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SHARP UPPER BOUNDS FOR GENERALIZED EDGE-CONNECTIVITY OF PRODUCT GRAPHS

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Abstract

The generalized k-connectivity $\kappa_k(G)$ of a graph G was introduced by Hager in 1985. As a natural counterpart of this concept, Li *et al.* in 2011 introduced the concept of generalized k-edge-connectivity which is defined as $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}$, where $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $S \subseteq V(T_i)$ for $1 \leq i \leq \ell$. In this paper, we study the generalized edge-connectivity of product graphs and obtain sharp upper bounds for the generalized 3-edge-connectivity of Cartesian product graphs and strong product graphs. Among our results, some special cases are also discussed.

Keywords: generalized edge-connectivity, Cartesian product, strong product, lexicographic product.

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1. Introduction

We refer to book [1] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$ be the set of vertices, the set of edges of G, respectively. For $X \subseteq V(G)$, we denote by $G \setminus X$ the subgraph obtained by deleting from G the vertices of X together with the edges incident with them. For a set S, we use $|S|$ to denote its size. We use P_n, C_m and K_ℓ to denote a path of order n, a cycle of order m and a complete graph of order ℓ, respectively.

Connectivity is one of the most basic concepts in graph theory, both in combinatorial sense and in algorithmic sense. The *connectivity* of G, written $\kappa(G)$, is the minimum size of a vertex set $X \subseteq V(G)$ such that $G \setminus X$ is disconnected or has only one vertex. This definition is called the *cut-version* definition of the connectivity. A well-known theorem of Menger provides an equivalent definition, which can be called the *path-version* definition of the connectivity. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x, y) : x, y \in V(G), x \neq y\}$ is defined to be the connectivity of G.

Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings and so people tried to generalize this concept.

The cut-version definition of the connectivity does not concern the number of components of $G \setminus X$. Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n-1}$ and the path P_n $(n \geq 3)$ are both trees of order n and therefore have connectivity 1, but the deletion of a cut-vertex from $K_{1,n-1}$ produces a graph with $n-1$ components while the deletion of a cut-vertex from P_n produces only two components. Chartrand *et al.* [2] generalized the cut-version definition of the connectivity as follows: For an integer k $(k \geq 2)$ and a graph G of order n $(n \geq k)$, the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G produces a graph with at least k components or a graph with fewer than k vertices. By definition, we clearly have $\kappa'_2(G) = \kappa(G)$. Thus, the concept of k-connectivity could be seen as a generalization of the classical connectivity. For more details about this topic, we refer to [2, 4, 18, 19, 26, 28, 29].

The generalized k-connectivity $\kappa_k(G)$ of a graph G which was mentioned by Hager [5] in 1985 is a natural generalization of the path-version definition of the connectivity. Let k be an integer with $2 \leq k \leq n$, when $n = |V(G)| \geq 2$ is the order of G. For a set S of k vertices of G, let $\kappa(S)$ denote the largest integer ℓ such that G contains ℓ edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ with $V(T_i) \cap V(T_j) = S$ for $1 \leq i < j \leq \ell$. Note that these trees must be vertex-disjoint in $G \setminus S$. A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called a *set of internally disjoint trees connecting S*. The *generalized k-connectivity* of G is defined as

$$\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G),\ |S| = k\}.$$

Hence, $\kappa_2(G) = \kappa(G)$, and $\kappa_k(G) = 0$ when G is disconnected. As a natural counterpart of the generalized connectivity, recently Li *et al.* [15] introduced the following concept of generalized edge-connectivity. Let $\lambda(S)$ denote the largest integer ℓ such that G contains ℓ pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ with $S \subseteq V(T_i)$ for $1 \leq i \leq \ell$. The *generalized k-edge-connectivity* of G is defined as

$$\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G),\ |S| = k\}.$$

Hence, $\lambda_2(G) = \lambda(G)$ is the usual edge-connectivity, and $\lambda_k(G) = 0$ when G is disconnected. Clearly, we have $\kappa_k(G) \leq \lambda_k(G)$. The generalized connectivity and edge-connectivity are also called *tree connectivities* in the literature. There are many results on this type of generalized connectivity, see ([3, 5, 9, 10, 12–16, 22–25, 29, 30]. The reader is also referred to a recent survey [11] on the state-of-the-art of research on tree connectivity and its applications.

Products of graphs occur naturally in discrete mathematics as tools in combinatorial constructions, they give rise to important classes of graphs and deep structural problems. Many researchers have investigated the topic of graph products in the past several decades, such as [6, 7, 8, 17, 20, 21, 31, 32].

The *Cartesian product* of two graphs G and H, denoted by $G\Box H$, is defined to have the vertex set $V(G) \times V(H)$ such that (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. The *strong product* of G and H is the graph $G\boxtimes H$ whose vertex set is $V(G)\times V(H)$ and whose edge set is the set of all pairs $(u, u')(v, v')$ such that either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$, or $uv \in E(G)$ and $u'v' \in E(H)$. Clearly, both of these two products are commutative, that is, $G\Box H = H\Box G$ and $G \boxtimes H = H \boxtimes G$. By definition, we also know that the graph $G\Box H$ is a spanning subgraph of the graph $G \boxtimes H$ for any two graphs G and H. The *lexicographic product* of two graphs G and H, written as $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (u, v) and (u', v') of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$.

For the Cartesian product graphs, the exact formula of $\kappa(G\Box H)$ was obtained.

Theorem 1 [17, 21]. *Let G and H be graphs on at least two vertices. Then*

$$\kappa(G\Box H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}.$$

This theorem was first stated by Liouville [17]. However, the proof never appeared. In the meantime, several partial results were obtained until Špacapan [21] provided the proof. Theorem 1 in particular implies the following result of Sabidussi [20].

Theorem 2 [20]. *Let G and H be connected graphs. Then*

$$\kappa(G\Box H) \geq \kappa(G) + \kappa(H).$$

Li, Li and Sun [10] investigated the generalized 3-connectivity of the Cartesian product graphs and obtain the following result which can be seen as an extension of Theorem 2.

Theorem 3 [10]**.** *Let G and H be connected graphs such that $\kappa_3(G) \geq \kappa_3(H)$.*

(a) *If $\kappa_3(G) < \kappa(G)$, then $\kappa_3(G\Box H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

(b) *If $\kappa_3(G) = \kappa(G)$, then $\kappa_3(G\Box H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp.*

Li and Mao derived a sharp upper bound for $\kappa_3(G\Box H)$.

Theorem 4 [13]**.** *Let G and H be two connected graphs. Then $\kappa_3(G\Box H) \leq \min\{\lfloor \frac{4}{3}\kappa_3(G) + r_1 - \frac{4}{3}\lceil \frac{r_1}{2}\rceil\rfloor |V(H)|, \lfloor \frac{4}{3}\kappa_3(H) + r_2 - \frac{4}{3}\lceil \frac{r_2}{2}\rceil\rfloor |V(G)|, \delta(G) + \delta(H)\}$, where $r_1 \equiv \kappa(G) \pmod 4$ and $r_2 \equiv \kappa(H) \pmod 4$. Moreover, the bound is sharp.*

In [23], we obtained the following result for the generalized 3-edge-connectivity of Cartesian product graph.

Theorem 5 [23]**.** *If G and H are connected graphs, then $\lambda_3(G\Box H) \geq \lambda_3(G) + \lambda_3(H)$. Moreover, the bound is sharp.*

For the strong product graphs, with a similar but more complicated argument, we obtained the following result for the generalized 3-edge-connectivity of the strong product graphs.

Theorem 6 [27]**.** *If G and H are two connected graphs, then $\lambda_3(G \boxtimes H) \geq \min\{2\lambda_3(G) + \lambda_3(H), \lambda_3(G) + 2\lambda_3(H)\}$. Moreover, the bound is sharp.*

For the lexicographic product graphs, Li and Mao obtained the following bounds for $\kappa_3(G \circ H)$.

Theorem 7 [13]**.** *Let G and H be two connected graphs. If G is non-trivial and non-complete, then $\kappa_3(G \circ H) \leq \lfloor \frac{4}{3}\kappa_3(G) + r - \frac{4}{3}\lceil \frac{r}{2}\rceil\rfloor |V(H)|$, where $r \equiv \kappa(G) \pmod 4$. Moreover, the bound is sharp.*

Theorem 8 [13]**.** *Let G and H be two connected graphs. Then $\kappa_3(G \circ H) \geq \kappa_3(G)|V(H)|$. Moreover, the bound is sharp.*

Li, Yue, and Zhao studied $\lambda_3(G \circ H)$ and provided both sharp lower and upper bounds.

Theorem 9 [16]**.** *Let G and H be two non-trivial graphs such that G is connected. Then $\lambda_3(H) + \lambda_3(G)|V(H)| \leq \lambda_3(G \circ H) \leq \min\left\{\left\lfloor \frac{4\lambda_3(G)+2}{3}\right\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)|\right\}$. Moreover, both bounds are sharp.*

In this paper, we continue the research on tree connectivities of product graphs and obtain sharp upper bounds for the generalized 3-edge-connectivity of Cartesian product graphs and strong product graphs (Theorems 14 and 16). In Section 4, we also discuss some special graph classes (Propositions 17, 18 and 19).

2. Cartesian Product Graphs

The following result shows that $\lambda_k(G)$ is monotonically decreasing with k for any connected graph G.

Lemma 10 [14]. *Let G be a connected graph of order n. Then $\lambda_k(G) \leq \lambda_{k-1}(G)$ for every integer k with $3 \leq k \leq n$.*

Li, Mao and Sun obtained a sharp lower bound for $\lambda_3(G)$.

Theorem 11 [15]. *Let G be a connected graph having order n and edge-connectivity $\lambda(G) = 4s + r$, where s and r are integers with $s \geq 0$ and $0 \leq r \leq 3$. Then $\lambda_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$ and the bound is sharp. In particular, $\lambda_3(G) \geq \frac{3\lambda(G)-2}{4}$.*

In [31], an exact formula for $\lambda(G \square H)$ was derived.

Theorem 12 [31]. *Let G and H be two graphs with at least two vertices. Then*

$$\lambda(G \square H) = \min\{\lambda(G)|V(H)|, \lambda(H)|V(G)|, \delta(G) + \delta(H)\}.$$

The following result deals with the Cartesian products of connected graphs with minimum degree 1 as well as some special graph classes.

Proposition 13 [24]. *Let G be a connected graph with $\delta(G) = 1$ and order $n \geq 3$.*
(a) *If H is a connected graph with $\delta(G) = 1$ and order $m \geq 3$, then $\lambda_3(G \square H) = 2$.*
(b) *If H is a cycle, then $\lambda_3(G \square H) = 2$.*
(c) *If H is a wheel graph, then $\lambda_3(G \square H) = 3$.*
(d) *If H is a complete graph with order $m \geq 3$, then $\lambda_3(G \square H) = m - 1$.*

The following theorem is one of our main results.

Theorem 14. *If G and H are graphs having order at least 2, then $\lambda_3(G \square H) \leq \min\big\{ \big\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3} \lceil \frac{r_1}{2} \rceil \big\rfloor |V(H)|, \big\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3} \lceil \frac{r_2}{2} \rceil \big\rfloor |V(G)|, \delta(G) + \delta(H) \big\}$, where $r_1 \equiv \lambda(G) \pmod 4$ and $r_2 \equiv \lambda(H) \pmod 4$. Moreover, the bound is sharp.*

Proof. By Theorem 11, if $\lambda(G) = 4s + r_1$, then $\lambda_3(G) \geq 3s + \lceil \frac{r_1}{2} \rceil$, where $r_1 \in \{0, 1, 2, 3\}$, and so

$$\lambda_3(G) \geq 3\frac{\lambda(G) - r_1}{4} + \left\lceil \frac{r_1}{2} \right\rceil = \frac{3}{4}\lambda(G) - \frac{3}{4}r_1 + \left\lceil \frac{r_1}{2} \right\rceil,$$

where $r_1 \equiv \lambda(G) \pmod 4$. Hence,

$$\lambda(G) \leq \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil.$$

Similarly, for a connected graph H, we have

$$\lambda(H) \leq \frac{4}{3}\lambda_3(H) + r_1 - \frac{4}{3}\left\lceil \frac{r_2}{2} \right\rceil,$$

where $r_2 \equiv \lambda(H) \pmod 4$.

Furthermore, by Lemma 10 and Theorem 12, we have

$$\lambda_3(G\square H) \leq \lambda(G\square H) = \min\{\lambda(G)|V(H)|, \lambda(H)|V(G)|, \delta(G)+\delta(H)\} \leq \lambda_0,$$

where $\lambda_0 = \min\left\{\left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2}\right\rceil\right\rfloor |V(H)|, \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2}\right\rceil\right\rfloor |V(G)|, \delta(G)+\delta(H)\right\}$.

For the sharpness of this bound, we consider two connected graphs G and H with $\delta(G) = \delta(H) = 1$ and orders at least 3. Clearly, $\lambda_3(G) = \lambda_3(H) = \lambda(G) = \lambda(H) = 1$, $r_1 = r_2 = 1$. Then $\lambda_3(G\square H) \leq \min\{\left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2}\right\rceil\right\rfloor |V(H)|, \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2}\right\rceil\right\rfloor |V(G)|, \delta(G)+\delta(H)\} \leq \min\{|V(H)|, |V(G)|, 2\} = 2$. By (a) of Proposition 13, the bound of our theorem is sharp. ■

Note that the minimum in Theorem 14 can be realized by any of three terms. From the example in Theorem 14, if G and H are connected graphs with $\delta(G) = \delta(H) = 1$ and orders at least 3, then

$$\min\left\{\left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2}\right\rceil\right\rfloor |V(H)|, \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2}\right\rceil\right\rfloor |V(G)|\right\} > \delta(G)+\delta(H)$$

in this case. Now let H be a complete graph of order at least 3, G be a connected graph with $V(G) = A \cup B$ and $|A|, |B| > 2$ such that $G[A]$ and $G[B]$ are complete graphs and there is exactly one edge between A and B. Now $\lambda_3(G) = \lambda(G) = 1$ and $r_1 = 1$. We then have

$$\begin{aligned}\left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2}\right\rceil\right\rfloor |V(H)| &= |V(H)| \\ &\leq \min\{|A|-1, |B|-1\} + (|V(H)|-1) - 1 \\ &= \delta(G) + \delta(H) - 1.\end{aligned}$$

In this case, we therefore have

$$\min\left\{\left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2}\right\rceil\right\rfloor |V(H)|, \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2}\right\rceil\right\rfloor |V(G)|\right\} < \delta(G)+\delta(H).$$

3. Strong Product Graphs

In [32], Yang and Xu obtained the exact formula for $\lambda(G \boxtimes H)$.

Theorem 15 [32]**.** *If both G and H are connected graphs, then* $\lambda(G \boxtimes H) = \min \{\lambda(G)(|V(H)| + 2|E(H)|), \lambda(H)(|V(G)| + 2|E(G)|), \delta(G) + \delta(H) + \delta(G)\delta(H)\}$.

Applying an argument similar to that of Theorem 14 for strong product graphs, we have the following result.

Theorem 16. *Let G and H be two connected graphs. Then* $\lambda(G \boxtimes H) \leq \min \left\{ \left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil \right\rfloor (|V(H)| + 2|E(H)|), \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2} \right\rceil \right\rfloor (|V(G)| + 2|E(G)|), \delta(G) + \delta(H) + \delta(G)\delta(H) \right\}$. *Moreover, the bound is sharp.*

Proof. According to the proof of Theorem 14, we find that for any two connected graphs G and H, we have

$$\lambda(G) \leq \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil, \lambda(H) \leq \frac{4}{3}\lambda_3(H) + r_1 - \frac{4}{3}\left\lceil \frac{r_2}{2} \right\rceil,$$

where $r_1 \equiv \lambda(G) \pmod 4$ and $r_2 \equiv \lambda(H) \pmod 4$.

Furthermore, by Lemma 10 and Theorem 15, we have $\lambda_3(G \boxtimes H) \leq \lambda(G \boxtimes H) = \min\{\lambda(G)(|V(H)| + 2|E(H)|), \lambda(H)(|V(G)| + 2|E(G)|), \delta(G) + \delta(H) + \delta(G)\delta(H)\} \leq \min \left\{ \left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil \right\rfloor (|V(H)| + 2|E(H)|), \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2} \right\rceil \right\rfloor (|V(G)| + 2|E(G)|), \delta(G) + \delta(H) + \delta(G)\delta(H) \right\}$,

For the sharpness of this bound, we consider two connected graphs G and H with $\delta(G) = \delta(H) = 1$ and orders at least 3. Clearly, $\lambda_3(G) = \lambda_3(H) = \lambda(G) = \lambda(H) = 1$, $r_1 = r_2 = 1$. Then $\lambda_3(G \boxtimes H) \leq \min \left\{ \left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil \right\rfloor (|V(H)| + 2|E(H)|), \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2} \right\rceil \right\rfloor (|V(G)| + 2|E(G)|), \delta(G) + \delta(H) + \delta(G)\delta(H) \right\} = 3$. By Theorem 6, $\lambda_3(G \boxtimes H) \geq \min\{2\lambda_3(G) + \lambda_3(H), \lambda_3(G) + 2\lambda_3(H)\} = 3$, so the bound of our theorem is sharp. ■

Note that the minimum in Theorem 16 can be realized by any of three terms. From the example in Theorem 16, if G and H are connected graphs with $\delta(G) = \delta(H) = 1$ and orders at least 3, then $\min \left\{ \left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil \right\rfloor (|V(H)| + 2|E(H)|), \left\lfloor \frac{4}{3}\lambda_3(H) + r_2 - \frac{4}{3}\left\lceil \frac{r_2}{2} \right\rceil \right\rfloor (|V(G)| + 2|E(G)|) \right\} > \delta(G) + \delta(H) + \delta(G)\delta(H)$ in this case. Now let H be a tree of order at least 3, G be a connected graph with $V(G) = A \cup B$ such that $G[A]$ and $G[B]$ are complete graphs and there is exactly one edge between A and B, where $|A|, |B| > \left\lceil \frac{3}{2}|V(H)| \right\rceil$. Now, $\lambda_3(G) = \lambda(G) = 1$ and $r_1 = 1$. We then have

$$\begin{aligned} &\left\lfloor \frac{4}{3}\lambda_3(G) + r_1 - \frac{4}{3}\left\lceil \frac{r_1}{2} \right\rceil \right\rfloor (|V(H)| + 2|E(H)|) = |V(H)| + 2|E(H)| \\ &= 3|V(H)| - 2 < \min\{2|A| - 1, 2|B| - 1\} = 2\delta(G) + 1 \\ &= \delta(G) + \delta(H) + \delta(G)\delta(H), \end{aligned}$$

that is, in this case we have $\min\left\{\left\lfloor\frac{4}{3}\lambda_3(G)+r_1-\frac{4}{3}\left\lceil\frac{r_1}{2}\right\rceil\right\rfloor(|V(H)|+2|E(H)|),\right.$ $\left.\left\lfloor\frac{4}{3}\lambda_3(H)+r_2-\frac{4}{3}\left\lceil\frac{r_2}{2}\right\rceil\right\rfloor(|V(G)|+2|E(G)|)\right\} < \delta(G)+\delta(H)+\delta(G)\delta(H)$.

4. Miscellaneous Results

In this section, we will obtain the exact values of generalized 3-connectivity and 3-edge-connectivity for products of some special graph classes.

Proposition 17. *Let G and H be two connected graphs. If $\delta(G)=1$, then*

$$\kappa_3(G\circ H)=|V(H)|.$$

Proof. If $\delta(G)=1$, then $\kappa_3(G)=\kappa(G)=1$ and we have $r=1$ since $r\equiv\kappa(G)$ (mod 4). Then by Theorem 7, we have $\kappa_3(G\circ H)\le\left\lfloor\frac{4}{3}\kappa_3(G)+r-\frac{4}{3}\left\lceil\frac{r}{2}\right\rceil\right\rfloor|V(H)|$ $=|V(H)|$. Furthermore, by Theorem 8, we have $\kappa_3(G\circ H)\ge\kappa_3(G)|V(H)|=$ $|V(H)|$. Thus, the result holds. ■

Proposition 18. *Let G and H be two connected graphs with minimum degree 1. Then*

$$\lambda_3(G\circ H)=1+|V(H)|.$$

Proof. If $\delta(G)=1$ and $\delta(H)=1$, we have $\lambda_3(G)=\lambda_3(H)=1$. Then by Theorem 9, we have $1+|V(H)|=\lambda_3(H)+\lambda_3(G)|V(H)|\le\lambda_3(G\circ H)\le$ $\min\left\{\left\lfloor\frac{4\lambda_3(G)+2}{3}\right\rfloor|V(H)|^2,\delta(H)+\delta(G)|V(H)|\right\}=1+|V(H)|$. Thus, the result holds. ■

The mappings $p_G:(u,v)\mapsto u$ and $p_H:(u,v)\mapsto v$ from $V(G\Box H)$ into $V(G)$ and $V(H)$, respectively, are weak homomorphisms from $G\Box H$ into factors G and H. These weak homomorphisms are called projections in [7, 8].

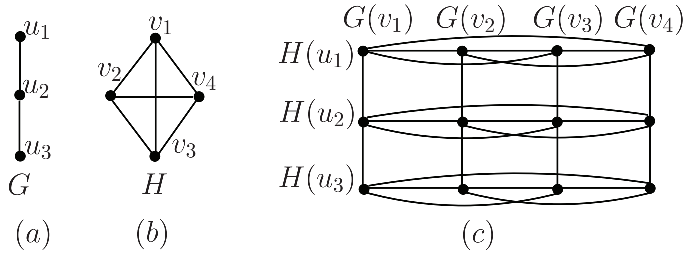

Figure 1. Graphs G, H and their Cartesian product.

Recall that $G\Box H$ is a spanning subgraph of $G\boxtimes H$ for any two graphs G and H. Let G and H be two connected graphs with $V(G)=\{u_i:1\le i\le n\}$

and $V(H) = \{v_j : 1 \le j \le m\}$. We write $G(v_j)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v_j) : 1 \le i \le n\}$ where $1 \le j \le m$, and use $H(u_i)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v_j) : 1 \le j \le m\}$ where $1 \le i \le n$. Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. For example, as shown in Figure 1, $G(v_j) \cong G$ for $1 \le j \le 4$ and $H(u_i) \cong H$ for $1 \le i \le 3$.

Proposition 19. *If $n, m \ge 5$, then $\kappa_3(P_n \boxtimes P_m) = 3$.*

Proof. Let $G \cong P_n : u_1, u_2, \ldots, u_n$ and $H \cong P_m : v_1, v_2, \ldots, v_m$. According to the proof of Theorem 16, for any two connected graphs G and H with $\delta(G) = \delta(H) = 1$, we have $\lambda_3(G \boxtimes H) = 3$. Then by the fact that $\kappa_k(G) \le \lambda_k(G)$, we have $\kappa_3(P_m \boxtimes P_n) \le 3$. Thus, it suffices to show that for any set $S = \{x, y, z\} \subseteq V(P_m \boxtimes P_n)$, we can find three internally disjoint trees connecting S.

Let $x \in V(G(v_\alpha)), y \in V(G(v_\beta)), z \in V(G(v_\gamma))$ for some $1 \le \alpha, \beta, \gamma \le m$. We now consider the case that α, β, γ are distinct and $p_G(x) = p_G(y) = p_G(z)$.

Suppose that two elements of S are adjacent, say $xy \in E(G \boxtimes H)$. Without loss of generality, we can assume that $x = (u_1, v_1), y = (u_1, v_2), z = (u_1, v_m)$. We use T_1 to denote the $x - z$ path in $H(u_1)$. Let T_2 and T_3 be the trees as shown in (a) of Figure 2.

For the case that any two elements of S are nonadjacent, without loss of generality, we can assume that $x = (u_1, v_1)$. We use T_1 to denote the $x - z$ path in $H(u_1)$. Let T_2 and T_3 be the trees as shown in (b) of Figure 2. There are other cases whose details are omitted because the arguments are similar. ■

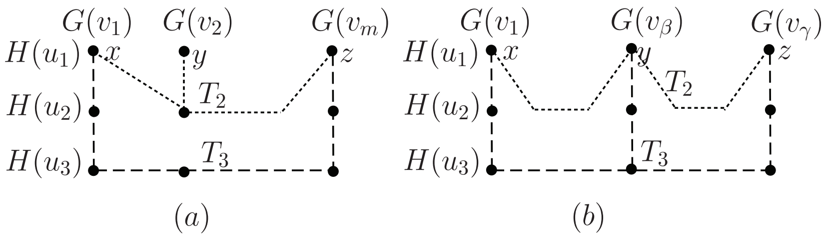

Figure 2. Graphs for Proposition 19.

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