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SOME RESULTS ON 4-TRANSITIVE DIGRAPHS

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Abstract

Let D be a digraph with set of vertices V and set of arcs A. We say that D is k-transitive if for every pair of vertices $u, v \in V$, the existence of a uv-path of length k in D implies that $(u, v) \in A$. A 2-transitive digraph is a transitive digraph in the usual sense.

A subset N of V is k-independent if for every pair of vertices $u, v \in N$, we have $d(u, v), d(v, u) \geq k$; it is l-absorbent if for every $u \in V \setminus N$ there exists $v \in N$ such that $d(u, v) \leq l$. A *k-kernel* of D is a k-independent and $(k-1)$-absorbent subset of V. The problem of determining whether a digraph has a k-kernel is known to be $\mathcal{NP}$-complete for every $k \geq 2$.

In this work, we characterize 4-transitive digraphs having a 3-kernel and also 4-transitive digraphs having a 2-kernel. Using the latter result, a proof of the Laborde-Payan-Xuong conjecture for 4-transitive digraphs is given. This conjecture establishes that for every digraph D, an independent set can be found such that it intersects every longest path in D. Also, Seymour's Second Neighborhood Conjecture is verified for 4-transitive digraphs and further problems are proposed.

Keywords: 4-transitive digraph, k-transitive digraph, 3-kernel, k-kernel, Laborde-Payan-Xuong Conjecture.

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1. Introduction

Since their introduction in [9], k-transitive digraphs have received a fair amount of attention. A good example is [15], where the following conjecture is proposed.

Conjecture 1. *Let $k \geq 2$ be an integer. If D is a k-transitive digraph such that none of its terminal components is isomorphic to C_k, then D has a $(k-1)$-kernel.*

If true, Conjecture 1 would have a very interesting consequence, the problem of determining whether a k-transitive digraph has a $(k-1)$-kernel could be solved in polynomial time (this problem is $\mathcal{NP}$-complete for general digraphs).

Unlike the case of undirected graphs, where a lot of different families with interesting properties exist which can be used to verify difficult problems or conjectures, there are only a few well known families of digraphs. There is still a lot to know about k-transitive digraphs in general, but the structure of 3-transitive digraphs is very well understood [11], and strong 4-transitive digraphs have been characterized [12]; there are even some general results on the structure of strong k-transitive digraphs [13]. The aim of the present work is contributing to the consolidation of 4-transitive digraphs as a well understood family by using its rich structure to solve problems that are usually difficult for general digraphs: the Laborde-Payan-Xoung Conjecture, Seymour's Second Neighborhood Conjecture, and characterizing 4-transitive digraphs having 2- and 3-kernels.

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1]. For a vertex $v \in V(D)$, we define the *out-neighborhood* of v in D, $N^+_D(v)$, as the set $N^+_D(v) = \{u \in V(D) \colon (v,u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript D. The elements of $N^+(v)$ are called the *out-neighbors* of v, and the *out-degree* of v, $d^+_D(v)$, is the number of out-neighbors of v. Definitions of *in-neighborhood, in-neighbors* and *in-degree* of v are analogously given. An arc $(u,v) \in A(D)$ is called *asymmetrical* (respectively symmetrical) if $(v,u) \notin A(D)$ (respectively $(v,u) \in A(D)$). We say that a vertex u *reaches* a vertex v in D if a directed uv-directed path (a path with initial vertex u and terminal vertex v) exists in D. The *distance* from vertex u to vertex v, $d_D(u,v)$, is the length of the shortest uv-path in D.

If D is a digraph and $X, Y \subseteq V(D)$, an XY*-arc* is an arc with initial vertex in X and terminal vertex in Y. If $X \cap Y = \emptyset$, $X \to Y$ will denote that $(x,y) \in A(D)$ for every $x \in X$ and $y \in Y$. Again, if X and Y are disjoint, $X \Rightarrow Y$ will denote that there are not YX-arcs in D. When $X \to Y$ and $X \Rightarrow Y$ we will simply write $X \mapsto Y$. If D_1, D_2 are subdigraphs of D, we will abuse notation to write $D_1 \to D_2$ or D_1D_2-arc, instead of $V(D_1) \to V(D_2)$ or $V(D_1)V(D_2)$-arc, respectively. Also, if $X = \{v\}$, we will abuse notation to write $v \to Y$ or vY-arc instead of $\{v\} \to Y$

or $\{v\}Y$-arc, respectively. Analogously if $Y = \{v\}$. The *distance* from X to Y, $d_D(X, Y)$, is defined as $\min\{d_D(x, y) \colon\ x \in X, y \in Y\}$. As before, we will write $d_D(u, Y)$ and $d_D(X, v)$ instead of $d_D(\{u\}, Y)$ and $d_D(X, \{v\})$.

A digraph is *strongly connected* (or *strong*) if for every $u, v \in V(D)$, there exists a uv-directed path, i.e., a directed path with initial vertex u and terminal vertex v. A *strong component* (or *component*) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of D, and $(S, T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly, $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d^+_{D^\star}(T) = 0$. An *initial component* of D is a strong component S of D such that $d^-_{D^\star}(S) = 0$.

The rest of the paper is ordered as follows. In Section 2, some basic lemmas that will be used through the rest of the paper are introduced, and 4-transitive digraphs having a 3-kernel are characterized. In Section 3, 4-transitive digraphs having a kernel are characterized. In Section 4, the characterization of the previous section is used to prove the Laborde-Payan-Xuong Conjecture for 4-transitive digraphs. The final section of this article is devoted to consider a brief summary of the contributions made, and propose further research directions. As an example of the potential of this familiy of digraphs, Seymour's Second Neighborhood Conjecture is also proved for 4-transitive digraphs in the final section.

2. 3-Kernels in 4-Transitive Digraphs

In this section we characterize the 4-transitive digraphs having a 3-kernel. Our next lemma is a simple property of 4-transitive digraphs having a directed 3-cycle extension as a subdigraph.

Lemma 2. *Let D be a 4-transitive digraph, and let $H \subseteq D$ be a 3-cycle extension with cyclic partition $\{V_0, V_1, V_2\}$. If (v_0, v) is an arc of D such that $v_0 \in V_0$ and $v \in V(D) \setminus V(H)$, then $V_0 \to v$.*

Proof. Let (v_0, v) be an arc of D with $v_0 \in V_0$ and $v \in V(D) \setminus V(H)$. If $V_0 = \{v_0\}$, then we trivialy have that $V_0 \to v$. Let us suppose that $|V_0| \geq 2$. Consider an arbitrary vertex $y \in V_0 \setminus \{v_0\}$. Recalling that H is a directed 3-cycle extension, we can find $v_1 \in V_1$ and $v_2 \in V_2$ such that (y, v_1, v_2, v_0, v) is a directed 4-path in D. But D is 4-transitive, thus we have that $(y, v) \in A(D)$. Since y was chosen arbitrarily, we conclude that $V_0 \to v$. ■

We will also use the following lemma found in [12].

Lemma 3. *Let $k \geq 2$ be an integer, let D be a k-transitive digraph and let C be a directed n-cycle, with $n \geq k$ and $(k-1, n) = 1$. If $v \in V(D) \setminus V(C)$ is such that a directed vC-path exists in D, then $v \to C$.*

Dually, we conclude that if v is such that a directed Cv-path exists, then $C \to v$. This fact will be used sometimes and, abusing notation, will be referred as Lemma 3. Let us observe that Lemma 3 is true, in particular, for $k = 4$, with $n = 4$ and $n = 5$. This is going to be very useful in the study of 4-transitive digraphs that contain cycles of length 4 or 5.

The next lemma, also proved in [12], tells us that there are only two possibilities for a 4-transitive digraph of circumference 2.

Lemma 4. *Let D be a strong 4-transitive digraph with circumference 2. Then D is the complete biorentation of the star $K_{1,n}$, or is the complete biorentation of the double star $D_{n,m}$.*

The following characterization of the strong 4-transitive digraphs is found in [12].

Theorem 5. *Let D be a strong 4-transitive digraph. Then exactly one of the next affirmations is true.*

(1) *D is a complete digraph.*
(2) *D is a directed 3-cycle extension.*
(3) *D has circumference 3, it contains a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\{V_0, V_1, V_2\}$. At least one symmetric arc $(v_i, v_{i+1}) \in A(D)$ exists in D, where $v_j \in V_j$ for $j \in \{i, i+1\} \pmod 3$ and $|V_i| = 1$ or $|V_{i+1}| = 1$.*
(4) *D has circumference 3 and $UG(D)$ is not 2-edge-connected. Consider $\{S_1, S_2, \ldots, S_n\}$ the vertex set of the maximal 2-edge-connected subgraphs of $UG(D)$. Then $S_i = \{u_i\}$ for every $2 \leq i \leq n$, and $D[S_1]$ contains a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\{V_0, V_1, V_2\}$. There exists a vertex $v_0 \in V_0$ such that $(v_0, u_j), (u_j, v_0) \in A(D)$ for every $2 \leq j \leq n$. Also, $|V_0| = 1$ and $D[S_1]$ has the structure described in (2) or (3), depending on the existence of symmetric arcs.*
(5) *D is a symmetrical 5-cycle.*
(6) *D is a complete biorentation of the star $K_{1,n}$, $n \geq 3$.*
(7) *D is the complete biorentation of the double star $D_{n,m}$.*
(8) *D is a strong digraph of order less than or equal to 4 not included in the previous families of digraphs.*

In Figure 1 we can see examples of digraphs that belong to families (3) and (4) described in Theorem 5.

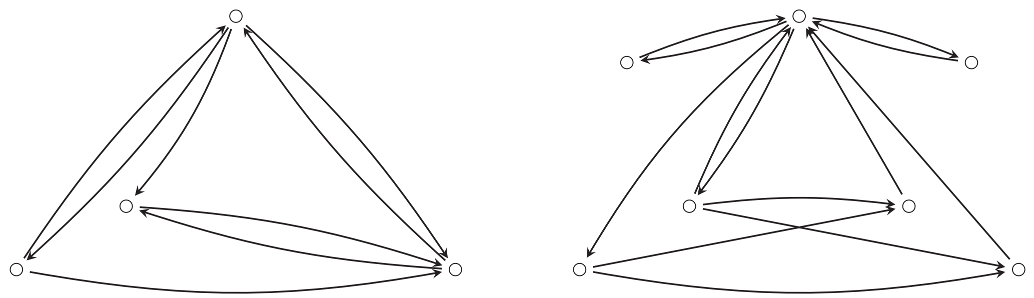

Figure 1. Digraphs of the families (3) and (4) described in Theorem 5.

The following theorem can be found in [15].

Theorem 6. *Let $k \geq 2$ be an integer. Every strong k-transitive digraph different from C_k has a $(k-1)$-kernel.*

We are ready to prove Conjecture 1 for $k = 4$.

Theorem 7. *If D is a 4-transitive digraph, then D has a 3-kernel if and only if none of its terminal components is isomorphic to C_4.*

Proof. Suppose that none of the terminal components of D is isomorphic to C_4. We will prove that D has a 3-kernel by induction on the number of strong components k of D. The case $k = 1$ is proved in Theorem 6.

Now, suppose that we can find a 3-kernel for every 4-transitive digraph D with $k-1$ strong components. Let D be a digraph with k strong components $D_1, D_2, \ldots, D_k$ and suppose without loss of generality that D_1 is an initial component of D. Using the inductive hypothesis we have that $D - D_1$ has a 3-kernel N.

If N is such that it 2-absorbs every vertex of D_1, then N is a 3-kernel for D. Suppose that there exists a vertex $x \in V(D_1)$ that is not 2-absorbed by N. Then $d(x, N) \geq 3$. Since D_1 is an initial component of D, we have that $N \cup \{x\}$ is a 3-independent set of D.

If D_1 is from the families (1), (5) or (6), using the previous observation we have that $N \cup \{x\}$ is a 3-kernel for D.

If D_1 is of type (2) or (3), then D contains a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\{V_0, V_1, V_2\}$. Suppose without loss of generality that $x \in V_0$. It follows from Lemma 2 that $N \cup V_0$ is a 3-independent set. Since clearly V_0 2-absorbs every vertex in V_1 and V_2, and N already is a 2-absorbent set in $D - D_1$, we have that $N \cup V_0$ is a 3-independent, 2-absorbent set, i.e., a 3-kernel of D.

Suppose that D_1 belongs to family (4). If $x = v_0$, then $N \cup \{x\}$ is a 3-kernel of D. If $x \in V_1$, an argument analogous to the one used in the previous case shows that there exists a subset $V_1' \subseteq V_1$ (depending on the existence of symmetric arcs between V_0 and V_1) such that $N \cup V_1'$ is a 3-kernel of D. So, let us suppose that v_0 and V_1 are already 2-absorbed by N. If u_i is 2-absorbed by N for every $2 \leq i \leq n$, then $x \in V_2$ and again a subset V_2' of V_2 exists such that $N \cup V_2'$ is a 3-kernel of D. If u_j is not 2-absorbed by N for some $2 \leq i \leq n$, then $N \cup \{u_i\}$ is a 3-kernel of D.

If D_1 is from the family (7), then D_1 is an orientation of $D_{n,m}$, a double star. Let u and v be the centers of $D_{n,m}$, and let S be the set of vertices of D_1 not 2-absorbed by N. If $\{u, v\} \cap S \neq \emptyset$, then $N \cup \{u\}$ or $N \cup \{v\}$ is a 3-kernel of D. Else, assume without loss of generality that $N(u) \cap S \neq \emptyset$, and choose any vertex $y \in N(u) \cup S$. If $N \cup \{y\}$ is a 3-kernel, we are done. Otherwise, there exists a vertex z in $N(v) \cup S$ which is not 2-absorbed by y. In this case, $N \cup \{y, z\}$ is a 3-kernel of D.

Finally, if D_1 is of type (8) and has order less than or equal to 3, then $N \cup \{x\}$ is a 3-kernel of D. Suppose that D_1 has order 4. If D has circumference 2, then D_1 belongs to the families (6) or (7) and, if D_1 has circumference 3, then D_1 is of type (2), (3) or (4).

If D_1 has circumference 4, then D_1 contains a directed 4-cycle as a subdigraph. But D_1 reaches some terminal component of D, and hence a vertex $v \in N$. Lemma 3 implies that $D_1 \to v$, and hence N is a 3-kernel of D. ■

3. Kernels in 4-Transitive Digraphs

As in the previous section, we begin by characterizing strong 4-transitive digraphs having a kernel, to then use this result to proceed by induction on the number of strong components of a general digraph D.

Lemma 8. *Let D be a strong 4-transitive digraph. Then D has a kernel if and only if D is not isomorphic to any of the following:*

a. *Directed 3-cycle extensions.*
b. *Strong semicomplete digraphs of order 4 without vertices of indegree 3.*
c. *Digraphs of the family (3) described in Theorem 5 with no kernel, i.e., those in which the number of symmetric arcs from V_i to V_{i+1} (mod 3) is less than $|V_{i+1}|$, whenever there is at least one symmetric arc from V_i to V_{i+1} (mod 3).*

Proof. Again we consider the notation used in Theorem 5 and analyze each possibility for D. Suppose that D is not isomorphic to any digraph of the type a, b or c.

If D is of the family (1), then any vertex in $V(D)$ is a kernel for D.

If D belongs to family (3), then D contains a directed 3-cycle extension as a subdigraph with cyclic partition $\{V_0, V_1, V_2\}$ and since D is not isomorphic to

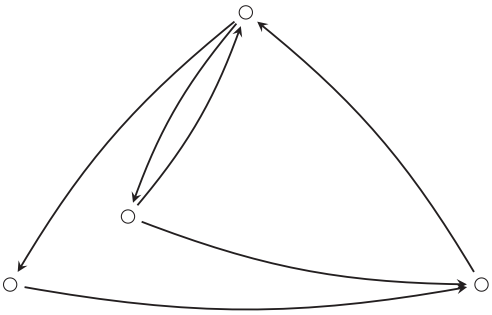

Figure 2. Digraph of type c. from Lemma 8.

any digraph of the type a, we have that classes V_i, V_{i+1} (mod 3) exists such that there are the same number of symmetric arcs from V_i to V_{i+1} that vertices in V_{i+1}. Then V_i absorbs V_{i-1} and V_{i+1} (mod 3), and therefore is a kernel for D.

If D belongs to family (4), then, with the notation of Theorem 5, the set $\{u_2, u_3, \ldots, u_n\} \cup V_2$ is a kernel for D.

If D belongs to any of the families (5), (6) or (7), then D is a symmetric digraph and hence any maximal independent set is a kernel for D.

For the case in which D belongs to family (8), we suppose that D has circumference 4, otherwise D belongs to one of the previous families. Hence, D has order 4 and contains a directed 4-cycle $(v_1, v_2, v_3, v_4, v_1)$. If D is not a semicomplete digraph, then there are vertices v_i and v_j such that $(v_i, v_j), (v_j, v_i) \notin A(D)$. This implies that $\{v_i, v_j\}$ is an independent set that absorbs the remaining two vertices.

If D is a semicomplete digraph, then by hypothesis there is a vertex v such that $d^-(v) = 3$ and therefore $\{v\}$ is a kernel for D. With no more possible cases, the result follows. ■

Theorem 9. *If D is a 4-transitive digraph, then D has a kernel if and only if D has no terminal components isomorphic to digraphs of the families described in Lemma* 8.

Proof. We will proceed by induction on the number of strong components k. If $k = 1$, then D is a strong digraph an the result follows from Lemma 8.

Let D be a 4-transitive digraph with $k > 1$ strong components $D_1, D_2, \ldots, D_k$. Without loss of generality assume that D_1 is an initial component of D; by the inductive hypothesis $D - D_1$ has a kernel N.

If N absorbs every vertex of D_1, then N is a kernel of D. Suppose that there is a vertex $x \in V(D_1)$ that is not absorbed by N. Since D_1 is an initial component, and x is not absorbed by N, we have that $N \cup \{x\}$ is an independent set of D.

With the notation used in Theorem 5 let us consider the different possibilities for D_1. If D_1 belongs to family (1), then $N \cup \{x\}$ is a kernel of D.

If D belongs to family (2) or (3), then D_1 contains a directed 3-cycle extension with cyclic partition $\{V_0, V_1, V_2\}$ as a spanning subdigraph. It follows from Lemma 2 that, for $i \in \{0, 1, 2\}$, either V_i is completely absorbed by N or no vertex in V_i is absorbed by N. Also, since D is 4-transitive, at least one vertex in D_1 must be absorbed by N. Suppose without loss of generality that V_0 is absorbed by N. If V_2 is also absorbed by N, then $N \cup V_1$ is a kernel of D. Otherwise, $N \cup V_1$ is a kernel of D.

In the case that D_1 is of type (4), let $S \subseteq \{u_2, u_3, \ldots, u_n\}$ be the subset of the u_i's not absorbed by N. If $S \neq \emptyset$, then S absorbs V_0. If V_2 is not absorbed by N, then $N \cup S \cup V_2$ is a kernel of D. If V_2 is absorbed by N but V_1 is not, then $N \cup S \cup V_1$ is a kernel for D. And, if both V_1 and V_2 are absorbed by N, then $N \cup S$ is a kernel for D. Now, suppose that $S = \emptyset$. Then u_i is absorbed for every $i \in \{2, 3, \ldots, n\}$. This fact, together with Lemma 2, implies that V_1 is absorbed by N. If V_0 is not absorbed by N, then $N \cup V_0$ is a kernel of D. Otherwise, $N \cup V_2$ is a kernel of D.

If we suppose that D_1 is from the family (5), then Lemma 3 implies that every vertex of D_1 is absorbed by N.

If D_1 is of type (6), then D_1 is a star; let $\{u\}$ be the center of D_1. If u is not absorbed by N, then $N \cup \{u\}$ is a kernel for D. Else, consider $S \subset V(D_1)$, the vertices of D_1 not absorbed by N. Clearly, $N \cup S$ is a kernel of D.

In the case that D_1 belongs to family (7), then D_1 is a double star with centers $\{u, v\}$. Let us consider $S \subseteq V(D_1)$, the vertices of D_1 that are not absorbed by N. If $u, v \notin S$, then S is an independent set and therefore $N \cup S$ is a kernel for D. If $u \in S$, then $u \in N(v) \cap S$, which is an independent set that absorbs every vertex of D_1 and therefore $N \cup (N(v) \cap S)$ is a kernel of D. Analogously if $v \in S$, then $N \cup (N(u) \cap S)$ is a kernel of D.

Since every strong 4-transitive digraph of order at most 4 and circumference 2 or 3 belongs to one of the previous families, if D_1 belongs to family (8), then D_1 has a 4-cycle as a spanning subdigraph. Hence, Lemma 3 implies that D_1 is completely absorbed by N. ■

4. The Laborde-Payan-Xuong Conjecture for 4-Transitive Digraphs

In [14], the following conjecture is proposed.

Conjecture 10. *For every digraph D there is an independent set that intersects every longest path in D.*

This conjecture is known as the Laborde-Payan-Xuong Conjecture and it remains as an open problem for general digraphs. However, it has been proved for several families of digraphs, e.g., quasi-transitive digraphs, line digraphs, arc-local tournaments, path mergeable digraphs, in- and out-semicomplete digraphs and semicomplete k-partite digraphs [3]; 3-quasi-transitive digraphs [16]; locally semicomplete digraphs and locally transitive digraphs which have directed paths of maximum length at most 4 [4].

In this section we will prove this conjecture for 4-transitive digraphs. We will need the following results, the first one of which is folklore.

Lemma 11. *Let D be a digraph with kernel N. For every longest path T in D, we have that $N \cap V(T) \neq \emptyset$.*

The following lemma can be thought as a set of directions to remove vertices from a 4-transitive digraph to obtain a 4-transitive digraph with a kernel. We will refer to the families of digraphs a, b and c of Lemma 8.

Lemma 12. *Let D be a 4-transitive digraph with terminal components $D_1, \ldots, D_r$. If $S = \bigcup_{i=1}^{r} S_i$, where $S_i \subseteq V(D_i)$ is defined as follows:*

- *$S_i = V_0$ if D_i is of type a or c, and has a cyclic partition $\{V_0, V_1, V_2\}$;*
- *$S_i = \{v_4\}$ if D_i is of type b and $v_4 \in V(D_i)$ is not absorbed by v_1, where $d^-(v_1) = 2$;*
- *$S_i = \emptyset$, otherwise,*

then $D - S$ has a kernel.

Proof. Let D_j be any terminal component of D. Note that if D_j is of type a or c, then D_j has a directed 3-cycle extension as a spanning subdigraph with cyclic partition $\{V_0, V_1, V_2\}$. And then $D_j - V_0$ has the kernel $N_j = V_2$.

If D_j is of type b, then D_j is a semicomplete digraph of order 4, such that $d^-(v) < 3$ for every $v \in V(D_j)$. Suppose, without loss of generality, that $V(D) = \{v_1, v_2, v_3, v_3\}$, $d^-(v_1) = 2$ and $N^-(v_1) = \{v_2, v_3\}$. It is clear that $D_j - \{v_4\}$ has the kernel $N_j = \{v_1\}$.

Furthermore, if N is a kernel for $D - S$ and D_j is a terminal component of D of type a or c with cyclic partition $\{V_0, V_1, V_2\}$, then $V_2 \subseteq N$ (because it is the only independent set that can absorb V_1 in $D_j - V_0$). Analogously, if D_j is terminal component of D of type b and v_4 is the vertex not absorbed by v_1, where $d^-(v_1) = 2$, then $v_1 \in N$. ■

Lemma 11 tells us that the kernel of a digraph is an independent set that intersects every longest path, fact that we will use to prove the Laborde-Payan-Xuong conjecture for 4-transitive digraphs.

Theorem 13. *For every* 4*-transitive digraph* D *there exists an independent set intersecting every longest path of* D.

Proof. Let D be a 4-transitive digraph. By Lemma 12, we can find a vertex subset S of the terminal components of D such that $D - S$ has a kernel N. We afirm that N is the independent set we are looking for.

Let T be a longest path in D. If $T \cap S = \emptyset$, then T is a longest path in $D - S$. It follows from Lemma 11 that $N \cap T \neq \emptyset$.

Now, let us supppose that $T \cap S \neq \emptyset$ and $T = (u_1, u_2, \ldots, u_m)$. Since $T \cap S \neq \emptyset$ and S is contained in the terminal components of D, we have that T reaches exactly one terminal component D_j; furthermore, the terminal vertex u_m of T is contained in $V(D_j)$.

Suppose that D_j is of type a or c. Then D_j has a directed 3-cycle extension as a spanning subdigraph with vertex partition $\{V_0, V_1, V_2\}$ and given the construction of S in the proof of Lemma 12 we have that $S \cap D_j = V_0$ and that $V_2 \subseteq N$, which gives us $T \cap S \subseteq V_0$. If $T \cap V_2 = \emptyset$, then $u_m \in V_0$ or $u_m \in V_1$. If $u_m \in V_1$, since T does not pass through V_2, then a vertex $v_2 \in V_2$ exists such that $T' = (u_1, u_2, \ldots, u_m, v_2)$ is a directed path in D longer than T, contradicting the choice of T.

If $u_m \in V_0$ and D_j is of type a, then T does not pass through V_1, because every directed V_1V_0-path passes through V_2. Hence, there is $v_1 \in V_1$ such that $T' = (u_1, u_2, \ldots, u_m, v_1)$ is a directed path in D longer than T, a contradiction.

If $u_m \in V_0$, D_j is of type c and there are no symmetric arcs from V_0 to V_1, then every directed V_1V_0-path passes through V_2. Analogously to the previous case, we can find a directed path T' longer than T.

If $u_m \in V_0$, D_j is of type c and there is at least one symmetric arc from V_0 to V_1, then $|V_0| = 1$ or $|V_1| = 1$. If $|V_0| = 1$ then $u_{m-1} \in V_1$, otherwise we would have that T does not intersect V_1 and with an argument similar to the above we could find a longer path than T. And since $u_{m-1} \in V_1$, then we can find $v_2 \in V_2$ such that $T' = (u_1, u_2, \ldots, u_{m-1}, v_2, u_m)$ is a directed path in D longer than T, which is impossible.

Now, if $|V_1| = 1$, then T can have at most two vertices of V_0. In this case we have that $u_{m-2}, u_m \in V_0$ and $u_{m-1} \in V_1$, but then a vertex $v_2 \in V_2$ exists such that $T' = (u_1, u_2, \ldots, u_{m-2}, u_{m-1}, v_2, u_m)$ is a directed path longer than T, a contradiction.

We conclude that $T \cap V_2 \neq \emptyset$ and therefore $T \cap N \neq \emptyset$.

If D_j is of type b, then given the construction of S in the proof of Lemma 12 we have that $S \cap D_j = \{v_4\}$ where v_4 is the vertex not absorbed by v_1, with $d^-(v_1) = 2$ and $v_1 \in N$. Since D_j is a strong semicomplete digraph, it has a Hamiltonian cycle $C = (v_4, v_3, v_2, v_1, v_4)$. Then any longest path that reaches D_j must use every vertex of D_j. Therefore, $T \cap N \neq \emptyset$. ■

5. Conclusions and Further Directions

In the previous section the Laborde-Payan-Xuong Conjecture was proved for 4-transitive digraphs. It is natural to consider other conjectures for this family of digraphs, such as Seymour's conjecture of the second out-neighborhood. This states that any asymmetric digraph D without loops contains a vertex v, such that $|N^{++}(v)| \geq |N^{+}(v)|$ where

$$N^{++}(v) = \bigcup_{u \in N^{+}(v)} N^{+}(u) \setminus N^{+}(v).$$

We call $N^{+}(v)$ the (first) out-neighborhood of v and $N^{++}(v)$ the second out-neighborhood of v. It is easy to see that this conjecture is true for 4-transitive digraphs.

Proposition 14. *Let D be a 4-transitive digraph with no loops or symmetric arcs. Then there exists $v \in V(D)$ such that $|N^{++}(v)| \geq |N^{+}(v)|$.*

Proof. Let D_j be a terminal component of D. Since D is asymmetrical, then there are only three possibilities for D_j, those are, D_j is an isolated vertex, a digraph of the family (2), or an asymmetric digraph of family (8) described in Theorem 5.

If $D_j = \{x\}$ is an isolated vertex, then x is a vertex with $d^{+}(x) = 0$ and therefore $0{=}|N^{++}(x)| \geq |N^{+}(x)| = 0$.

If D_j is of the family (2), then D_j is a directed 3-cycle extension, with cyclic partition $\{V_0, V_1, V_2\}$. Let V_i be the largest set of the partition. Hence, for any vertex in $x \in V_{i-2}$ (mod 3) we have that $|N^{+}(x)| = |V_{i-1}| \leq |V_i| = |N^{++}(x)|$.

Finally, if D_j is of the family (8), then D_j is an asymmetric digraph of order less than 4. As in previous arguments, we may assume that D_j has order and circumference 4. This implies that D_j is either C_4 or C_4 with one or two diagonals. One can easily find a vertex x that satisfies $|N^{++}(x)| \geq |N^{+}(x)|$. ∎

The search of algorithms that find kernels and other structures in families of digraphs is another problem that has been studied in the past. As it has been already mentioned, in [2], Chvátal shows that the problem of determining whether a given digraph has a kernel is $\mathcal{NP}$-complete. Recently Hell and Hernández-Cruz proved in [10] that the problem of finding 3-kernels in digraphs is also an $\mathcal{NP}$-complete problem.

On the other hand, we know that there are several algorithms that find the strong components of a digraph in linear time. Also, to verify that a terminal component is not isomorphic to any of the families of type a, b and c can be done in polynomial time. So, the problem of finding a kernel in the family of 4-transitive digraphs can be solved in polynomial time. In the same way, we can conclude that the problem of finding a 3-kernel for 4-transitive digraphs can be solved in polynomial time.

Like every time a "well-behaved" family of digraphs is found, it is pertinent to ask what other problems that are usually difficult, are "easy" to solve for 4-transitive digraphs. Also, after 3- and 4-transitive digraphs have been analyzed, it seems to be a good idea to find general results for k-transitive digraphs, like Conjecture 1 proposes. In this direction, we propose the following problem.

Problem 15. For each integer $2 \leq n \leq k-1$, determine the complexity of determining whether a k-transitive digraph has an n-kernel.

If true, Conjecture 1 would show that determining whether a k-transitive digraph has an $(k-1)$-kernel can be done in polynomial time. The results of the present paper and those found in [11] solve the problem for $k = 3$ and $k = 4$; in all cases the answer is that the n-kernel problem, which is usually $\mathcal{NP}$-complete, becomes polynomial in these families of digraphs.

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