# SOME RESULTS ON 4-TRANSITIVE DIGRAPHS 

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#### Abstract

Let $D$ be a digraph with set of vertices $V$ and set of $\operatorname{arcs} A$. We say that $D$ is $k$-transitive if for every pair of vertices $u, v \in V$, the existence of a $u v$-path of length $k$ in $D$ implies that $(u, v) \in A$. A 2-transitive digraph is a transitive digraph in the usual sense.

A subset $N$ of $V$ is $k$-independent if for every pair of vertices $u, v \in N$, we have $d(u, v), d(v, u) \geq k$; it is $l$-absorbent if for every $u \in V \backslash N$ there exists $v \in N$ such that $d(u, v) \leq l$. A $k$-kernel of $D$ is a $k$-independent and $(k-1)$-absorbent subset of $V$. The problem of determining whether a digraph has a $k$-kernel is known to be $\mathcal{N} \mathcal{P}$-complete for every $k \geq 2$.

In this work, we characterize 4-transitive digraphs having a 3 -kernel and also 4 -transitive digraphs having a 2 -kernel. Using the latter result, a proof of the Laborde-Payan-Xuong conjecture for 4 -transitive digraphs is given. This conjecture establishes that for every digraph $D$, an independent set can be found such that it intersects every longest path in $D$. Also, Seymour's Second Neighborhood Conjecture is verified for 4-transitive digraphs and further problems are proposed.


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## 1. Introduction

Since their introduction in [9], $k$-transitive digraphs have received a fair amount of attention. A good example is [15], where the following conjecture is proposed.

Conjecture 1. Let $k \geq 2$ be an integer. If $D$ is a $k$-transitive digraph such that none of its terminal components is isomorphic to $C_{k}$, then D has a $(k-1)$-kernel.

If true, Conjecture 1 would have a very interesting consequence, the problem of determining whether a $k$-transitive digraph has a $(k-1)$-kernel could be solved in polynomial time (this problem is $\mathcal{N} \mathcal{P}$-complete for general digraphs).

Unlike the case of undirected graphs, where a lot of different families with interesting properties exist which can be used to verify difficult problems or conjectures, there are only a few well known families of digraphs. There is still a lot to know about $k$-transitive digraphs in general, but the structure of 3 -transitive digraphs is very well understood [11], and strong 4-transitive digraphs have been characterized [12]; there are even some general results on the structure of strong $k$-transitive digraphs [13]. The aim of the present work is contributing to the consolidation of 4 -transitive digraphs as a well understood family by using its rich structure to solve problems that are usually difficult for general digraphs: the Laborde-Payan-Xoung Conjecture, Seymour's Second Neighborhood Conjecture, and characterizing 4 -transitive digraphs having 2 - and 3 -kernels.

In this work, $D=(V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1]. For a vertex $v \in$ $V(D)$, we define the out-neighborhood of $v$ in $D, N_{D}^{+}(v)$, as the set $N_{D}^{+}(v)=$ $\{u \in V(D):(v, u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript $D$. The elements of $N^{+}(v)$ are called the out-neighbors of $v$, and the out-degree of $v, d_{D}^{+}(v)$, is the number of out-neighbors of $v$. Definitions of in-neighborhood, in-neighbors and in-degree of $v$ are analogously given. An arc $(u, v) \in A(D)$ is called asymmetrical ( respectively symmetrical) if $(v, u) \notin A(D)$ ( respectively $(v, u) \in A(D)$ ). We say that a vertex $u$ reaches a vertex $v$ in $D$ if a directed $u v$-directed path (a path with initial vertex $u$ and terminal vertex $v$ ) exists in $D$. The distance from vertex $u$ to vertex $v, d_{D}(u, v)$, is the length of the shortest $u v$-path in $D$.

If $D$ is a digraph and $X, Y \subseteq V(D)$, an $X Y$-arc is an arc with initial vertex in $X$ and terminal vertex in $Y$. If $X \cap Y=\emptyset, X \rightarrow Y$ will denote that $(x, y) \in A(D)$ for every $x \in X$ and $y \in Y$. Again, if $X$ and $Y$ are disjoint, $X \Rightarrow Y$ will denote that there are not $Y X$-arcs in $D$. When $X \rightarrow Y$ and $X \Rightarrow Y$ we will simply write $X \mapsto Y$. If $D_{1}, D_{2}$ are subdigraphs of $D$, we will abuse notation to write $D_{1} \rightarrow D_{2}$ or $D_{1} D_{2}$-arc, instead of $V\left(D_{1}\right) \rightarrow V\left(D_{2}\right)$ or $V\left(D_{1}\right) V\left(D_{2}\right)$-arc, respectively. Also, if $X=\{v\}$, we will abuse notation to write $v \rightarrow Y$ or $v Y-\operatorname{arc}$ instead of $\{v\} \rightarrow Y$
or $\{v\} Y$-arc, respectively. Analogously if $Y=\{v\}$. The distance from $X$ to $Y$, $d_{D}(X, Y)$, is defined as $\min \left\{d_{D}(x, y): x \in X, y \in Y\right\}$. As before, we will write $d_{D}(u, Y)$ and $d_{D}(X, v)$ instead of $d_{D}(\{u\}, Y)$ and $d_{D}(X,\{v\})$.

A digraph is strongly connected (or strong) if for every $u, v \in V(D)$, there exists a $u v$-directed path, i.e., a directed path with initial vertex $u$ and terminal vertex $v$. A strong component (or component) of $D$ is a maximal strong subdigraph of $D$. The condensation of $D$ is the digraph $D^{\star}$ with $V\left(D^{\star}\right)$ equal to the set of all strong components of $D$, and $(S, T) \in A\left(D^{\star}\right)$ if and only if there is an $S T$-arc in $D$. Clearly, $D^{\star}$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A terminal component of $D$ is a strong component $T$ of $D$ such that $d_{D^{\star}}^{+}(T)=0$. An initial component of $D$ is a strong component $S$ of $D$ such that $d_{D^{\star}}^{-}(S)=0$.

The rest of the paper is ordered as follows. In Section 2, some basic lemmas that will be used through the rest of the paper are introduced, and 4 -transitive digraphs having a 3 -kernel are characterized. In Section 3, 4-transitive digraphs having a kernel are characterized. In Section 4, the characterization of the previous section is used to prove the Laborde-Payan-Xuong Conjecture for 4 -transitive digraphs. The final section of this article is devoted to consider a brief summary of the contributions made, and propose further research directions. As an example of the potential of this familiy of digraphs, Seymour's Second Neighborhood Conjecture is also proved for 4 -transitive digraphs in the final section.

## 2. 3 -Kernels in 4 -Transitive Digraphs

In this section we characterize the 4 -transitive digraphs having a 3 -kernel. Our next lemma is a simple property of 4 -transitive digraphs having a directed 3 -cycle extension as a subdigraph.

Lemma 2. Let $D$ be a 4-transitive digraph, and let $H \subseteq D$ be a 3-cycle extension with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$. If $\left(v_{0}, v\right)$ is an arc of $D$ such that $v_{0} \in V_{0}$ and $v \in V(D) \backslash V(H)$, then $V_{0} \rightarrow v$.

Proof. Let $\left(v_{0}, v\right)$ be an arc of $D$ with $v_{0} \in V_{0}$ and $v \in V(D) \backslash V(H)$. If $V_{0}=\left\{v_{0}\right\}$, then we trivialy have that $V_{0} \rightarrow v$. Let us suppose that $\left|V_{0}\right| \geq 2$. Consider an arbitrary vertex $y \in V_{0} \backslash\left\{v_{0}\right\}$. Recalling that $H$ is a directed 3-cycle extension, we can find $v_{1} \in V_{1}$ and $v_{2} \in V_{2}$ such that $\left(y, v_{1}, v_{2}, v_{0}, v\right)$ is a directed 4 -path in $D$. But $D$ is 4 -transitive, thus we have that $(y, v) \in A(D)$. Since $y$ was chosen arbitrarily, we conclude that $V_{0} \rightarrow v$.

We will also use the following lemma found in [12].
Lemma 3. Let $k \geq 2$ be an integer, let $D$ be a $k$-transitive digraph and let $C$ be a directed $n$-cycle, with $n \geq k$ and $(k-1, n)=1$. If $v \in V(D) \backslash V(C)$ is such that a directed $v C$-path exists in $D$, then $v \rightarrow C$.

Dually, we conclude that if $v$ is such that a directed $C v$-path exists, then $C \rightarrow v$. This fact will be used sometimes and, abusing notation, will be referred as Lemma 3. Let us observe that Lemma 3 is true, in particular, for $k=4$, with $n=4$ and $n=5$. This is going to be very useful in the study of 4 -transitive digraphs that contain cycles of length 4 or 5 .

The next lemma, also proved in [12], tells us that there are only two possibilities for a 4 -transitive digraph of circumference 2 .
Lemma 4. Let $D$ be a strong 4-transitive digraph with circumference 2 . Then $D$ is the complete biorentation of the star $K_{1, n}$, or is the complete biorentation of the double star $D_{n, m}$.

The following characterization of the strong 4-transitive digraphs is found in [12].

Theorem 5. Let $D$ be a strong 4-transitive digraph. Then exactly one of the next affirmations is true.
(1) $D$ is a complete digraph.
(2) $D$ is a directed 3 -cycle extension.
(3) D has circumference 3, it contains a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$. At least one symmetric arc ( $v_{i}$, $\left.v_{i+1}\right) \in A(D)$ exists in $D$, where $v_{j} \in V_{j}$ for $j \in\{i, i+1\}(\bmod 3)$ and $\left|V_{i}\right|=1$ or $\left|V_{i+1}\right|=1$.
(4) D has circumference 3 and $U G(D)$ is not 2 -edge-connected. Consider $\left\{S_{1}, S_{2}\right.$, $\left.\ldots, S_{n}\right\}$ the vertex set of the maximal 2 -edge-connected subgraphs of $U G(D)$. Then $S_{i}=\left\{u_{i}\right\}$ for every $2 \leq i \leq n$, and $D\left[S_{1}\right]$ contains a directed 3 -cycle extension as a spanning subdigraph with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$. There exists a vertex $v_{0} \in V_{0}$ such that $\left(v_{0}, u_{j}\right),\left(u_{j}, v_{0}\right) \in A(D)$ for every $2 \leq j \leq n$. Also, $\left|V_{0}\right|=1$ and $D\left[S_{1}\right]$ has the structure described in (2) or (3), depending on the existence of symmetric arcs.
(5) $D$ is a symmetrical 5 -cycle.
(6) $D$ is a complete biorentation of the star $K_{1, n}, n \geq 3$.
(7) $D$ is the complete biorentation of the double star $D_{n, m}$.
(8) $D$ is a strong digraph of order less than or equal to 4 not included in the previous families of digraphs.
In Figure 1 we can see examples of digraphs that belong to families (3) and (4) described in Theorem 5.


Figure 1. Digraphs of the families (3) and (4) described in Theorem 5.

The following theorem can be found in [15].
Theorem 6. Let $k \geq 2$ be an integer. Every strong $k$-transitive digraph different from $C_{k}$ has a $(k-1)$-kernel.

We are ready to prove Conjecture 1 for $k=4$.
Theorem 7. If $D$ is a 4-transitive digraph, then $D$ has a 3 -kernel if and only if none of its terminal components is isomorphic to $C_{4}$.

Proof. Suppose that none of the terminal components of $D$ is isomorphic to $C_{4}$. We will prove that $D$ has a 3 -kernel by induction on the number of strong components $k$ of $D$. The case $k=1$ is proved in Theorem 6 .

Now, suppose that we can find a 3 -kernel for every 4 -transitive digraph $D$ with $k-1$ strong components. Let $D$ be a digraph with $k$ strong components $D_{1}, D_{2}, \ldots, D_{k}$ and suppose without loss of generality that $D_{1}$ is an initial component of $D$. Using the inductive hypothesis we have that $D-D_{1}$ has a 3 -kernel $N$.

If $N$ is such that it 2-absorbs every vertex of $D_{1}$, then $N$ is a 3 -kernel for $D$. Suppose that there exists a vertex $x \in V\left(D_{1}\right)$ that is not 2 -absorbed by $N$. Then $d(x, N) \geq 3$. Since $D_{1}$ is an initial component of $D$, we have that $N \cup\{x\}$ is a 3 -independent set of $D$.

If $D_{1}$ is from the families (1), (5) or (6), using the previous observation we have that $N \cup\{x\}$ is a 3 -kernel for $D$.

If $D_{1}$ is of type (2) or (3), then $D$ contains a directed 3 -cycle extension as a spanning subdigraph with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$. Suppose without loss of generality that $x \in V_{0}$. It follows from Lemma 2 that $N \cup V_{0}$ is a 3 -independent set. Since clearly $V_{0}$ 2-absorbs every vertex in $V_{1}$ and $V_{2}$, and $N$ already is a 2 -absorbent set in $D-D_{1}$, we have that $N \cup V_{0}$ is a 3 -independent, 2 -absorbent set, i.e., a 3 -kernel of $D$.

Suppose that $D_{1}$ belongs to family (4). If $x=v_{0}$, then $N \cup\{x\}$ is a 3-kernel of $D$. If $x \in V_{1}$, an argument analogous to the one used in the previous case shows
that there exists a subset $V_{1}^{\prime} \subseteq V_{1}$ (depending on the existence of symmetric arcs between $V_{0}$ and $V_{1}$ ) such that $N \cup V_{1}^{\prime}$ is a 3 -kernel of $D$. So, let us suppose that $v_{0}$ and $V_{1}$ are already 2 -absorbed by $N$. If $u_{i}$ is 2 -absorbed by $N$ for every $2 \leq i \leq n$, then $x \in V_{2}$ and again a subset $V_{2}^{\prime}$ of $V_{2}$ exists such that $N \cup V_{2}^{\prime}$ is a 3-kernel of $D$. If $u_{j}$ is not 2-absorbed by $N$ for some $2 \leq i \leq n$, then $N \cup\left\{u_{i}\right\}$ is a 3 -kernel of $D$.

If $D_{1}$ is from the family (7), then $D_{1}$ is an orientation of $D_{n, m}$, a double star. Let $u$ and $v$ be the centers of $D_{n, m}$, and let $S$ be the set of vertices of $D_{1}$ not 2-absorbed by $N$. If $\{u, v\} \cap S \neq \emptyset$, then $N \cup\{u\}$ or $N \cup\{v\}$ is a 3-kernel of $D$. Else, assume without loss of generality that $N(u) \cap S \neq \emptyset$, and choose any vertex $y \in N(u) \cup S$. If $N \cup\{y\}$ is a 3 -kernel, we are done. Otherwise, there exists a vertex $z$ in $N(v) \cup S$ which is not 2 -absorbed by $y$. In this case, $N \cup\{y, z\}$ is a 3 -kernel of $D$.

Finally, if $D_{1}$ is of type (8) and has order less than or equal to 3 , then $N \cup\{x\}$ is a 3 -kernel of $D$. Suppose that $D_{1}$ has order 4 . If $D$ has circumference 2 , then $D_{1}$ belongs to the families (6) or (7) and, if $D_{1}$ has circumference 3 , then $D_{1}$ is of type (2), (3) or (4).

If $D_{1}$ has circumference 4 , then $D_{1}$ contains a directed 4 -cycle as a subdigraph. But $D_{1}$ reaches some terminal component of $D$, and hence a vertex $v \in N$. Lemma 3 implies that $D_{1} \rightarrow v$, and hence $N$ is a 3-kernel of $D$.

## 3. Kernels in 4-Transitive Digraphs

As in the previous section, we begin by characterizing strong 4-transitive digraphs having a kernel, to then use this result to proceed by induction on the number of strong components of a general digraph $D$.

Lemma 8. Let $D$ be a strong 4-transitive digraph. Then $D$ has a kernel if and only if $D$ is not isomorphic to any of the following:
a. Directed 3-cycle extensions.
b. Strong semicomplete digraphs of order 4 without vertices of indegree 3 .
c. Digraphs of the family (3) described in Theorem 5 with no kernel, i.e., those in which the number of symmetric arcs from $V_{i}$ to $V_{i+1}(\bmod 3)$ is less than $\left|V_{i+1}\right|$, whenever there is at least one symmetric arc from $V_{i}$ to $V_{i+1}(\bmod 3)$.

Proof. Again we consider the notation used in Theorem 5 and analyze each possibility for $D$. Suppose that $D$ is not isomorphic to any digraph of the type $a, b$ or $c$.

If $D$ is of the family (1), then any vertex in $V(D)$ is a kernel for $D$.
If $D$ belongs to family (3), then $D$ contains a directed 3 -cycle extension as a subdigraph with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$ and since $D$ is not isomorphic to


Figure 2. Digraph of type $c$. from Lemma 8.
any digraph of the type $a$, we have that classes $V_{i}, V_{i+1}(\bmod 3)$ exists such that there are the same number of symmetric arcs from $V_{i}$ to $V_{i+1}$ that vertices in $V_{i+1}$. Then $V_{i}$ absorbs $V_{i-1}$ and $V_{i+1}(\bmod 3)$, and therefore is a kernel for $D$.

If $D$ belongs to family (4), then, with the notation of Theorem 5 , the set $\left\{u_{2}, u_{3}, \ldots, u_{n}\right\} \cup V_{2}$ is a kernel for $D$.

If $D$ belongs to any of the families (5), (6) or (7), then $D$ is a symmetric digraph and hence any maximal independent set is a kernel for $D$.

For the case in which $D$ belongs to family (8), we suppose that $D$ has circumference 4 , otherwise $D$ belongs to one of the previous families. Hence, $D$ has order 4 and contains a directed 4 -cycle ( $v_{1}, v_{2}, v_{3}, v_{4}, v_{1}$ ). If $D$ is not a semicomplete digraph, then there are vertices $v_{i}$ and $v_{j}$ such that $\left(v_{i}, v_{j}\right),\left(v_{j}, v_{i}\right) \notin A(D)$. This implies that $\left\{v_{i}, v_{j}\right\}$ is an independent set that absorbs the remaining two vertices.

If $D$ is a semicomplete digraph, then by hypothesis there is a vertex $v$ such that $d^{-}(v)=3$ and therefore $\{v\}$ is a kernel for $D$. With no more possible cases, the result follows.

Theorem 9. If $D$ is a 4-transitive digraph, then $D$ has a kernel if and only if $D$ has no terminal components isomorphic to digraphs of the families described in Lemma 8.

Proof. We will proceed by induction on the number of strong components $k$. If $k=1$, then $D$ is a strong digraph an the result follows from Lemma 8 .

Let $D$ be a 4 -transitive digraph with $k>1$ strong components $D_{1}, D_{2}, \ldots, D_{k}$. Without loss of generality assume that $D_{1}$ is an initial component of $D$; by the inductive hypothesis $D-D_{1}$ has a kernel $N$.

If $N$ absorbs every vertex of $D_{1}$, then $N$ is a kernel of $D$. Suppose that there is a vertex $x \in V\left(D_{1}\right)$ that is not absorbed by $N$. Since $D_{1}$ is an initial component, and $x$ is not absorbed by $N$, we have that $N \cup\{x\}$ is an independent set of $D$.

With the notation used in Theorem 5 let us consider the different possibilities for $D_{1}$. If $D_{1}$ belongs to family (1), then $N \cup\{x\}$ is a kernel of $D$.

If $D$ belongs to family (2) or (3), then $D_{1}$ contains a directed 3 -cycle extension with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$ as a spanning subdigraph. It follows from Lemma 2 that, for $i \in\{0,1,2\}$, either $V_{i}$ is completely absorbed by $N$ or no vertex in $V_{i}$ is absorbed by $N$. Also, since $D$ is 4 -transitive, at least one vertex in $D_{1}$ must be absorbed by $N$. Suppose without loss of generality that $V_{0}$ is absorbed by $N$. If $V_{2}$ is also absorbed by $N$, then $N \cup V_{1}$ is a kernel of $D$. Otherwise, $N \cup V_{1}$ is a kernel of $D$.

In the case that $D_{1}$ is of type (4), let $S \subseteq\left\{u_{2}, u_{3}, \ldots, u_{n}\right\}$ be the subset of the $u_{i}$ 's not absorbed by $N$. If $S \neq \emptyset$, then $S$ absorbs $V_{0}$. If $V_{2}$ is not absorbed by $N$, then $N \cup S \cup V_{2}$ is a kernel of $D$. If $V_{2}$ is absorbed by $N$ but $V_{1}$ is not, then $N \cup S \cup V_{1}$ is a kernel for $D$. And, if both $V_{1}$ and $V_{2}$ are absorbed by $N$, then $N \cup S$ is a kernel for $D$. Now, suppose that $S=\emptyset$. Then $u_{i}$ is absorbed for every $i \in\{2,3, \ldots, n\}$. This fact, together with Lemma 2, implies that $V_{1}$ is absorbed by $N$. If $V_{0}$ is not absorbed by $N$, then $N \cup V_{0}$ is a kernel of $D$. Otherwise, $N \cup V_{2}$ is a kernel of $D$.

If we suppose that $D_{1}$ is from the family (5), then Lemma 3 implies that every vertex of $D_{1}$ is absorbed by $N$.

If $D_{1}$ is of type (6), then $D_{1}$ is a star; let $\{u\}$ be the center of $D_{1}$. If $u$ is not absorbed by $N$, then $N \cup\{u\}$ is a kernel for $D$. Else, consider $S \subset V\left(D_{1}\right)$, the vertices of $D_{1}$ not absorbed by $N$. Clearly, $N \cup S$ is a kernel of $D$.

In the case that $D_{1}$ belongs to family (7), then $D_{1}$ is a double star with centers $\{u, v\}$. Let us consider $S \subseteq V\left(D_{1}\right)$, the vertices of $D_{1}$ that are not absorbed by $N$. If $u, v \notin S$, then $S$ is an independent set and therefore $N \cup S$ is a kernel for $D$. If $u \in S$, then $u \in N(v) \cap S$, which is an independent set that absorbs every vertex of $D_{1}$ and therefore $N \cup(N(v) \cap S)$ is a kernel of $D$. Analogously if $v \in S$, then $N \cup(N(u) \cap S)$ is a kernel of $D$.

Since every strong 4-transitive digraph of order at most 4 and circumference 2 or 3 belongs to one of the previous families, if $D_{1}$ belongs to family (8), then $D_{1}$ has a 4 -cycle as a spanning subdigraph. Hence, Lemma 3 implies that $D_{1}$ is completely absorbed by $N$.

## 4. The Laborde-Payan-Xuong Conjecture for 4-Transitive Digraphs

In [14], the following conjecture is proposed.
Conjecture 10. For every digraph $D$ there is an independent set that intersects every longest path in $D$.

This conjecture is known as the Laborde-Payan-Xuong Conjecture and it remains as an open problem for general digraphs. However, it has been proved for several families of digraphs, e.g., quasi-transitive digraphs, line digraphs, arclocal tournaments, path mergeable digraphs, in- and out-semicomplete digraphs and semicomplete $k$-partite digraphs [3]; 3-quasi-transitive digraphs [16]; locally semicomplete digraphs and locally transitive digraphs which have directed paths of maximum length at most 4 [4].

In this section we will prove this conjecture for 4 -transitive digraphs. We will need the following results, the first one of which is folklore.

Lemma 11. Let $D$ be a digraph with kernel $N$. For every longest path $T$ in $D$, we have that $N \cap V(T) \neq \emptyset$.

The following lemma can be thought as a set of directions to remove vertices from a 4 -transitive digraph to obtain a 4 -transitive digraph with a kernel. We will refer to the families of digraphs $a, b$ and $c$ of Lemma 8 .

Lemma 12. Let $D$ be a 4-transitive digraph with terminal components $D_{1}, \ldots$, $D_{r}$. If $S=\bigcup_{i=1}^{r} S_{i}$, where $S_{i} \subseteq V\left(D_{i}\right)$ is defined as follows:

- $S_{i}=V_{0}$ if $D_{i}$ is of type a or $c$, and has a cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$;
- $S_{i}=\left\{v_{4}\right\}$ if $D_{i}$ is of type $b$ and $v_{4} \in V\left(D_{i}\right)$ is not absorbed by $v_{1}$, where $d^{-}\left(v_{1}\right)=2$;
- $S_{i}=\emptyset$, otherwise,
then $D-S$ has a kernel.
Proof. Let $D_{j}$ be any terminal component of $D$. Note that if $D_{j}$ is of type $a$ or $c$, then $D_{j}$ has a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$. And then $D_{j}-V_{0}$ has the kernel $N_{j}=V_{2}$.

If $D_{j}$ is of type $b$, then $D_{j}$ is a semicomplete digraph of order 4 , such that $d^{-}(v)<3$ for every $v \in V\left(D_{j}\right)$. Suppose, without loss of generality, that $V(D)=$ $\left\{v_{1}, v_{2}, v_{3}, v_{3}\right\}, d^{-}\left(v_{1}\right)=2$ and $N^{-}\left(v_{1}\right)=\left\{v_{2}, v_{3}\right\}$. It is clear that $D_{j}-\left\{v_{4}\right\}$ has the kernel $N_{j}=\left\{v_{1}\right\}$.

Furthermore, if $N$ is a kernel for $D-S$ and $D_{j}$ is a terminal component of $D$ of type $a$ or $c$ with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$, then $V_{2} \subseteq N$ (because it is the only independent set that can absorb $V_{1}$ in $D_{j}-V_{0}$ ). Analogously, if $D_{j}$ is terminal component of $D$ of type $b$ and $v_{4}$ is the vertex not absorbed by $v_{1}$, where $d^{-}\left(v_{1}\right)=2$, then $v_{1} \in N$.

Lemma 11 tells us that the kernel of a digraph is an independent set that intersects every longest path, fact that we will use to prove the Laborde-PayanXuong conjecture for 4 -transitive digraphs.

Theorem 13. For every 4-transitive digraph $D$ there exists an independent set intersecting every longest path of $D$.

Proof. Let $D$ be a 4 -transitive digraph. By Lemma 12, we can find a vertex subset $S$ of the terminal components of $D$ such that $D-S$ has a kernel $N$. We afirm that $N$ is the independent set we are looking for.

Let $T$ be a longest path in $D$. If $T \cap S=\emptyset$, then $T$ is a longest path in $D-S$. It follows from Lemma 11 that $N \cap T \neq \emptyset$.

Now, let us supppose that $T \cap S \neq \emptyset$ and $T=\left(u_{1}, u_{2}, \ldots, u_{m}\right)$. Since $T \cap S \neq \emptyset$ and $S$ is contained in the terminal components of $D$, we have that $T$ reaches exactly one terminal component $D_{j}$; furthermore, the terminal vertex $u_{m}$ of $T$ is contained in $V\left(D_{j}\right)$.

Suppose that $D_{j}$ is of type $a$ or $c$. Then $D_{j}$ has a directed 3-cycle extension as a spanning subdigraph with vertex partition $\left\{V_{0}, V_{1}, V_{2}\right\}$ and given the construction of $S$ in the proof of Lemma 12 we have that $S \cap D_{j}=V_{0}$ and that $V_{2} \subseteq N$, which gives us $T \cap S \subseteq V_{0}$. If $T \cap V_{2}=\emptyset$, then $u_{m} \in V_{0}$ or $u_{m} \in V_{1}$. If $u_{m} \in V_{1}$, since $T$ does not pass through $V_{2}$, then a vertex $v_{2} \in V_{2}$ exists such that $T^{\prime}=\left(u_{1}, u_{2}, \ldots, u_{m}, v_{2}\right)$ is a directed path in $D$ longer than $T$, contradicting the choice of $T$.

If $u_{m} \in V_{0}$ and $D_{j}$ is of type $a$, then $T$ does not pass through $V_{1}$, because every directed $V_{1} V_{0}$-path passes through $V_{2}$. Hence, there is $v_{1} \in V_{1}$ such that $T^{\prime}=\left(u_{1}, u_{2}, \ldots, u_{m}, v_{1}\right)$ is a directed path in $D$ longer than $T$, a contradiction.

If $u_{m} \in V_{0}, D_{j}$ is of type $c$ and there are no symmetric arcs from $V_{0}$ to $V_{1}$, then every directed $V_{1} V_{0}$-path passes through $V_{2}$. Analogously to the previous case, we can find a directed path $T^{\prime}$ longer than $T$.

If $u_{m} \in V_{0}, D_{j}$ is of type $c$ and there is at least one symmetric arc from $V_{0}$ to $V_{1}$, then $\left|V_{0}\right|=1$ or $\left|V_{1}\right|=1$. If $\left|V_{0}\right|=1$ then $u_{m-1} \in V_{1}$, otherwise we would have that $T$ does not intersect $V_{1}$ and with an argument similar to the above we could find a longer path than $T$. And since $u_{m-1} \in V_{1}$, then we can find $v_{2} \in V_{2}$ such that $T^{\prime}=\left(u_{1}, u_{2}, \ldots, u_{m-1}, v_{2}, u_{m}\right)$ is a directed path in $D$ longer than $T$, which is impossible.

Now, if $\left|V_{1}\right|=1$, then $T$ can have at most two vertices of $V_{0}$. In this case we have that $u_{m-2}, u_{m} \in V_{0}$ and $u_{m-1} \in V_{1}$, but then a vertex $v_{2} \in V_{2}$ exists such that $T^{\prime}=\left(u_{1}, u_{2}, \ldots, u_{m-2}, u_{m-1}, v_{2}, u_{m}\right)$ is a directed path longer than $T$, a contradiction.

We conclude that $T \cap V_{2} \neq \emptyset$ and therefore $T \cap N \neq \emptyset$.
If $D_{j}$ is of type $b$, then given the construction of $S$ in the proof of Lemma 12 we have that $S \cap D_{j}=\left\{v_{4}\right\}$ where $v_{4}$ is the vertex not absorbed by $v_{1}$, with $d^{-}\left(v_{1}\right)=2$ and $v_{1} \in N$. Since $D_{j}$ is a strong semicomplete digraph, it has a Hamiltonian cycle $C=\left(v_{4}, v_{3}, v_{2}, v_{1}, v_{4}\right)$. Then any longest path that reaches $D_{j}$ must use every vertex of $D_{j}$. Therefore, $T \cap N \neq \emptyset$.

## 5. Conclusions and Further Directions

In the previous section the Laborde-Payan-Xuong Conjecture was proved for 4transitive digraphs. It is natural to consider other conjectures for this family of digraphs, such as Seymour's conjecture of the second out-neighborhood. This states that any asymmetric digraph $D$ without loops contains a vertex $v$, such that $\left|N^{++}(v)\right| \geq\left|N^{+}(v)\right|$ where

$$
N^{++}(v)=\bigcup_{u \in N^{+}(v)} N^{+}(u) \backslash N^{+}(v) .
$$

We call $N^{+}(v)$ the (first) out-neighborhood of $v$ and $N^{++}(v)$ the second outneighborhood of $v$. It is easy to see that this conjecture is true for 4 -transitive digraphs.

Proposition 14. Let $D$ be a 4-transitive digraph with no loops or symmetric arcs. Then there exists $v \in V(D)$ such that $\left|N^{++}(v)\right| \geq\left|N^{+}(v)\right|$.

Proof. Let $D_{j}$ be a terminal component of $D$. Since $D$ is asymmetrical, then there are only three possibilities for $D_{j}$, those are, $D_{j}$ is an isolated vertex, a digraph of the family (2), or an asymmetric digraph of family (8) described in Theorem 5.

If $D_{j}=\{x\}$ is an isolated vertex, then $x$ is a vertex with $d^{+}(x)=0$ and therefore $0=\left|N^{++}(x)\right| \geq\left|N^{+}(x)\right|=0$.

If $D_{j}$ is of the family (2), then $D_{j}$ is a directed 3 -cycle extension, with cyclic partition $\left\{V_{0}, V_{1}, V_{2}\right\}$. Let $V_{i}$ be the largest set of the partition. Hence, for any vertex in $x \in V_{i-2}(\bmod 3)$ we have that $\left|N^{+}(x)\right|=\left|V_{i-1}\right| \leq\left|V_{i}\right|=\left|N^{++}(x)\right|$.

Finally, if $D_{j}$ is of the family (8), then $D_{j}$ is an asymmetric digraph of order less than 4. As in previous arguments, we may assume that $D_{j}$ has order and circumference 4. This implies that $D_{j}$ is either $C_{4}$ or $C_{4}$ with one or two diagonals. One can easily find a vertex $x$ that satisfies $\left|N^{++}(x)\right| \geq\left|N^{+}(x)\right|$.

The search of algorithms that find kernels and other structures in families of digraphs is another problem that has been studied in the past. As it has been already mentioned, in [2], Chvátal shows that the problem of determining whether a given digraph has a kernel is $\mathcal{N} \mathcal{P}$-complete. Recently Hell and Hernández-Cruz proved in [10] that the problem of finding 3 -kernels in digraphs is also an $\mathcal{N P}$ complete problem.

On the other hand, we know that there are several algorithms that find the strong components of a digraph in linear time. Also, to verify that a terminal component is not isomorphic to any of the families of type $a, b$ and $c$ can be done in polynomial time. So, the problem of finding a kernel in the family of 4 -transitive digraphs can be solved in polynomial time. In the same way, we can
conclude that the problem of finding a 3-kernel for 4 -transitive digraphs can be solved in polynomial time.

Like every time a "well-behaved" family of digraphs is found, it is pertinent to ask what other problems that are usually difficult, are "easy" to solve for 4transitive digraphs. Also, after 3 - and 4 -transitive digraphs have been analyzed, it seems to be a good idea to find general results for $k$-transitive digraphs, like Conjecture 1 proposes. In this direction, we propose the following problem.

Problem 15. For each integer $2 \leq n \leq k-1$, determine the complexity of determining whether a $k$-transitive digraph has an $n$-kernel.

If true, Conjecture 1 would show that determining whether a $k$-transitive digraph has an $(k-1)$-kernel can be done in polynomial time. The results of the present paper and those found in [11] solve the problem for $k=3$ and $k=4$; in all cases the answer is that the $n$-kernel problem, which is usually $\mathcal{N} \mathcal{P}$-complete, becomes polynomial in these families of digraphs.

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