[S]$ is connected on λ vertices and $G[T]$ is OL-AP of order $n - \lambda$.

Definition 1.2 [5]**.** A graph G is said to be *recursively arbitrarily partitionable* (R-AP for short) if and only if one of the following two conditions holds.

- G is isomorphic to K_1.
- G is connected and for every sequence $\tau = (n_1, \dots, n_p)$ admissible for it, there exists a partition $(V_1, \dots, V_p)$ of $V(G)$ into p parts such that each V_i induces a R-AP subgraph of G on n_i vertices.

Observe that the notion of OL-AP graphs (R-AP graphs, respectively) can be used to deal with our network sharing problem taking issue 1 (issue 2, respectively) pointed out above into account. It has to be known that the two properties of being OL-AP and R-AP are actually quite similar to each other. Indeed, previous investigations have shown that every R-AP graph is also OL-AP [5] and that, in the context of some classes of graphs (like trees and so-called suns), there only exist a few OL-AP graphs being not R-AP [4, 5].

Theorem 1.3 [5]**.** *Every* R-AP *graph is also* OL-AP, *but the contrary does not necessarily hold.*

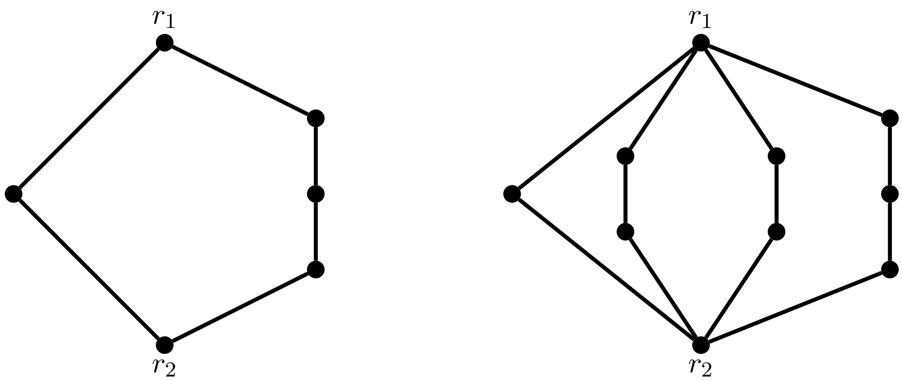

Figure 1. The balloons $B(1,3)$ and $B(1,2,2,3)$.

We here focus on the class of balloon graphs introduced in [5].

Definition 1.4. Let $b_1, \dots, b_k$ be $k \geq 1$ non-negative integers such that at most one of them is 0, and consider k paths on $b_1 + 2, \dots, b_k + 2$ vertices, respectively, where the endvertices of the ith path are denoted by u_i and v_i. The *balloon with k branches* $B(b_1, \dots, b_k)$ (sometimes called *k-balloon* for short) is the graph obtained by identifying all the u_i's together and all the v_i's together.

Two examples of such graphs are given in Figure 1. Partitionable balloons are interesting regarding our problem because their structure is closely related to the one of partitionable graphs with connectivity 2.

Observation 1.5. *Let G be a graph with connectivity* 2*, u and v be two vertices forming a cut pair of G, and $b_1, \dots, b_k$ be the numbers of vertices of the $k \geq 2$ connected components of $G - \{u, v\}$. If G is* AP, OL-AP *or* R-AP*, then $B(b_1, \dots, b_k)$ is* AP, OL-AP *or* R-AP*, respectively.*

Indeed, observe that no graph of order n is easier to partition than the path on n vertices. Hence, a realization of a sequence τ in $B(b_1, \dots, b_k)$ can be directly deduced from a realization of τ in G. By Observation 1.5, we get that some properties holding for AP, OL-AP or R-AP balloons also hold for AP, OL-AP or R-AP graphs with connectivity 2, respectively. In particular, if $c \geq 1$ is an upper bound on the number of branches in an AP balloon, then c is also an upper bound on the number of components resulting from the deletion of a cut pair in an AP graph. One can also deduce an upper bound on the number of vertices in some of these components from a bound on the order of the corresponding branches in an AP balloon. The interesting thing is that these statements also hold for OL-AP and R-AP balloons and OL-AP and R-AP graphs with connectivity 2. A more general approach for exhibiting structural properties of AP graphs with given connectivity was already used in [3].

In this article such upper bounds on the structure of OL-AP and R-AP balloons are exhibited. After having provided in Section 2 some preliminary definitions and tools necessary to understand our results, we then show, in Section 3, that an OL-AP or R-AP balloon has at most five branches. In Section 4 is exhibited an infinite family of OL-AP and R-AP 5-balloons showing that the number of these graphs is not bounded and that our previous bound is tight. We then give constant upper bounds on the order of the smallest branches in an OL-AP or R-AP 4- or 5-balloon in Section 5. Finally, concluding Section 6 outlines some structural properties of OL-AP or R-AP graphs with connectivity 2 that can be derived from our results on OL-AP and R-AP balloons.

2. Terminology and Preliminary Results

2.1. Terminology and notation

Let $x \geq 1$ be an integer. Throughout this paper, we denote by x^+ or $x+$ (x^- or $x-$, respectively) an arbitrary integer y such that $y \geq x$ ($y \leq x$, respectively).

We deal with connected, non-oriented and simple graphs, using mainly the terminology of [8]. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively, or simply by V and E when no ambiguity is possible. The order of G, commonly denoted by n, is its number of vertices. Given a subset of vertices $S \subseteq V$ of G, we denote by $G[S]$ the subgraph of G induced by the vertices of S. We additionally denote by $G - S$ the subgraph of G induced by all the vertices of $V \setminus S$. If $F \subseteq E$ is a subset of edges of G, we denote by $G - F$ the subgraph of G on the same vertex set, obtained by removing all the edges of F from G.

We denote by P_n the path of order n. Given $k \geq 1$ strictly positive integers $a_1, \dots, a_k$, the *k-pode* $P(a_1, \dots, a_k)$ is the tree obtained by joining one central node to one extremity of each one of k disjoint paths on $a_1, \dots, a_k$ vertices, respectively. A k-pode is equivalently obtained by performing subdivisions in the star with k edges. Since previous investigations [5, 7], a 3-pode $P(1, a_2, a_3)$ is often referenced as a *caterpillar* and is denoted by $\mathrm{Cat}(a_2 + 1, a_3 + 1)$ for convenience[2]. Hence, throughout this paper, every mention of caterpillars actually refers to caterpillars of the form $\mathrm{Cat}(a, b)$.

We now give more notations associated with the notion of balloon graphs. Let $B = B(b_1, \dots, b_k)$ be a k-balloon. The vertices r_1 and r_2 of degree k in B are called the *roots of* B, while the path of order b_i connecting them is said to be the *ith branch of* B. For every $i \in \{1, \dots, k\}$, the vertices of the ith branch of B are denoted by $v_1^i, \dots, v_{b_i}^i$ in such a way that $v_1^i r_1, v_{b_i}^i r_2 \in E$ and $v_j^i v_{j+1}^i \in E$

[2] Observe that $\mathrm{Cat}(a, b)$ has order $a + b$.

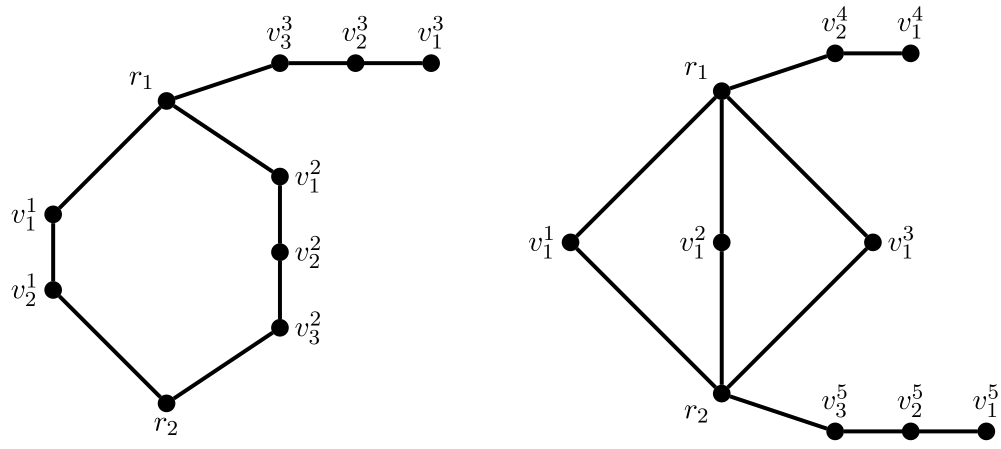

Figure 2. The partial balloons $B(2,3,\overline{3})$ and $B(1,1,1,\overline{2},\underline{3})$.

for every $j \in \{1,\dots,b_i-1\}$. Finally, we denote by $b_i(B)$ the number of vertices composing the ith branch of B.

We note that a balloon B may contain exactly one branch with no internal vertices (that is $B = B(0,b_1,\dots,b_k)$ with $b_1,\dots,b_k > 0$). In such a situation, the edge r_1r_2 is called an *empty branch.* In case $B = B(b_1,\dots,b_k)$ is a balloon with no empty branch, we denote by B^+ the graph $B + r_1r_2$, which corresponds to $B(0,b_1,\dots,b_k)$. Since we consider simple graphs only, a balloon can have at most one empty branch.

Throughout this paper, we deal with connected subgraphs of balloons, which we call *partial balloons*, obtained by removing edges from balloons. Formally, given an $(x+y+z)$-balloon $B = B(b_1,\dots,b_{x+y+z})$, by

$$B(b_1,\dots,b_x,\overline{b_{x+1}},\dots,\overline{b_{x+y}},\underline{b_{x+y+1}},\dots,\underline{b_{x+y+z}})$$

we refer to the *partial* $(x+y+z)$*-balloon* being

$$B' = B - \left(\bigcup_{i=x+1}^{x+y} \left\{v_{b_i}^i r_2\right\}, \bigcup_{i=x+y+1}^{x+y+z} \left\{v_1^i r_1\right\} \right).$$

Typically, B' is obtained by removing $y+z$ edges from non-empty branches of B, without disconnecting the graph. In this paper, the notations introduced above for balloons are used in an analogous way for partial balloons. For convenience, the vertices of a *hanging branch* of order b_i of a partial balloon B (that is, a branch attached to only one root of B) are successively denoted by $v_1^i,\dots,v_{b_i}^i$, where v_1^i is the degree-1 vertex of the branch and $v_{b_i}^i$ is the vertex adjacent to one root of B. Refer to Figure 2 for two examples of partial balloons and their associated notations.

We note that a balloon may be considered as a partial balloon without hanging branches. This is basically why, throughout, the notation $B(\dots)$ refers both to balloons and partial balloons. Nevertheless, no confusion is possible, as, in the notation $B(\dots)$, the number of vertices in a hanging branch is always either overlined (to emphasize that the hanging branch is attached to r_1) or underlined (otherwise).

2.2. Some properties of OL-AP and R-AP graphs

The next two theorems give a complete characterization of OL-AP and R-AP trees.

Theorem 2.1 [10]**.** *A tree is* OL-AP *if and only if it is either isomorphic to a path, to a caterpillar* $\mathrm{Cat}(a,b)$ *with* a *and* b *given in Table* 1, *or to the* 3*-pode* $P(2,4,6)$.

a	b
2, 4	$\equiv 1 \bmod 2$
3	$\equiv 1, 2 \bmod 3$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \bmod 6$
7	8, 9, 11, 13, 15
8	11, 19
9, 10	11
11	12

Table 1. Values a and b, with $b \geq a$, such that $\mathrm{Cat}(a,b)$ is OL-AP.

Theorem 2.2 [5]**.** *A tree is* R-AP *if and only if it is either isomorphic to a path, to a caterpillar* $\mathrm{Cat}(a,b)$ *with* a *and* b *given in Table* 2, *or to the* 3*-pode* $P(2,4,6)$.

Theorems 2.1 and 2.2 were proved using the following two observations, which provide two alternative methods to check whether a graph is OL-AP or R-AP.

Observation 2.3. *A graph* G *is* R-AP *if and only if for every* $\lambda \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$ *there exists a partition* (S,T) *of* V *into two parts such that* $G[S]$ *and* $G[T]$ *are* R-AP *graphs on* λ *and* $n-\lambda$ *vertices, respectively.*

Observation 2.4. *The property of being* OL-AP (R-AP, *respectively*) *is closed under edge-additions: if a graph has an* OL-AP (*a* R-AP, *respectively*) *spanning subgraph, then it is* OL-AP (R-AP, *respectively*).

a	b
2, 4	$\equiv 1 \bmod 2$
3	$\equiv 1, 2 \bmod 3$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 2. Values a and b, with $b \geq a$, such that $\mathrm{Cat}(a, b)$ is R-AP.

2.3. Some properties of AP, OL-AP and R-AP balloons

Some properties of OL-AP and R-AP balloons are given or recalled in the next two sections. The first properties, in Section 2.3.1, are quite general, and were already pointed out in previous works. The new properties from Section 2.3.2 are about the importance of empty branches in partitionable balloons. A whole section is dedicated to the latter properties, as the associated proofs are rather non-trivial.

2.3.1. General properties of AP, OL-AP and R-AP balloons

First, notice that every path is AP, OL-AP and R-AP. For this reason, it follows that every *traceable graph*[3] is AP, OL-AP and R-AP too. Since 1-, 2- and 3-balloons are traceable, we are mainly interested in balloons with at least four branches in this work.

We additionally introduce the following results on the orders of the branches in an AP balloon. Since every OL-AP or R-AP graph is also AP, these results also hold when considering OL-AP and R-AP balloons.

Observation 2.5 [5]**.** *Let B be an* AP *balloon. If n is odd, then B has at most three branches of odd order. If n is even, then B has at most two branches of odd order.*

Lemma 2.6 [3, 5]**.** *Let $B(b_1, \dots, b_k)$ be an* AP *balloon with $1 \leq b_1 \leq \dots \leq b_k$. For every $i \in \{2, \dots, k\}$, we have $2b_i \geq \sum_{j=1}^{i-1} b_j$.*

2.3.2. On OL-AP and R-AP balloons with an empty branch

It was previously proved in [5] that the vertex set of an AP balloon with an empty branch can always be partitioned in such a way that if the partition is non-trivial (has at least two parts), then the roots of the balloon belong to two different parts of the partition. In other words, any balloon $B(0, b_1, \dots, b_k)$ is AP if and

[3]A graph is *traceable* if it has a Hamiltonian path.

only if $B(b_1, \dots, b_k)$ is AP. We prove here the analogous result for OL-AP and R-AP balloons, so that we can, in the rest of the paper, focus on balloons with no empty branch. While this fact is rather easy to prove for AP balloons, it turns out to be more challenging in the context of OL-AP and R-AP balloons.

We first need to introduce an additional notation. For integers $j, k, l \geq 0$, we denote by $\mathcal{B}_l^k(j)$ the set of partial balloons without an empty branch having j ordinary (non-hanging) branches, k hanging branches attached to their root r_1, and l hanging branches attached to their root r_2. We below freely make use of the notation $\mathcal{B}_l^k(j)$ with j, k, l being of the form $x+$. Furthermore, given a partial balloon

$$B(b_1, \dots, b_j, \overline{b_{j+1}}, \dots, \overline{b_{j+k}}, \underline{b_{j+k+1}}, \dots, \underline{b_{j+k+l}}) \in \mathcal{B}_l^k(j),$$

it is assumed, throughout this section, that the sequences $\beta_1 = (b_1, \dots, b_j)$, $\beta_2 = (b_{j+1}, \dots, b_{j+k})$ and $\beta_3 = (b_{j+k+1}, \dots, b_{j+k+l})$ are non-decreasing, and that either $\beta_2 = \beta_3$ or β_2 is lexicographically smaller than β_3. As any two sets $\mathcal{B}_l^k(j)$ and $\mathcal{B}_k^l(j)$ contain the same graphs, in our upcoming proofs we will implicitly assume that $l \leq k$.

When partitioning a balloon in an OL-AP or R-AP way, one part can induce a partial balloon, so we need to know which ones are OL-AP or R-AP, and which ones are not. For this reason, it is necessary to first study some base cases (that is, small values of j, k, l), so that we can get our conclusion. We below focus on the case of OL-AP balloons, as the case of R-AP balloons can be conducted in a similar way. Recall that B^+ is the partial balloon obtained by adding an empty branch to the partial balloon B, assuming B does not already have one.

Lemma 2.7. *A graph $B \in \mathcal{B}_0^{0+}(1)$ is* OL-AP *if and only if B^+ is* OL-AP.

Proof. We note that if $B \in \mathcal{B}_0^0(1) \cup \mathcal{B}_0^1(1)$, then both B and B^+ are traceable, hence OL-AP. Now, if $B \in \mathcal{B}_0^{2+}(1)$, then $n = |V(B)| \geq 5$. If $n = 5$, then B is isomorphic to $\mathrm{Cat}(2, 3)$, and hence OL-AP according to Theorem 2.1. According to Observation 2.4, B^+ is also OL-AP.

Now suppose that $B = B(b_1, \overline{b_2}, \dots, \overline{b_{k+1}}) \in \mathcal{B}_0^{2+}(1)$, where $k \geq 2$ and $n = |V(B)| \geq 6$, and that the result holds for every partial balloon $B' \in \mathcal{B}_0^{2+}(1)$ with $|V(B')| \leq n-1$. Due to Observation 2.4, to get our conclusion, it suffices to show that if B^+ is OL-AP, then so is B.

For every $i \in \{1, \dots, k+1\}$, we denote by B_i the set (of cardinality b_i) containing the vertices of the ith non-empty branch of B^+. Let $\lambda \in \{1, \dots, n-1\}$, and (S, T) be a partition of $V(B^+)$ such that $B^+[S]$ is a connected graph on λ vertices, while $B^+[T]$ is an OL-AP graph on $n - \lambda$ vertices. To prove the claim, we need to show that, from (S, T), we can always deduce a similar partition (S^-, T^-) of $V(B)$ for B. As we can clearly choose $(S^-, T^-) = (S, T)$ whenever

r_1r_2 is not used by the partition, that is when r_1 and r_2 belong to different sets of (S,T), we can focus on the cases where $r_1, r_2 \in S$ and $r_1, r_2 \in T$.

• If $r_1, r_2 \in S$, then, because $B^+[T]$ is OL-AP and hence connected, it follows that $B^+[T]$ is isomorphic to a path, and hence there is an $i \in \{1, \ldots, k+1\}$ such that $T \subseteq B_i$. If $i \geq 2$, then, by the connectedness of $B^+[S]$, part T includes the degree-1 vertex of B_i. Furthermore, r_1 and r_2 are joined by two vertex-disjoint paths in $B^+[S]$, namely by the edge r_1r_2 and the branch B_1. So we can here set $S^- = S$ and $T^- = T$. Now, if $i = 1$, then note that we can freely "slide" T along B_1 to obtain T^- including the $n - \lambda$ vertices of B_1 in the direction from r_1 to r_2. Then $B[S^-]$ is connected and $B[T^-]$ is a path isomorphic to $B^+[T]$.

• If $r_1, r_2 \in T$, then, due to the connectedness of $B^+[S]$, there is an $i \in \{1, \ldots, k+1\}$ such that $S \subseteq B_i$ and $B^+[S]$ is a path. We conduct the same case distinction as above. First, if $i \geq 2$, then, because $B^+[T]$ is connected, necessarily S contains λ consecutive vertices of B_i, starting from the degree-1 vertex. So $B^+[T] \in \mathcal{B}_0^{0+}(1)$ is a partial balloon with less vertices than B^+, and, by the induction hypothesis, we deduce that $B[T]$ is OL-AP since $B^+[T]$ is OL-AP. Hence (S^-, T^-) is a satisfying partition of B. Finally, if $i = 1$, then $B^+[T]$, which is OL-AP, is a tree with r_1 being of degree $k+1$ or $k+2$. According to Theorem 2.1, necessarily $k = 2$, and S contains the first λ consecutive vertices of B_1, including the neighbour of r_1 in B_1. Here again, the partition (S,T) can be modified by sliding S along B_1, so that it includes r_2 and the last $\lambda - 1$ vertices of B_1 including the neighbour of r_2. The resulting partition still induces the same graphs, and is hence still correct, while r_1 and r_2 now belong to different parts of (S,T), a case we know how to handle. ■

Lemma 2.8. *A graph $B \in \mathcal{B}_1^{0+}(1)$ is* OL-AP *if and only if B^+ is* OL-AP.

Proof. We prove the result quite similarly as Lemma 2.7. Therefore, we make use of the same terminology, and omit the proof of the analogous easy cases. According to the traceability argument, we may focus on balloons in $\mathcal{B}_1^{2+}(1)$, hence on balloons B with $n = |V(B)| \geq 6$. If $n = 6$, then $B = (1, \overline{1}, \overline{1}, \underline{1})$, and neither B nor B^+ can be partitioned as required for $\lambda = 2$, since, for every choice of S forcing T to induce a connected subgraph, necessarily T induces $\mathrm{Cat}(2,2)$, which is not OL-AP.

Now consider a partial balloon $B = B(b_1, \overline{b_2}, \ldots, \overline{b_{k+1}}, \underline{b_{k+2}}) \in \mathcal{B}_1^{2+}(1)$, where $k \geq 2$ and $n = |V(G)| \geq 7$, and assume the claim holds for partial balloons in $\mathcal{B}_1^{2+}(1)$ with at most $n-1$ vertices. We prove the claim for B. Again, we just need to show that if (S,T) is a partition of B^+ with the same properties as described in Definition 1.1, for some $\lambda \in \{1, \ldots, n-1\}$, then we can deduce a similar partition (S^-, T^-) for B. We again focus on the cases where r_1 and r_2 belong to the same part of (S,T).

• If $r_1, r_2 \in S$, then $B^+[T]$ is a path, and there is an $i \in \{1, \ldots, k+2\}$ such that $T \subseteq B_i$. Similarly as in the proof of Lemma 2.7, if $i \geq 2$, then T includes $n - \lambda$ consecutive vertices of B_i, starting from the degree-1 vertex. So, again, r_1 and r_2 are joined by two paths in $B^+[S]$, hence $r_1 r_2$ is not necessary for the connectivity of $B^+[S]$, and we can set $S^- = S$ and $T^- = T$. Now, if $i = 1$, then we can slide (the result of sliding of) T along B_{k+2} to obtain T^- containing the degree-1 vertex of B_{k+2}. Then $B[S^-]$ is connected and $B[T^-]$ is a path isomorphic to $B^+[T]$.

• If $r_1, r_2 \in T$, then there is an $i \in \{1, \ldots, k+2\}$ such that $S \subseteq B_i$. If $i \geq 2$, then, again by the connectedness of $B^+[T]$, the part S contains the degree-1 vertex of B_i. Then $B^+[T]$ is an OL-AP partial balloon with less vertices than B, that is a balloon from $\mathcal{B}_0^{2+}(1)$ or from $\mathcal{B}_1^{1+}(1)$. So $B[T]$ is OL-AP according to Lemma 2.7, or by the induction hypothesis, respectively. It follows that (S, T) is a satisfying partition of B as well. Now, if $i = 1$, then $B^+[T]$ is an OL-AP tree in which r_1 is of degree $k+2$ or $k+1$, while r_2 is of degree 2 or 3. According to Theorem 2.1, necessarily $k = 2$, $\lambda = b_1$, and $B^+[T]$ is an OL-AP 3-pode of the form $P(b_2, b_3, b_4 + 1)$. In this situation, by sliding S along B_1 towards r_2 and then along B_4 to obtain S^- containing the degree-1 vertex of B_4, we finish with the pair (S^-, T^-) in which S^- and T^- induce the same graphs as S and T do, respectively. ∎

Lemma 2.9. *If $B \in \mathcal{B}_2^2(1)$, then B^+ is not* OL-AP.

Proof. Let $B = B(b_1, \overline{b_2}, \overline{b_3}, \underline{b_4}, \underline{b_5})$. Set $n = |V(B)| \geq 7$. First, if $n = 7$, then $B = B(1, \overline{1}, \overline{1}, \underline{1}, \underline{1})$, and neither B nor B^+ are OL-AP, again because, as in the proof of Lemma 2.8, they cannot be partitioned as required for $\lambda = 2$.

Now assume that $n \geq 8$, and assume the claim is true for all partial balloons in $\mathcal{B}_2^2(1)$ with at most $|V(B)| - 1$ vertices. The claim can be proved by showing that there is always a $\lambda \in \{1, \ldots, n-1\}$ such that, for every partition (S, T) of $V(B^+)$ with $|S| = \lambda$ and $B^+[S]$, $B^+[T]$ being connected, one of the following conditions is necessarily met.

1. $B^+[T] \in \mathcal{B}_2^2(1)$ — hence $B^+[T]$ is not OL-AP by the induction hypothesis;
2. $B^+[T] \in \mathcal{B}_1^2(1) \cup \mathcal{B}_2^1(1)$, and $B[T]$ is a tree not listed in Theorem 2.1 — hence $B^+[T]$ is not OL-AP by Lemma 2.8;
3. $r_1, r_2 \in T$, and $B^+[T]$ is a tree with $\Delta(B^+[T]) = 4$, or in which both r_1 and r_2 have degree at least 3 — hence not OL-AP according to Theorem 2.1.

This can be essentially proved by case distinction, but the choice of λ is highly dependent on b_1, b_2, b_3, b_4, b_5. Due to the number of cases to consider, we move this part of the proof to Appendix A. ∎

Lemma 2.10. *If $B \in \mathcal{B}_{2+}^{3+}(1)$, then B^+ is not* OL-AP.

Proof. We have $n = |V(B)| \geq 8$. If $n = 8$, then $B = B(1, \overline{1}, \overline{1}, \overline{1}, \underline{1}, \underline{1})$, and B^+ cannot be partitioned as required for $\lambda = 2$, according to the same arguments as earlier. The claim is hence true for B.

Now assume $B = B(b_1, \overline{b_2}, \ldots, \overline{b_{k+1}}, \underline{b_{k+2}}, \ldots, \underline{b_{k+l+1}})$ is a partial balloon of order $n = |V(B)| \geq 9$ with $k \geq 3$ and $l \geq 2$, and suppose the claim is true for every partial balloon in $\mathcal{B}_{2+}^{3+}(1)$ having less vertices than B.

Let (S, T) be a partition of $V(B^+)$ as described in Definition 1.1 for $\lambda = 1$. Since $B^+[T]$ is connected, there are only two possibilities for S.

- First, S may contain a degree-1 vertex v of B^+. But it can be seen that $B^+[T]$ is then either a partial balloon from $\mathcal{B}_{2+}^{3+}(1)$, from $\mathcal{B}_1^{3+}(1)$, or from $\mathcal{B}_{2+}^{2}(1)$. Hence $B^+[T]$ cannot be OL-AP, according either to the induction hypothesis, Lemma 2.8 and Theorem 2.1, or Lemma 2.9.
- Second, S may contain a vertex from B_1. But, in such a situation, $B^+[T]$ is then isomorphic to a tree with maximum degree at least 4, and is hence not OL-AP.

In both cases, we get that $B^+[T]$ cannot be OL-AP, hence that the partition (S, T) does not exist for $\lambda = 1$. Therefore, B^+ cannot be OL-AP. ■

Corollary 2.11. *A graph $B \in \mathcal{B}_{0+}^{0+}(1)$ is* OL-AP *if and only if B^+ is* OL-AP.

Proof. We may assume that $B \in \mathcal{B}_l^k(1)$ with $k \geq l$. If $l \leq 1$, then we are done according to Lemmas 2.7 and 2.8. Now, if $l \geq 2$, then B^+ cannot be OL-AP according to Lemmas 2.9 and 2.10. The same is also true for B, according to Observation 2.4. ■

We now get to our conclusion.

Theorem 2.12. *A graph $B \in \mathcal{B}_{0+}^{0+}(1+)$ is* OL-AP *if and only if B^+ is* OL-AP.

Proof. If $B \in \mathcal{B}_{0+}^{0+}(1)$, then the claim follows from Corollary 2.11. So assume $B \in \mathcal{B}_{0+}^{0+}(2+)$. Then $n = |V(B)| \geq 4$. If $n = 4$, then $B = B(1, 1)$ and both B and B^+ are traceable, hence OL-AP.

So now suppose that $n \geq 5$, and that every partial balloon B' in $\mathcal{B}_{0+}^{0+}(1+)$, having less vertices than B, is OL-AP if and only if B'^+ is OL-AP. We now prove the claim for B. Again, due to Observation 2.4 and Definition 1.1, it suffices to show that if, for any $\lambda \in \{1, \ldots, n-1\}$, there exists a partition (S, T) of $V(B^+)$, such that $B^+[S]$ and $B^+[T]$ are connected on λ vertices, and OL-AP on $n - \lambda$ vertices, respectively, then a similar partition (S^-, T^-) of $V(B)$ also exists. We actually prove below that letting even $S^- = S$ and $T^- = T$ is valid.

In case r_1 and r_2 belong to different parts of the partition, the edge r_1r_2 is useless for the partition, and we directly get our conclusion. So we consider the two remaining cases.

- If $r_1, r_2 \in S$, then, since $B^+[T]$ is connected, $B^+[T]$ is a path containing consecutive vertices from a single branch, including the one with degree 1, if any (since $B^+[S]$ is connected). So both $B^+[T]$ and $B[T]$ are OL-AP. Now, since B has at least two ordinary (non-hanging) branches, we note that, even if vertices of T belong to a non-hanging branch, we have that $B^+[S] - r_1r_2$ is connected. So $B[S]$ is connected, and (S, T) is a correct partition of B.
- If $r_1, r_2 \in T$, then $B[S] = B^+[S]$ is connected. Now, we have that $B^+[T] \in \mathcal{B}_{0+}^{0+}(1+)$ is OL-AP, hence that $B[T]$ is also OL-AP, according either to the induction hypothesis or to Corollary 2.11. So (S, T) is a correct partition, concluding the proof. ∎

We note that a similar approach can be conducted for R-AP graphs, due to the symmetric form of the definition of R-AP graphs given in Observation 2.3. More precisely, one can prove the following.

Theorem 2.13. *A graph $B \in \mathcal{B}_{0+}^{0+}(1+)$ is* R-AP *if and only if B^+ is* R-AP*.*

Also note that every graph $B \in \mathcal{B}_l^k(0)$ is disconnected, hence neither OL-AP nor R-AP. However, we have that B^+ is a tree, and may hence be OL-AP or R-AP, respectively, according to Theorems 2.1 and 2.2, respectively. So the sets

$$\left\{B \in \mathcal{B}_l^k(0) : B^+ \text{ is OL-AP, but } B \text{ is not OL-AP}\right\},$$

and

$$\left\{B \in \mathcal{B}_l^k(0) : B^+ \text{ is R-AP, but } B \text{ is not R-AP}\right\},$$

are infinite.

3. OL-AP Balloons Cannot Have More Than Five Branches

It is already known that a R-AP ballon has at most five branches [5]. In this section, we prove a more general statement by showing the same upper bound on the number of branches in an OL-AP balloon. By Theorem 1.3, this result also holds for R-AP balloons.

Theorem 3.1. *An* OL-AP *balloon cannot have more than five branches.*

Proof. The proof is by contradiction. Let $\mathcal{B}$ be the set of OL-AP balloons with at least six branches, and let B denote a k-balloon of $\mathcal{B}$ with the least order.

By Definition 1.1, B is OL-AP if and only if for every $\lambda \in \{1, \ldots, n-1\}$ we can partition V into two parts S and T such that $B[S]$ is connected on λ vertices and $B[T]$ is OL-AP on $n - \lambda$ vertices. Observe that, because of the minimality of B, the subgraph $B[T]$ cannot be a partial k'-balloon with $k' \geq 6$ since otherwise there would exist a balloon of $\mathcal{B}$ with less vertices than B (Observation 2.4). It follows that $B[T]$ is either an OL-AP 5-balloon or an OL-AP tree (see Theorem 2.1).

We claim that B has branches with 1, 2, 3, 4, 5 and 6 vertices. Let us suppose that $\lambda \in \{1, \dots, 6\}$ and that B does not have a branch of order λ. We show that it is not possible to partition V into two parts S and T of cardinalities λ and $n - \lambda$, respectively, satisfying the above conditions.

- $\lambda \in \{1, 2, 3\}$: for every choice of S, the subgraph $B[T]$ is either a partial k-balloon having less vertices than B or a tree with maximum degree at least 4. In both cases, the subgraph $B[T]$ is not OL-AP.

- $\lambda = 4$: so far, we have shown that B necessarily has branches of order 1, 2 and 3. Similarly as in the previous case, observe that for every choice of S, the subgraph $B[T]$ is either a partial k-balloon, a partial $(k-1)$-balloon having less vertices than B, or a tree having maximum degree at least 3. Hence, the only possibility here is to choose S in such a way that $B[T]$ is a tree with maximum degree 3, but this is only possible when $B = B(1, 1, 1, 2, 3, \dots)$. According to Observation 2.5, such a balloon is not AP, and thus is not OL-AP.

- $\lambda = 5$: by the previous cases, we know that B has branches composed of 1, 2, 3 and 4 vertices. For the same reasons as before, S must be chosen in such a way that $B[T]$ is either a path or an OL-AP 3-pode. Hence, since B has at least six branches, S must contain one root of B and all the vertices of at least $k - 3$ of its branches. Observe that S can only be chosen in this way when $k = 6$ and $B = B(1, 1, 1, 2, 3, 4)$. It follows that B has four branches of odd order, and thus that it is not AP according to Observation 2.5. It hence cannot be OL-AP.

- $\lambda = 6$: we know that B has branches with 1, 2, 3, 4 and 5 vertices. Moreover, since $k \geq 6$, the balloon B has an additional branch of order b_i. If $b_i \leq 7$, then B is not AP by Lemma 2.6, and thus is not OL-AP. Hence, $b_i \geq 8$ but, again, we cannot exhibit a subset S for which $B[T]$ would be an OL-AP tree, that is with maximum degree at most 3. Hence B is not OL-AP.

Finally, B is isomorphic to $B(1, 2, 3, 4, 5, 6, \dots)$ which is not AP by Lemma 2.6; it thus cannot be OL-AP. ∎

4. There are Infinitely Many OL-AP and R-AP 4- or 5-Balloons

By Theorem 3.1, we know that the number of branches in an OL-AP or R-AP balloon is bounded by 5. In what follows, we prove that this bound is tight by exhibiting two infinite families of R-AP balloons with four and five branches, respectively. We use the following two lemmas for this purpose.

Lemma 4.1. *The partial balloon* $B(1,1,2,\overline{k})$ *is* R-AP *for every* $k \geq 1$.

Proof. Observe that this claim is true whenever $k = 1$, $k = 2$ or $k = 3$ since the corresponding partial balloons are spanned by $\mathrm{Cat}(2,5)$, $\mathrm{Cat}(3,5)$ and $\mathrm{Cat}(4,5)$, respectively.

Suppose now that this claim holds for every k up to $i-1$ and consider the partial balloon $B = B(1,1,2,\overline{i})$. By Observation 2.3, we know that B is R-AP if it can be partitioned, for every $\lambda \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, into two R-AP subgraphs on λ and $n - \lambda$ vertices, respectively. One can consider the following partitions.

- $\lambda = 1$: P_1 and $B(1,2,\overline{i})$ (traceable).
- $\lambda = 2$: P_2 and $B(1,1,\overline{i})$ (traceable).
- $\lambda \in \{3,4\}$: P_λ and $P_{n-\lambda}$.
- $\lambda = 5$: $\mathrm{Cat}(2,3)$ and P_{i+1}.
- $\lambda = 6$: $B(1,1,2)$ (traceable) and P_i.
- $\lambda \geq 7$: $B(1,1,2,\overline{\lambda-6})$ (induction hypothesis) and $P_{i-\lambda+6}$. ■

Lemma 4.2. *The partial balloon* $B(1,2,3,\overline{k})$ *is* R-AP *for every* $k \geq 1$.

Proof. The proof is by induction on k. If we first suppose that $k = 1$, $k = 2$, or $k = 3$, then observe that the corresponding partial 4-balloons are R-AP since they are spanned by the R-AP caterpillars $\mathrm{Cat}(2,7)$, $\mathrm{Cat}(3,7)$ and $\mathrm{Cat}(4,7)$, respectively.

Let us secondly suppose that this lemma holds for every $k \leq i-1$, and consider the partial balloon $B = B(1,2,3,\overline{i})$. Once again, by Observation 2.3, it is sufficient to show, to prove that B is R-AP, that we can partition it into two R-AP subgraphs on λ and $n-\lambda$ vertices, respectively, for every $\lambda \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We show that these partitions exist for every λ.

- $\lambda = 1$: P_1 and $B(2,3,\overline{i})$ (traceable).
- $\lambda = 2$: P_2 and $B(1,3,\overline{i})$ (traceable).
- $\lambda = 3$: P_3 and $B(1,2,\overline{i})$ (traceable).
- $\lambda \in \{4,5,6\}$: P_λ and $P_{n-\lambda}$.
- $\lambda = 7$: $\mathrm{Cat}(3,4)$ and P_{i+1}.
- $\lambda = 8$: $B(1,2,3)$ (traceable) and P_i.
- $\lambda \geq 9$: $B(1,2,3,\overline{\lambda-8})$ (induction hypothesis) and $P_{n-\lambda+8}$. ■

Observe that, according to Observation 2.4, Lemmas 4.1 and 4.2 directly imply that there exist infinitely many R-AP 4-balloons (and, thus, infinitely many OL-AP 4-balloons, see Theorem 1.3).

Corollary 4.3. *The* 4*-balloons* $B(1,1,2,k)$ *and* $B(1,2,3,k)$ *are* OL-AP *and* R-AP *for every* $k \geq 1$.

We now prove that there exists an infinite family of R-AP 5-balloons.

Theorem 4.4. *The partial balloon* $B(1,1,2,3,\overline{2k})$ *is* R-AP *for every* $k \geq 1$.

Proof. Let $B = B(1,1,2,3,\overline{2k})$ with $k \geq 1$. Recall that, according to Observation 2.3, the partial balloon B is R-AP if we can partition it into two R-AP subgraphs $B[S]$ and $B[T]$ having order λ and $n-\lambda$, respectively, for every $\lambda \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$. One can consider the following partitions for the first values of λ.

- $\lambda = 1$: P_1 and $B(1,2,3,\overline{2k})$ (Lemma 4.2).
- $\lambda = 2$: P_2 and $B(1,1,3,\overline{2k})$ (spanned by $\mathrm{Cat}(2, 5+2k)$).
- $\lambda = 3$: P_3 and $B(1,1,2,\overline{2k})$ (Lemma 4.1).
- $\lambda = 4$: P_4 and $\mathrm{Cat}(4, 2k+1)$.
- $\lambda = 5$: $\mathrm{Cat}(2,3)$ and P_{2k+4}.
- $\lambda = 6$: P_6 and $\mathrm{Cat}(2, 2k+1)$.
- $\lambda = 7$: $\mathrm{Cat}(3,4)$ and P_{2k+2}.

By now, it should be clear that the proposition holds for every partial balloon $B(1,1,2,3,\overline{2k})$ such that $n \leq 15$ (that is, for each $k \in \{1,2,3\}$). Let us suppose, as an induction hypothesis, that the claim is true for every $k \leq i-1$, and consider the partition of $B = B(1,1,2,3,\overline{2i})$ into two R-AP subgraphs for the remaining values of λ, that is for every $\lambda \in \{8,\dots,\lfloor \frac{n}{2} \rfloor\}$.

- $\lambda \geq 8$, λ even: observe that $\lambda \leq 2i$ since $\lambda \leq \lfloor \frac{n}{2} \rfloor$ and we handled the cases where $k \leq 3$. We can thus partition B into P_λ and either $B(1,1,2,3)$ (when $i = 4$) or $B(1,1,2,3,\overline{2i-\lambda})$ (when $i > 4$). These graphs are R-AP according to Lemma 4.2 and by the induction hypothesis (since $2i - \lambda$ is even), respectively.
- $\lambda = 9$: $B(1,1,2,3)$ (spanned by $\mathrm{Cat}(2,7)$) and P_{2i}.
- $\lambda > 9$, λ odd: $B(1,1,2,3,\overline{\lambda-9})$ (induction hypothesis since $\lambda - 9$ is even) and $P_{n-\lambda+9}$. ■

Combining Observation 2.4 and Theorem 4.4 we get that the 5-balloon $B(1,1,2,3,2k)$ is R-AP for every $k \geq 1$. Since every R-AP graph is also OL-AP (Theorem 1.3), we deduce the following.

Corollary 4.5. *The* 5*-balloon* $B(1,1,2,3,2k)$ *is* OL-AP *and* R-AP *for every* $k \geq 1$.

5. Constant Upper Bounds on the Order of the Smallest Branches in OL-AP or R-AP 4- or 5-Balloons

Let $B = (b_1,\dots,b_k)$ be a k-balloon on n vertices with $b_1 \leq \dots \leq b_k$. In this section, we prove the following result.

Theorem 5.1. *If* $B = B(b_1,\dots,b_k)$ *is an* OL-AP 4- *or* 5*-balloon with* $b_1 \leq \dots \leq b_k$*, then* $b_1 \leq 11$.

The proof of this claim reads as follows. When considering the partition of B according to Definition 1.1 for $\lambda = 1$ (that is, a partition of B into two connected subgraphs $B[S]$ and $B[T]$ that are an isolated vertex and an OL-AP graph on $n-1$ vertices, respectively), the possible ways for choosing $B[S]$ are actually quite limited. Moreover, it turns out that, under the assumption that $b_1 \geq 12$, none of these possibilities leads to a correct partition of B. Hence, B cannot be OL-AP for such a value of b_1. Lemmas 5.2 to 5.5 below list the graph structures that cannot be obtained while partitioning B.

Lemma 5.2. *The graph* $B(12^+, 12^+, \overline{x}, \underline{y})$ *is not* OL-AP *for every* $x, y \geq 1$.

Proof. We prove this claim by induction on $x + y$. As a base case, consider $x = y = 1$ and the partial balloon $B = B(12^+, 12^+, \overline{1}, \underline{1})$. By Definition 1.1, recall that B is OL-AP if and only if, for every $\lambda \in \{1, \ldots, n-1\}$, there exists a partition (S, T) of V such that $B[S]$ and $B[T]$ are connected on λ vertices and OL-AP on $n - \lambda$ vertices, respectively. In particular, observe here that B cannot be partitioned in this way for $\lambda = 2$. Indeed, every possible choice of S makes $B[T]$ being either disconnected, a caterpillar $\mathrm{Cat}(13^+, 13^+)$ or $\mathrm{Cat}(11^+, 15^+)$, or a tree with two degree-3 vertices. Since none of these graphs is OL-AP (Theorem 2.1), B is not OL-AP.

To complete the base case, let us now suppose that $x + y = 3$ and denote by B the partial balloon $B(12^+, 12^+, \overline{1}, \underline{2})$. As in the previous base case, one has to observe that B is not OL-AP since it cannot be partitioned in the way specified by Definition 1.1 for $\lambda = 3$. In particular, observe that for every possible choice of S, the graph $B[T]$ is not OL-AP for it is disconnected, a non-caterpillar 3-pode different from $P(2, 4, 6)$, a caterpillar $\mathrm{Cat}(10^+, 16^+)$ or $\mathrm{Cat}(13^+, 13^+)$, or a tree with two degree-3 vertices.

Consider now that the claim holds for every partial balloon $B(12^+, 12^+, \overline{x}, \underline{y})$ such that $x + y \leq k - 1$ for some $k \geq 4$. We now prove that it is also true for a partial balloon $B = B(12^+, 12^+, \overline{x}, \underline{y})$ with $x + y = k$. There are two cases to consider.

- $x > 1$ and $y > 1$: B is not OL-AP since we cannot partition its vertex set as explained above for $\lambda = 1$. Indeed, we must consider $S = \{v_1^3\}$ or $S = \{v_1^4\}$ since otherwise $B[T]$ would be either disconnected, or isomorphic to a large 3-pode or a tree with two degree-3 vertices. But for these two choices of S, the remaining graph $B[T]$ is isomorphic to a partial balloon $B(12^+, 12^+, \overline{x'}, \underline{y'})$ with $x' + y' = x + y - 1 \leq k - 1$, which is not OL-AP by the induction hypothesis.

- $x = 1$ and $y > 2$: we want to partition B as previously for $\lambda = 2$. For the same reasons as above, we have to consider $S = \{v_1^4, v_2^4\}$. But then $B[T]$ is isomorphic to $B(12^+, 12^+, \overline{x}, \underline{y-2})$, which is not OL-AP according to the induction hypothesis. Hence, B is not OL-AP. These arguments hold analogously when $x > 2$ and $y = 1$. ■

Since the proofs of Lemmas 5.3 to 5.5 are quite similar to the one of Lemma 5.2, the reader is referred to Appendix B for in-depth proofs on these statements.

Lemma 5.3. *The graph* $B(12^+, 12^+, 12^+, \overline{x}, \underline{y})$ *is not* OL-AP *for every* $x, y \geq 1$.

Lemma 5.4. *The graph* $B(12^+, 12^+, 12^+, \overline{x})$ *is not* OL-AP *for every* $x \geq 1$.

Lemma 5.5. *The graph* $B(12^+, 12^+, 12^+, 12^+, \overline{x})$ *is not* OL-AP *for every* $x \geq 1$.

Using Lemmas 5.2 to 5.5, we now prove Theorem 5.1.

***Proof of Theorem* 5.1.** Let $B = B(12^+, 12^+, 12^+, 12^+)$ be a 4-balloon. B is not OL-AP since its vertex set cannot be partitioned in the way specified by Definition 1.1 for $\lambda = 1$. Indeed, for every choice of S, the graph $B[T]$ is not OL-AP since it is either a tree with maximum degree 4 or a partial balloon which is not OL-AP by Lemma 5.3 or 5.4. It follows that an OL-AP 4-balloon must have a branch of order at most 11.

Now let $B = B(12^+, 12^+, 12^+, 12^+, 12^+)$ be a 5-balloon. Similarly as before, B is not OL-AP since there does not exist a partition of its vertex set respecting Definition 1.1 for $\lambda = 1$. Indeed, every possible choice of S makes $B[T]$ being either a tree with maximum degree 5, a partial balloon which is not OL-AP according to Lemma 5.5, or a partial 6-balloon. In the latter case, observe that $B[T]$ cannot be OL-AP since otherwise there would exist, by Observation 2.4, an OL-AP 6-balloon contradicting Theorem 3.1. Hence, a 5-balloon cannot be OL-AP when its smallest branch has order at least 12. ■

Since every OL-AP graph is also R-AP (Theorem 1.3), Theorem 5.1 directly implies that R-AP balloons with four or five branches have their smallest branch of order at most 11 too. However, using the fact that R-AP caterpillars are generally smaller than OL-AP caterpillars (see Theorems 2.1 and 2.2), one can easily derive Lemmas 5.2 to 5.5 above for R-AP partial balloons to get a better upper bound on the order of the smallest branch of a R-AP 4- or 5-balloon. The proof of this statement is omitted in this work since it is very similar to the proof of Theorem 5.1.

Theorem 5.6. *If* $B = B(b_1, \ldots, b_k)$ *is a* R-AP 4- *or* 5-*balloon with* $b_1 \leq \cdots \leq b_k$, *then* $b_1 \leq 7$.

One can also get a similar constant upper bound on the order of the second smallest branch in a R-AP 4- or 5-balloon B, that is that $b_2 \leq 39$. The main argument in our proof of Theorem 5.6 is that partial 4-balloons of the form $B(8^+, 8^+, \overline{x}, \underline{y})$ are generally not R-AP. Because of that fact, plenty of other partial balloons cannot be R-AP too, and the result follows. Such a statement is also true when considering that $b_1 \leq 7$ and $b_2 \geq 40$. Indeed, most of partial balloons $B(7^-, 40^+, \overline{x}, \underline{y})$ cannot be R-AP because they cannot be partitioned into two R-AP subgraphs of order $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ according to Observation 2.3. In general, one of these two subgraphs has to be isomorphic to a caterpillar $\mathrm{Cat}(a, b)$ with a and b being greater than the values given by Theorem 2.2. Once again, the proof of the following claim is omitted here, but it can be obtained by deriving Lemmas 5.2 to 5.5 adequately.

Theorem 5.7. *If* $B = B(b_1, \ldots, b_k)$ *is a* R-AP 4- *or* 5-*balloon with* $b_1 \le \cdots \le b_k$, *then* $b_2 \le 39$.

Unfortunately, we did not manage to derive a similar constant upper bound on b_2 when B is an OL-AP 4- or 5-balloon. When partitioning such a graph regarding Definition 1.1, the recursive property only concerns one of the two induced subgraphs. Hence, it appears tricky to find a constant value $c \ge 1$ such that partial balloons of the form $B(11^-, c^+, \overline{x}, \underline{y})$ are generally not OL-AP. Because we cannot find such a class of non OL-AP partial balloons, we cannot derive our proof scheme as it was done before and get a bound on b_2 for OL-AP 4- or 5-balloons, if such exists. So we leave the following question unanswered.

Question 5.8. *Is there a positive constant* $c \ge 11$ *such that, for every* OL-AP 4- *or* 5-*balloon* $B = B(b_1, \ldots, b_k)$ *with* $b_1 \le \cdots \le b_k$, *we have* $b_2 \le c$?

Something more can be deduced from the previous bounds. In what follows, let us denote by $LP(G)$ the length of the longest path in a given graph G. Observe that if $B = B(b_1, \ldots, b_k)$ is a balloon with $k \ge 4$ such that $b_1 \le \cdots \le b_k$, then $LP(B) = b_k + b_{k-1} + b_{k-2} + 1$. Therefore, thanks to Theorems 5.1, 5.6 and 5.7, we get the following result on OL-AP or R-AP balloons.

Corollary 5.9. *Let* B *be a* k-*balloon of order* n.

- *If* B *is* OL-AP *and* $k = 4$, *then* $LP(B) \ge n - 12$.
- *If* B *is* R-AP *and* $k = 4$, *then* $LP(B) \ge n - 8$.
- *If* B *is* R-AP *and* $k = 5$, *then* $LP(B) \ge n - 47$.

Such a result relates to the following question, considered in [6].

Question 5.10. *Is there an absolute constant* $c \ge 1$ *such that, for every* OL-AP (*or* R-AP) *graph* G, *we have* $LP(G) \ge |V(G)| - c$?

In other words, we wonder whether there is a $c \ge 1$, such that the longest path of any OL-AP or R-AP graph goes through almost all vertices, that is, all but at most c. This was disproved for R-AP (and, hence, OL-AP) graphs in [6]. Corollary 5.9 implies that such a c exists for the classes of OL-AP 4-balloons, and R-AP 4- and 5-balloons. If the bound c on b_2 mentioned in Question 5.8 exists, the same conclusion would also hold for the case of OL-AP 5-balloons.

6. Structural Properties of OL-AP or R-AP Graphs with Connectivity 2

As mentioned in the introduction section, properties of OL-AP or R-AP graphs with connectivity 2 can be deduced from properties of OL-AP or R-AP balloons, respectively. In particular, the results on OL-AP or R-AP balloons we pointed out along Sections 3 to 5 are extendable in the following way.

Corollary 6.1. *Let G be a graph with connectivity 2, u and v be two vertices forming a cut pair of G, and $b_1 \leq \cdots \leq b_k$ be the numbers of vertices of the $k \geq 2$ connected components of $G - \{u, v\}$. If G is* OL-AP *or* R-AP*, then the following conditions hold:*

- *$k \leq 5$,*
- *b_k can be arbitrarily large,*
- *if G is* R-AP *and $k \in \{4, 5\}$, then $b_1 \leq 7$ and $b_2 \leq 39$,*
- *if G is* OL-AP *and $k \in \{4, 5\}$, then $b_1 \leq 11$.*

The first item of Corollary 6.1 follows directly from Theorem 3.1, the second item is derived from Corollaries 4.3 and 4.5, while the third and fourth items result from Theorems 5.1, 5.6 and 5.7.

Since 2- and 3-balloons are always OL-AP and R-AP because they are traceable, the only structural property we can derive is that if G is disconnected into only two or three components after the removal of u and v, then these components can all be arbitrarily large.

Notice that Corollary 5.9 cannot be extended to OL-AP or R-AP graphs with connectivity 2 as previously. Indeed, there is no direct analogy between the longest path in a balloon $B = B(b_1, \ldots, b_k)$ with $b_1 \leq \cdots \leq b_k$, and the longest path in a graph G that has a cut pair $\{u, v\}$ whose removal leads to k components of order $b_1, \ldots, b_k$, respectively. As an illustration, suppose that the component of order b_k in $G - \{u, v\}$ is a $(b_k - 2)$-balloon $B(1, \ldots, 1)$ whose roots are connected to u and v in G. Depending on the structure of G, the longest path of G may not pass through its component of order b_k. In comparison, the longest path of B generally has to go along its kth branch.

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A. Details for the Proof of Lemma 2.9

Let $B = B(b_1, \overline{b_2}, \overline{b_3}, \underline{b_4}, \underline{b_5})$, where $b_2 \leq b_3$, and $b_2 \leq b_4 \leq b_5$. In case $b_2 = b_4$, we assume $b_3 \leq b_5$. We end the proof of Lemma 2.9 by showing that, no matter what are b_1, b_2, b_3, b_4, b_5, there is always a $\lambda \in \{1, \dots, n-1\}$ such that, for every partition (S, T) of $V(B^+)$ with $|S| = \lambda$ and $B^+[S]$, $B^+[T]$ being connected, one of the following conditions is necessarily met.

1. $B^+[T] \in \mathcal{B}_2^2(1)$ – hence $B^+[T]$ is not OL-AP by the induction hypothesis;
2. $B^+[T] \in \mathcal{B}_1^2(1) \cup \mathcal{B}_2^1(1)$, and $B[T]$ is a tree not listed in Theorem 2.1 – hence $B^+[T]$ is not OL-AP by Lemma 2.8;
3. $r_1, r_2 \in T$, and $B^+[T]$ is a tree with $\Delta(B^+[T]) = 4$, or in which both r_1 and r_2 have degree at least 3 – hence not OL-AP according to Theorem 2.1.

We basically proceed by case analysis, that is consider the possible values of b_1, b_2, b_3, b_4, b_5. For every case, we indicate below the appropriate choices of λ only, as well as, when applicable, the sets of graphs in $\mathcal{B}_1^2(1) \cup \mathcal{B}_2^1(1)$ (second situation above) which appear as $B^+[T]$ for the indicated values of λ.

(1) $b_2 \geq 2$: $\lambda = 1$.

(2) $b_2 = 1$.

(2.1) $b_3 = 1$.

(2.1.1) $b_4 \geq 3$: $\lambda = 2$.

(2.1.2) $b_4 = 2$.

(2.1.2.1) $b_5 \in \{4, 6\}$.

(2.1.2.1.1) $b_1 = 8 - b_5$: $\lambda = 3 \to P(2, 8 - b_5, b_5) \in \{P(2, 4, 4), P(2, 2, 6)\}$.

(2.1.2.1.2) $b_1 \neq 8 - b_5$: $\lambda = 1 \to P(2, b_1 + 2, b_5) \neq P(2, 4, 6)$.

(2.1.2.2) $b_5 \notin \{4, 6\}$: $\lambda = 1 \to P(2, b_1 + 2, b_5) \neq P(2, 4, 6)$.

(2.1.3) $b_4 = 1$.

(2.1.3.1) $b_5 \neq 2$: $\lambda = 2$.

(2.1.3.2) $b_5 = 2$.

(2.1.3.2.1) $b_1 \equiv 1 \pmod 2$: $\lambda = 2 \to \mathrm{Cat}(2, b_1 + 3)$.

(2.1.3.2.2) $b_1 \equiv 0 \pmod 2$.

(2.1.3.2.2.1) $b_1 = 6p + 2$: $\lambda = 3 \to \mathrm{Cat}(3, 6p + 3)$.

(2.1.3.2.2.2) $b_1 = 6p + 4$: $\lambda = 5 \to \mathrm{Cat}(2, 6p + 4)$, $\mathrm{Cat}(3, 6p + 3)$.

(2.1.3.2.2.3) $b_1 = 6p + 6$: $\lambda = 7 \to \mathrm{Cat}(2, 6p + 4)$, $\mathrm{Cat}(3, 6p + 3)$.

(2.2) $b_3 = 2$.

(2.2.1) $b_4 = 1$.

(2.2.1.1) $b_5 \geq 4$: $\lambda = 3$.

(2.2.1.2) $b_5 = 3$.

(2.2.1.2.1) $b_1 \equiv 1 \pmod 2$: $\lambda = 4 \to \text{Cat}(4, b_1 + 1)$.

(2.2.1.2.2) $b_1 \equiv 0 \pmod 2$.

(2.2.1.2.2.1) $b_1 = 6p + 2$: $\lambda = 5 \to \text{Cat}(3, 6p + 3)$, $\text{Cat}(4, 6p + 2)$.

(2.2.1.2.2.2) $b_1 = 6p + 4$: $\lambda = 7 \to \text{Cat}(3, 6p + 3)$, $\text{Cat}(4, 6p + 2)$.

(2.2.1.2.2.3) $b_1 = 6p + 6$: $\lambda = 3 \to \text{Cat}(3, 6p + 9)$.

(2.2.1.3) $b_5 = 2$: $\lambda = 3$.

(2.2.2) $b_4 = 2$.

(2.2.2.1) $b_5 \geq 4$: $\lambda = 3$.

(2.2.2.2) $b_5 = 3$.

(2.2.2.2.1) $b_1 \geq 2$: $\lambda = 4 \to P(2, 3, b_1)$.

(2.2.2.2.2) $b_1 = 1$: $\lambda = 2 \to P(2, 3, 3)$, $\text{Cat}(3, 6)$.

(2.2.2.3) $b_5 = 2$: $\lambda = 3$.

(2.2.3) $b_4 \geq 3$: $\lambda = 1 \to P(b_4, b_5, b_1 + 3)$ with $\min\{b_4, b_5, b_1 + 3\} \geq 3$.

(2.3) $b_3 \geq 3$.

(2.3.1) $b_4 = 1$: $\lambda = 2$.

(2.3.2) $b_4 = 2$.

(2.3.2.1) $b_5 \geq 4$.

(2.3.2.1.1) $b_3 \geq 4$: $\lambda = 3$.

(2.3.2.1.2) $b_3 = 3$.

(2.3.2.1.2.1) $b_5 \geq 5$: $\lambda = 4$.

(2.3.2.1.2.2) $b_5 = 4$.

(2.3.2.1.2.2.1) $b_1 = 2$: $\lambda = 2 \to \text{Cat}(4, 8)$.

(2.3.2.1.2.2.2) $b_1 \neq 2$: $\lambda = 1 \to P(2, 4, b_1 + 4)$.

(2.3.2.2) $b_5 \in \{2, 3\}$: $\lambda = 1 \to P(2, b_1 + 4, b_5)$.

(2.3.3) $b_4 \geq 3$: $\lambda = 1 \to P(b_1 + 1 + b_3, b_4, b_5)$.

B. PROOFS OF LEMMAS 5.3 TO 5.5

This appendix gathers all the proofs of Lemmas 5.3 to 5.5 of Section 5, as well as some intermediate lemmas needed to prove these statements. It is worth mentioning that these proofs often make implicit use of the full characterization of OL-AP trees (Theorem 2.1) and the two sufficient conditions for a graph to be OL-AP given by Observations 2.3 and 2.4. In all these proofs, it is assumed that

$x \leq y$ or $x \leq y \leq z$ generally holds when the corresponding elements have been introduced. Given a graph G and an integer $\lambda \in \{1, \ldots, n-1\}$, an OL-AP-*partition of* G *for* λ is a partition (S, T) of V such that $G[S]$ and $G[T]$ are connected on λ vertices and OL-AP on $n - \lambda$ vertices, respectively. According to Definition 1.1, the graph G is OL-AP if and only if either G is an isolated vertex or G admits an OL-AP-partition for every $\lambda \in \{1, \ldots, n-1\}$.

Lemma B.1. *The graph* $B(12^+, 12^+, \overline{x}, \overline{y})$ *is not* OL-AP *for every* $x \geq 1$ *and* $y \geq 10$.

Proof. Let us prove this claim by induction on $x+y$ as we did to prove Lemma 5.2. As a base case, let us consider the graph $B = B(12^+, 12^+, \overline{1}, \overline{10})$. Observe that B is not OL-AP since it does not admit an OL-AP-partition for 11. Indeed, every possible choice of 11 vertices as S inducing a connected subgraph of B makes $B[T]$ being either disconnected, a caterpillar $\mathrm{Cat}(11, 15^+)$, or a tree with maximum degree 4. For similar reasons, observe that neither $B(12^+, 12^+, \overline{1}, \overline{11})$ nor $B(12^+, 12^+, \overline{2}, \overline{10})$ are OL-AP since they do not admit an OL-AP-partition for 12 and 11, respectively.

Let us now suppose that this lemma holds whenever $x + y \leq k - 1$ for some $k \geq 13$, and consider a graph $B = B(12^+, 12^+, \overline{x}, \overline{y})$ such that $x + y = k$. We claim that there exists a $\lambda \in \{1, \ldots, n-1\}$ such that B does not admit an OL-AP-partition for λ, and thus that B is not OL-AP.

- $x > 1$ and $y > 10$: under these conditions, there does not exist an OL-AP-partition of B for 1. Indeed, every possible choice for S which does not make $B[T]$ being disconnected makes this subgraph being isomorphic to either a non-caterpillar 3-pode different from $P(2, 4, 6)$, a tree with maximum degree 4, or a graph not OL-AP according to the induction hypothesis.
- $x = 1$ and $y > 11$: observe that there does not exist an OL-AP-partition of B for 2, since every coherent choice for S makes $B[T]$ being disconnected or isomorphic to either a caterpillar $\mathrm{Cat}(13^+, 13^+)$, a tree with maximum degree 4, or a partial balloon which is not OL-AP by the induction hypothesis.
- $x > 2$ and $y = 10$: once again, B does not admit an OL-AP-partition for 11 since every choice of 11 vertices as S inducing a connected subgraph of B makes $B[T]$ being either disconnected, a tree with maximum degree 4, or a non-OL-AP 3-pode. ∎

Lemma B.2. *The graph* $B(12^+, 12^+, \overline{x}, \overline{y}, \underline{z})$ *is not* OL-AP *for every* $x, y, z \geq 1$.

Proof. We prove this claim by induction on $x + y + z$. Let us first suppose that $x = y = z = 1$ and consider the associated graph $B = B(12^+, 12^+, \overline{1}, \overline{1}, \underline{1})$. Once again, B is not OL-AP since there does not exist an OL-AP-partition of B for 2. Indeed, every possible choice for S makes $B[T]$ being either disconnected, or isomorphic to either a tree with maximum degree 4 or a tree having two degree-3 vertices.

To complete the base case, observe that $B(12^+, 12^+, \overline{1}, \overline{2}, \underline{1})$ and $B(12^+, 12^+, \overline{1}, \overline{1}, \underline{2})$ are not OL-AP since they do not admit an OL-AP-partition for 3. Indeed, for every coherent choice of S, the subgraph $B[T]$ is disconnected, or isomorphic to either a tree with maximum degree 4, a tree having two degree-3 vertices, or a non-caterpillar 3-pode different from $P(2, 4, 6)$.

Suppose now that this claim holds whenever $x+y+z \leq k-1$ for some $k \geq 5$, and consider a balloon $B = B(12^+, 12^+, \overline{x}, \overline{y}, \underline{z})$ where $x+y+z = k$. Once again, we consider two main cases.

- $z > 1$: in this case, B is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, observe that removing one vertex from B makes the remaining subgraph being disconnected, isomorphic to a tree with maximum degree 4 or two degree-3 vertices, or isomorphic to a partial balloon which is not OL-AP according to the induction hypothesis or Lemma 5.2.
- $z = 1$: once again, B is not OL-AP under this condition since it cannot be OL-AP-partitioned for 2. Indeed, for every coherent choice as S, the remaining graph $B[T]$ is indeed not connected, a tree with maximum degree 4 or two degree-3 vertices, or a partial balloon which is not OL-AP according to our induction hypothesis or previous Lemma 5.2. ■

***Proof of Lemma* 5.3.** Once more, let us prove this claim by induction on $x+y$. Consider first that $x = y = 1$ and let $B = B(12^+, 12^+, 12^+, \overline{1}, \underline{1})$. Observe that B is not OL-AP because it cannot be OL-AP-partitioned for 2. Indeed, every possible choice for S makes $B[T]$ being disconnected, isomorphic to a tree with maximum degree 4, to a partial balloon which is not OL-AP by Lemma B.2, or to a partial 6-balloon. In the latter case, such a graph cannot be OL-AP since otherwise there would exist an OL-AP 6-balloon contradicting Theorem 3.1.

Additionally, observe that $B = B(12^+, 12^+, 12^+, \overline{1}, \underline{2})$ cannot be OL-AP-partitioned for 3. Indeed, for every coherent choice for S, the subgraph $B[T]$ is not OL-AP for the same reasons as in the previous case. Hence, B is not OL-AP.

We now suppose that this claim holds for every $x+y \leq k-1$ for some $k \geq 4$, and consider a partial balloon $B = B(12^+, 12^+, 12^+, \overline{x}, \underline{y})$ where $x+y = k$. Let us take the following two cases in consideration to show that B is not OL-AP.

- $x > 1$ and $y > 1$: notice that, in this situation, B cannot be OL-AP-partitioned for 1. Indeed, for some similar reasons as the ones we used for the base cases, we have to consider $S = \{v_1^4\}$ or $S = \{v_1^5\}$. But in both cases, $B[T]$ cannot be OL-AP by the induction hypothesis.
- $x = 1$ and $y > 2$: once again, observe that B cannot be OL-AP-partitioned for 2. Indeed, observe that we must consider $S = \{v_1^5, v_2^5\}$ since otherwise there would exist an OL-AP 6-balloon, an OL-AP tree having maximum

degree 4, or a graph contradicting Lemma B.2. But for this choice of S, we have $B[T] = B(12^+, 12^+, 12^+, \overline{1}, \underline{y-2})$ which is not OL-AP according to our induction hypothesis. ■

***Proof of Lemma* 5.4.** Once again, this claim is proved by induction on x. Let us first suppose that $x = 1$ and let B be the partial balloon $B(12^+, 12^+, 12^+, \overline{1})$. This time, B is not OL-AP since it cannot be OL-AP-partitioned for 2. Indeed, for every possible choice of S, the remaining graph $B[T]$ is not OL-AP since it is disconnected, isomorphic to a tree with maximum degree 4, to a non-caterpillar 3-pode different from $P(2,4,6)$ or to a partial balloon which is not OL-AP by Lemma 5.2, B.1 or B.2.

Let us now suppose that this claim holds for every $x \leq k-1$ and some $k \geq 2$. To complete the proof, observe that a graph $B = B(12^+, 12^+, 12^+, \overline{k})$ is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every choice of S, the subgraph $B[T]$ is not OL-AP according to the induction hypothesis, or because of one reason used for the base case. ■

Lemma B.3. *The graph* $B(12^+, 12^+, \overline{x}, \overline{y}, \overline{z})$ *is not* OL-AP *for every* $x, y, z \geq 1$.

Proof. We prove this claim by induction on $x+y+z$. First, let us suppose that $x = y = z = 1$ and consider the partial balloon $B = B(12^+, 12^+, \overline{1}, \overline{1}, \overline{1})$. Notice that B is not OL-AP since it cannot be OL-AP-partitioned for 2. Indeed, every choice of S implies that $B[T]$ is either disconnected or isomorphic to a tree with maximum degree at least 4. Analogously, observe that neither $B(12^+, 12^+, \overline{1}, \overline{1}, \overline{2})$ nor $B(12^+, 12^+, \overline{1}, \overline{2}, \overline{2})$ are OL-AP since they cannot be OL-AP-partitioned for 3, and that $B(12^+, 12^+, \overline{1}, \overline{1}, \overline{3})$ is not OL-AP as it cannot be OL-AP-partitioned for 2.

Suppose now that this claim holds by induction whenever $x+y+z \leq k-1$ for a $k \geq 6$, and consider a partial balloon $B = B(12^+, 12^+, \overline{x}, \overline{y}, \overline{z})$ where $x+y+z = k$. We distinguish the following two main cases depending on x, y and z.

- $x > 1$, $y > 1$ and $z > 1$: suppose we want to OL-AP-partition B for 1. Then, we must consider $S = \{v_1^3\}$, $S = \{v_1^4\}$ or $S = \{v_1^5\}$ since, for every other choice of S, the remaining graph $B[T]$ is either disconnected or isomorphic to a tree having maximum degree at least 4. But in any of these three choices for S, the subgraph $B[T]$ is not OL-AP by the induction hypothesis. Thus, B is not OL-AP.
- $x = 1$: let $\alpha = \min(\{2,3,4\} \setminus \{y, z\})$. In this situation, B cannot be OL-AP for the same reason as above but for an OL-AP-partition of B for α. Indeed, for every coherent choice of S, the remaining graph $B[T]$ is not OL-AP either according to the induction hypothesis, or because it is isomorphic to a non-connected graph or a tree with maximum degree at least 4. ■

Lemma B.4. *The graph* $B(12^+, 12^+, 12^+, \overline{x}, \overline{y})$ *is not* OL-AP *for every* $x, y \geq 1$.

Proof. Once again, we prove this claim by induction on $x+y$. We first suppose that $x=y=1$ and let $B=B(12^+,12^+,12^+,\overline{1},\overline{1})$. Similarly as in the proofs of the previous lemmas, B is not OL-AP because it cannot be OL-AP-partitioned for 2. Indeed, for every possible choice of S, the remaining graph $B[T]$ is not connected, a tree with maximum degree 5, a partial balloon not OL-AP according to Lemma B.2 or B.3, or a partial 6-balloon. For the latter case, recall that a partial 6-balloon cannot be OL-AP since otherwise there would exist a 6-balloon contradicting Theorem 3.1. Similarly, observe that $B(12^+,12^+,12^+,\overline{1},\overline{2})$ is not OL-AP since it cannot be OL-AP-partitioned for 3.

We finally suppose that the induction hypothesis is true whenever $x+y \leq k-1$ for some $k \geq 4$, and consider a partial balloon $B=B(12^+,12^+,12^+,\overline{x},\overline{y})$ where $x+y=k$. We distinguish two main cases, depending on the values of x and y, to prove that B is not OL-AP.

- $x>1$ and $y>1$: in this situation, B is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every choice of S, the remaining graph $B[T]$ is not OL-AP either for one of the reasons used for the base cases or according to the induction hypothesis.
- $x=1$ and $y>2$: the above arguments hold to prove that B cannot be OL-AP-partitioned for 2. Thus, B is not OL-AP. ∎

***Proof of Lemma* 5.5.** Let us prove this claim by induction on x. We first suppose that $x=1$ and consider the OL-AP-partition of $B=B(12^+,12^+,12^+,12^+,\overline{1})$ for 2. Such a partition does not exist since for every choice of S, the remaining graph $B[T]$ cannot be OL-AP. Indeed, this subgraph is either not connected, a tree with maximum degree at least 4, a partial balloon which cannot be OL-AP according to Lemma 5.3 or B.4, or a partial 6-balloon. In the latter case, such a graph cannot be OL-AP since otherwise there would exist a graph contradicting Theorem 3.1.

Suppose now that $B(12^+,12^+,12^+,12^+,\overline{x})$ is not OL-AP for every $x \leq k-1$ and some $k \geq 2$, and consider a graph $B=B(12^+,12^+,12^+,12^+,\overline{k})$. Once again, B cannot be OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every possible choice for S, the graph $B[T]$ cannot be OL-AP either according to the induction hypothesis or because of one of the reasons used for the base case. ∎