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WORM COLORINGS OF PLANAR GRAPHS

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Abstract

Given three planar graphs F, H, and G, an (F, H)-WORM coloring of G is a vertex coloring such that no subgraph isomorphic to F is rainbow and no subgraph isomorphic to H is monochromatic. If G has at least one (F, H)-WORM coloring, then $W^-_{F,H}(G)$ denotes the minimum number of colors in an (F, H)-WORM coloring of G. We show that

(a) $W^-_{F,H}(G) \leq 2$ if $|V(F)| \geq 3$ and H contains a cycle,

(b) $W^-_{F,H}(G) \leq 3$ if $|V(F)| \geq 4$ and H is a forest with $\Delta(H) \geq 3$,

(c) $W^-_{F,H}(G) \leq 4$ if $|V(F)| \geq 5$ and H is a forest with $1 \leq \Delta(H) \leq 2$.

The cases when both F and H are nontrivial paths are more complicated; therefore we consider a relaxation of the original problem. Among others, we prove that any 3-connected plane graph (respectively outerplane graph) admits a 2-coloring such that no facial path on five (respectively four) vertices is monochromatic.

Keywords: plane graph, monochromatic path, rainbow path, WORM coloring, facial coloring.

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1. Introduction and Notations

All graphs considered in this paper are simple planar graphs. We use a standard graph theory terminology according to Bondy and Murty [4]. However, we recall some more important notions.

A *plane* graph is a particular drawing of a planar graph in the Euclidean plane. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. Faces of G are open 2-cells. The boundary of a face f is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face f.

The size of a face f, denoted by $\deg(f)$, is the length of its boundary walk. If f is of size k, then we call it a *k-gonal face*. A k-gonal face is called *odd-gonal* or *even-gonal* if k is odd or even, respectively. Let f be a k-gonal face having a boundary walk $v_0v_1 \cdots v_k = v_0$ with $v_0 \in V(G)$ and $v_iv_{i+1} \in E(G)$, $i = 0, \ldots, k-1$. A *facial path* of f is any path of the form $v_mv_{m+1} \cdots v_{n-1}v_n$, indices modulo k. A *k-path* is a path on k vertices.

A *k-vertex* is a vertex of degree k. A k-vertex is called *odd-vertex* or *even-vertex* if k is odd or even, respectively.

A *vertex coloring* of G is an assignment of colors to the vertices of G, one color to each vertex. If adjacent vertices are assigned distinct colors, then the coloring is a *proper coloring*.

In a vertex colored graph, a subgraph is *rainbow* if its vertices have pairwise distinct colors and it is *monochromatic* if its vertices have the same color. Vertex colorings avoiding monochromatic subgraphs have been studied extensively, see a survey of Tuza [28]. Vertex colorings that avoid rainbow subgraphs have been studied for example by Bujtás *et al.* [7, 8].

Given three graphs F, H, and G, an (F, H)-WORM coloring of G is a vertex coloring such that no subgraph isomorphic to F is rainbow and no subgraph isomorphic to H is monochromatic. If G has at least one (F, H)-WORM coloring, then $W^-_{F,H}(G)$ denotes the minimum number of colors and $W^+_{F,H}(G)$ denotes the maximum number of colors in an (F, H)-WORM coloring of G.

The concept of (F, F)-WORM (or simply F-WORM) coloring was introduced by Goddard, Wash, and Xu in [20, 21] but the idea of colorings with both rainbow and monochromatic constraints is due to Voloshin [29]. The name WORM comes as the abbreviation of WithOut Rainbow and Monochromatic. In [21] the authors explored this new concept and established some basic properties. They focused on the fundamental results and the case that $F = H$ is the path on three vertices. Further results where a cycle or clique is forbidden are given in [20]. Bujtás and

Tuza [10] studied (K_3, K_3)-WORM colorings, where K_3 denotes the complete graph on three vertices. There is only one paper [9] on (F, F)-WORM colorings in which F is not a particular graph (path, cycle, complete graph) but an arbitrary 2-connected graph.

The decision problem of (F, F)-WORM colorability is proved to be NP-complete for $F = P_3$ in [21], for $F = K_3$ in [20], and for every 2-connected graph F in [9].

Lovász [25] proved (in a different language) that $W^-_{P_3,P_3}(G) \leq 2$ for any subcubic graph G. Goddard, Wash, and Xu improved this result; they showed that $W^-_{P_3,P_3}(G) \leq 2$ holds for any graph G with at least one (P_3, P_3)-WORM coloring, see [20]. Bujtás and Tuza [9] showed that for every 2-connected graph F and a positive integer k there exists a graph G such that $W^-_{F,F}(G) = k$.

There are several papers that study "WORM" colorings of plane graphs where constraints are given by faces. In this type of coloring it is required that there is neither a rainbow nor a monochromatic face, see e.g. [14, 23, 24].

Motivated by the above mentioned papers and by talks of Bujtás and Goddard at the conference CID 2015, we show (in Section 2) that $W^-_{F,H}(G) \leq 4$ for every planar graph G and every F of order at least five or H not being a path. Particular attention is paid to the cases when H is a path. Then we introduce (in Section 3) a *facial* (P_k, P_ℓ)-WORM *coloring* as a vertex coloring of a plane graph G having neither a rainbow facial k-path nor a monochromatic facial ℓ-path. If G has at least one facial (P_k, P_ℓ)-WORM coloring, then $W^-_{k,\ell}(G)$ denotes the minimum number of colors used in a facial (P_k, P_ℓ)-WORM coloring of G. The problem is to determine $W^-_{k,\ell}(G)$ for given k, ℓ and G. Among others, we prove that any 3-connected plane graph (respectively outerplane graph) admits a 2-coloring such that no facial path on five (respectively four) vertices is monochromatic. Note that the facial (P_k, P_ℓ)-WORM coloring is not hereditary.

2. WORM Colorings of Planar Graphs

Let G, F, and H be planar graphs. Any coloring of G with at least two colors involves a rainbow P_2, therefore in the following we will assume that F has at least three vertices.

2.1. H contains a cycle

The following result is due to Broersma *et al.* [6]. For the sake of completeness, we provide its proof.

Theorem 1. *Let G, F, and H be planar graphs. If $|V(F)| \geq 3$ and H contains a cycle, then $W^-_{F,H}(G) \leq 2$.*

Proof. Consider a plane embedding of G. First we extend G to a plane triangulation T by adding some edges. Since $W^-_{F,H}(G) \leq W^-_{F,H}(T)$, it suffices to show that $W^-_{F,H}(T) = 2$.

Assume that H contains a 3-cycle. By the Four Color Theorem [1], T has a proper coloring with at most four colors, say a, b, c, d. Note that every 3-cycle uses three different colors. If we assign 1 to all the vertices colored with a, b and assign 2 to all the vertices colored with c, d, then we obtain a 2-coloring of T such that no 3-cycle is monochromatic. Consequently, T contains no monochromatic copy of H.

Now assume that H contains no 3-cycle. Thomassen [28] proved that any 2-coloring of the vertices of the outer triangle of any triangulation can be extended to a 2-coloring of the whole triangulation such that there is no monochromatic cycle of length greater than three. So T admits a 2-coloring such that among all of its cycles only 3-cycles can be monochromatic. Hence, T cannot contain any monochromatic copy of H.

The colorings described above use only two colors, therefore no subgraph on at least three vertices is rainbow. ■

Corollary 2 [10]**.** *For any planar graph G it holds $W^-_{K_3,K_3}(G) \leq 2$.*

2.2. *H* is a forest with maximum degree at least three

The linear vertex arboricity of a graph G is the minimum number of subsets into which the vertex set of G can be partitioned so that every subset induces a linear forest (i.e., a forest in which all vertices have degree at most two).

Theorem 3. *Let G, F, and H be planar graphs. If $|V(F)| \geq 4$ and H is a forest with maximum degree at least three, then $W^-_{F,H}(G) \leq 3$. Moreover, it is NP-hard to decide whether $W^-_{F,H}(G) = 2$ if H is a tree.*

Proof. Poh [26] and independently Goddard [19] proved that the linear vertex arboricity of any planar graph is at most three. So G can be colored with at most three colors such that each its monochromatic component is a path.

Clearly, $W^-_{F,H}(G) = 2$ if and only if G has a 2-coloring without monochromatic H. Broersma *et al.* [6] proved that for every tree T with at least two edges it is NP-hard to decide whether a planar graph has a 2-coloring without monochromatic T. ■

2.3. *H* is a forest with maximum degree at most two

Let $\chi(G)$ denote the chromatic number of G.

Theorem 4. *Let G be a planar graph. If $k \leq \chi(G)$, then G admits no (P_k, P_2)-WORM coloring.*

Proof. Every (P_k, P_2)-WORM coloring of G is a proper coloring since G cannot contain any monochromatic P_2. Fung [17] proved that every proper coloring of any graph G with $\chi(G)$ colors involves a rainbow path on $\chi(G)$ vertices. The same assertion holds when a proper coloring uses more than $\chi(G)$ colors. Let c be a proper vertex coloring of G. Define an auxiliary digraph D (which is an orientation of G) in the following way. Let uv be an edge of G. If $c(u) < c(v)$, then uv is a directed edge in D, otherwise vu. Roy [27] proved that any orientation of any graph G contains a directed path on $\chi(G)$ vertices. So, D contains a directed path $v_1 \cdots v_{\chi(G)}$. By the construction of D we have $c(v_1) < \cdots < c(v_{\chi(G)})$, consequently, $v_1 \cdots v_{\chi(G)}$ is a rainbow path in G. ∎

Theorem 5. *Let G, F, and H be planar graphs. If $|V(F)| \geq 5$ and $H = P_2$, then $W^-_{F,H}(G) = \chi(G) \leq 4$. Moreover, if G is non-bipartite, then determining whether G has an (F, H)-WORM coloring with three colors is NP-hard.*

Proof. Clearly, $W^-_{F,P_2}(G) \geq \chi(G)$. On the other hand, the Four Color Theorem implies that $\chi(G) \leq 4$. Therefore, no proper coloring of G with $\chi(G)$ colors involves a rainbow F.

Determining whether a non-bipartite graph admits a proper 3-coloring is NP-hard, even when restricted to planar graphs, see [18]. ∎

Theorem 6. *Let G, F, and H be planar graphs. If $|V(F)| \geq 5$ and H is a linear forest with $|E(H)| \geq 1$, then $W^-_{F,H}(G) \leq 4$. Moreover, it is NP-hard to decide whether G has an (F, P_n)-WORM coloring with three (respectively two) colors, $n \geq 3$.*

Proof. Any proper coloring of G is also an (F, H)-WORM coloring.

Broersma *et al.* [6] proved that it is NP-hard to decide whether a planar graph has a 3-coloring (respectively 2-coloring) without monochromatic P_n, $n \geq 3$. ∎

2.4. Open problems

There are three open problems when considering WORM colorings of planar graphs.

Problem 1. Determine $W^-_{F,H}(G)$ when $F = P_3$ and H is a tree.

Problem 2. Determine $W^-_{F,H}(G)$ when $F = P_4$ and $H = P_n$.

Problem 3. Determine $W^-_{F,H}(G)$ when $F = K_{1,3}$ and $H = P_n$.

In the rest of this section we discuss the present situation concerning the above mentioned problems.

Theorem 7. *Let G, F, and H be planar graphs with $\chi(G) = 4$. If $|V(F)| \geq 4$ and G contains a matching M such that every 3-cycle of G contains an edge of M, then $W^-_{F,P_n}(G) \leq 3$ for any $n \geq 3$.*

Proof. If we subdivide every edge in M with a 2-vertex, then we obtain a triangle-free planar graph, say H. By Grötzsch's theorem, H admits a proper coloring with at most three colors. This coloring of H induces a coloring of G. Clearly, if two adjacent vertices u, v of G have the same color, then uv belongs to M. Since M is a matching, any monochromatic path in G has at most two vertices. ■

Esperet and Joret [15] proved that for every $\Delta \geq 2$ there exists a constant $f(\Delta)$ such that every planar graph with maximum degree Δ has a 3-coloring in which each monochromatic component has size at most $f(\Delta)$. This result implies that every planar graph G with maximum degree Δ admits a $(K_{1,3}, P_n)$- and a (P_4, P_n)-WORM 3-coloring for $n \geq f(\Delta)$.

On the other hand, Chartrand, Geller, and Hedetniemi [11] proved that for every positive integer n, there exists a plane triangulation G such that any its 3-coloring involves a monochromatic path of length n.

If $\chi(G) = 3$, then trivially $W^-_{P_4,P_n}(G) \leq 3$ and $W^-_{K_{1,3},P_n}(G) \leq 3$. So the natural question is the following: Is it true that $W^-_{P_4,P_n}(G) \leq 2$ or $W^-_{K_{1,3},P_n}(G) \leq 2$ when G is 3-colorable?

The answer is negative in general. Axenovich, Ueckerdt, and Weiner [2] constructed for every $t \geq 2$ a planar graph G_t of girth 4 (triangle-free planar graphs are 3-colorable, see [22]) such that in any 2-coloring of G_t there is a monochromatic path of length at least t.

On the other hand, Borodin, Kostochka, and Yancey [5] proved that the vertices of each planar graph of girth at least 7 can be 2-colored so that each monochromatic component has at most 2 vertices. Axenovich, Ueckerdt, and Weiner [2] proved a similar result for planar graphs of girth 6. They showed that the vertices of each planar graph of girth at least 6 can be 2-colored so that each monochromatic component is a path of length at most 14.

Problem 4 [2]**.** Is there a positive integer n such that any planar graph G of girth 5 admits a 2-coloring such that each monochromatic component is a path of length at most n?

Note that there are planar graphs and integers n that admit no (P_n, P_3)-WORM coloring. For instance, the graph of the octahedron admits no (P_3, P_3)-WORM coloring. Actually, determining if a planar graph has a (P_3, P_3)-WORM coloring is NP-complete, since (P_3, P_3)-WORM coloring is equivalent to defective (2,1)-coloring (i.e., a 2-coloring such that every vertex has at most one neighbor of the same color) and this is NP-complete in planar graphs, see [13].

3. Facial Worm Colorings of Plane Graphs

From the discussion in Subsection 2.4 it follows that the cases when both F and

H are paths are more complicated. Therefore we have introduced the facial (P_k, P_ℓ)-WORM coloring of a plane graph, which is a relaxation of the original problem. Recall that a facial (P_k, P_ℓ)-WORM coloring of a plane graph G is a vertex coloring having neither a rainbow facial k-path nor a monochromatic facial ℓ-path.

3.1. Facial (P_k, P_2)-WORM colorings

Clearly, any facial (P_k, P_ℓ)-WORM coloring is also a facial $(P_k, P_{\ell+1})$-WORM coloring. Therefore $W^-_{k,\ell+1}(G) \leq W^-_{k,\ell}(G) \leq W^-_{k,2}(G)$ for $\ell \geq 2$.

Let G be a connected plane graph. Theorem 5 implies that $W^-_{k,2}(G) \leq 4$ for any $k \geq 5$. On the other hand, for every plane graph with chromatic number 4 it holds $W^-_{k,2}(G) \geq 4$. So the bound is tight. Now consider a wheel on $2n$, $n \geq 2$, vertices. It is easy to see that its chromatic number is four. From Theorem 4 it follows that there is no (P_4, P_2)-WORM coloring of such a wheel. On the other hand, we show that every plane graph admits a facial (P_4, P_2)-WORM coloring.

Theorem 8. *Let G be a connected plane graph. Then $W^-_{4,2}(G) = \chi(G) \leq 4$.*

Proof. The inequality $W^-_{4,2}(G) \geq \chi(G)$ trivially holds.

Now assume that $\chi(G) = k$. If $k \leq 3$, then any proper k-coloring of G is a facial (P_4, P_2)-WORM coloring, i.e., $W^-_{4,2}(G) \leq k = \chi(G)$.

So assume that $\chi(G) = 4$. Insert into each face f of G a new vertex v_f and join it by an edge with every vertex on the boundary of f. The obtained graph is a plane semitriangulation T (i.e., a plane multigraph triangulating the plane), therefore it has a proper coloring with at most four colors. This coloring of T induces a coloring of G such that on the vertices of any face f there are used at most three colors. Hence, it is a facial (P_4, P_2)-WORM coloring. Consequently, $W_{4,2}(G) \leq 4$. ■

It is easy to see that a plane graph is bipartite if and only if every its face has an even size (one implication is trivial and the second one can be proved by induction on the number of faces). Observe that no plane graph with an odd-gonal face admits a facial (P_3, P_2)-WORM coloring. Moreover, for every bipartite graph G it holds $W^-_{3,2}(G) = 2$. Notice that no connected graph on at least two vertices has a facial (P_2, P_2)-WORM coloring.

3.2. Facial (P_3, P_ℓ)-WORM colorings

Observe that any facial (P_k, P_ℓ)-WORM coloring is a facial (P_{k+1}, P_ℓ)-WORM coloring too. Hence $W^-_{k+1,\ell}(G) \leq W^-_{k,\ell}(G) \leq W^-_{3,\ell}(G)$ for $k \geq 3$.

First we show that not every plane graph admits a facial (P_3, P_3)-WORM coloring.

Theorem 9. *For any integer $n \geq 12$ there exists a connected plane graph on n vertices having no facial (P_3, P_3)-WORM coloring.*

Proof. First we take a plane drawing of the cycle $v_1v_2 \cdots v_{12}$. Then we insert the diagonals v_1v_5, v_5v_9, v_9v_1 into the inner face and we insert the diagonals v_iv_{i+2}, $i = 1, 3, 5, 7, 9, 11$, into the outer face, where $v_{13} := v_1$. In such a way we obtain a plane graph G_{12}, see Figure 1 for illustration.

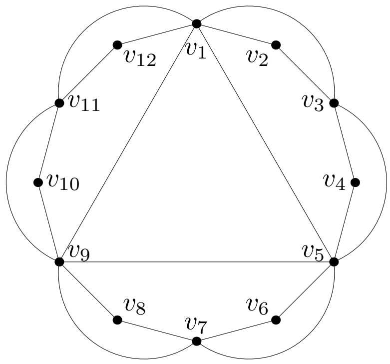

Figure 1. The graph G_{12}.

Suppose that G_{12} admits a facial (P_3, P_3)-WORM coloring c. The face f determined by the vertices v_1, v_5, v_9 has size 3. The fact that no facial 3-path is rainbow implies that there is a pair of adjacent vertices on the boundary of f that have the same color. Without loss of generality we can assume that $c(v_1) = c(v_5) = a$. Then $a \notin \{c(v_2), c(v_3), c(v_4)\}$, otherwise G_{12} contains a monochromatic facial 3-path. Let $c(v_3) = b$. Then necessarily $c(v_2) = b$ because $v_1v_2v_3$ is a facial 3-path. Similarly, the color of v_4 must be b since $v_3v_4v_5$ is a facial 3-path. So $c(v_2) = c(v_3) = c(v_4) = b$, a contradiction.

If $n > 12$, we take G_{12}, then add together $n - 12$ vertices inside triangular faces different from the 3-face $v_1v_5v_9$, and finally extend this graph to a connected plane graph by adding some edges. ■

Lovász [25] showed that every cubic graph admits a 2-coloring such that every vertex has at most one neighbor of the same color. This implies that any connected subcubic plane graph has a facial (P_3, P_3)-WORM coloring with two colors. On the other hand there are plane graphs with maximum degree six that have no facial (P_3, P_3)-WORM coloring. In the light of Theorem 9 the following problems seem to be interesting.

Problem 5. Are there plane graphs with maximum degree four (respectively five) with no facial (P_3, P_3)-WORM coloring?

Problem 6. Which plane graphs admit a facial (P_3, P_3)-WORM coloring?

The following result is a corollary of Theorem 1.

Theorem 10. *Let G be a plane triangulation. Then $W^-_{3,3}(G) = 2$.*

Let us recall that the (geometric) *dual* $G^* = (V^*, E^*, F^*)$ of the plane graph $G = (V, E, F)$ can be defined as follows (see [4], pp. 252): There is a vertex f^* of G^* corresponding to each face f of G, and there is an edge e^* of G^* corresponding to each edge e of G; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it).

Theorem 11. *Let G be a plane graph with $2n$ odd-gonal faces and let G^* be its dual. If G^* has n vertex-disjoint paths with odd-vertices as ends, then $W^-_{3,4}(G) \leq 2$, i.e., there is a 2-coloring of G without a monochromatic facial 4-path.*

Proof. Let $P_i = x^*_i - y^*_i$, $i = 1, 2, \ldots, n$, be vertex-disjoint paths in G^*, where x^*_i, y^*_i are odd-vertices.

First we create an auxiliary graph H from G by subdividing some of its edges. For every edge e^* of P_i in G^* we subdivide the corresponding edge e in G with a new 2-vertex. We employ this procedure for every path P_i.

Observe that the resulting plane graph H has only even-gonal faces, hence it is bipartite and admits a proper coloring with two colors.

Notice that any former odd-gonal face of G is now, in H, incident with exactly one new 2-vertex, and any former even-gonal face is incident with exactly two new 2-vertices or with no new 2-vertex.

The coloring of H induces a coloring of G. Evidently no rainbow facial 3-path appears in G and all monochromatic facial paths have length at most two. ∎

Corollary 12. *Let G be a plane graph whose dual G^* is hamiltonian. Then $W^-_{3,4}(G) \leq 2$.*

Chartrand, Geller, and Hedetniemi [12] proved that for every positive integer n, there exists an outerplanar graph G such that any its 2-coloring involves a monochromatic path of length n. On the other hand, we prove the following.

Theorem 13. *Let G be an outerplane graph. Then $W^-_{3,4}(G) \leq 2$, i.e., G admits a 2-coloring without a monochromatic facial 4-path.*

Proof. Suppose there is a counterexample to Theorem 13. Let G be a counterexample with the minimum number of edges among all counterexamples. First, we prove several structural properties of G.

Claim 1. *G does not contain any bridge.*

Proof. Let uv be a bridge of G. Let $G \setminus \{uv\}$ be the graph obtained from G by deleting the edge uv. Clearly, $G \setminus \{uv\}$ consists of two outerplane graphs G_u and G_v, where G_u, G_v contains the vertex u, v, respectively. The graphs G_u, G_v have fewer edges than G, therefore they admit facial (P_3, P_4)-WORM colorings with colors $1, 2$. We may assume u and v receive distinct colors. Then the colorings of G_u and G_v induce a facial (P_3, P_4)-WORM coloring of G, a contradiction. □

Claim 2. *G does not contain adjacent 2-vertices.*

Proof. Let uv be an edge such that $\deg(u) = \deg(v) = 2$. Let $G' = G \setminus \{u, v\}$ be the graph obtained from G by deleting the vertices u and v. Since G' has fewer edges than G, it has a facial (P_3, P_4)-WORM coloring c' with at most two colors. We extend the coloring c' of G' to the required coloring of G in the following way.

Assume that u is adjacent to u_1 and v is adjacent to v_1 in G ($u_1 = v_1$ is possible). If $c'(u_1) = c'(v_1)$, then we color the vertices u and v with the same color different from $c'(u_1) = c'(v_1)$. Otherwise, we put $c(u) = c'(v_1)$ and $c(v) = c'(u_1)$. □

Claim 3. *G contains neither adjacent 3-faces nor a 3-face adjacent to a 4-face.*

Proof. Let f_1 be a 3-face and let f_2 be a k-face, $k \in \{3, 4\}$. Let e be a common edge of f_1 and f_2. The graph $G' = G \setminus \{e\}$ has fewer edges than G, therefore, it has a facial (P_3, P_4)-WORM coloring. Observe that this coloring induces a facial (P_3, P_4)-WORM coloring of G. □

Claim 4. *G contains at least one of the configurations depicted in Figure 2 (or symmetric one), where x_1, x_2, x_3 are 2-vertices, y_1, y_2 are 3-vertices, and z is a 4-vertex.*

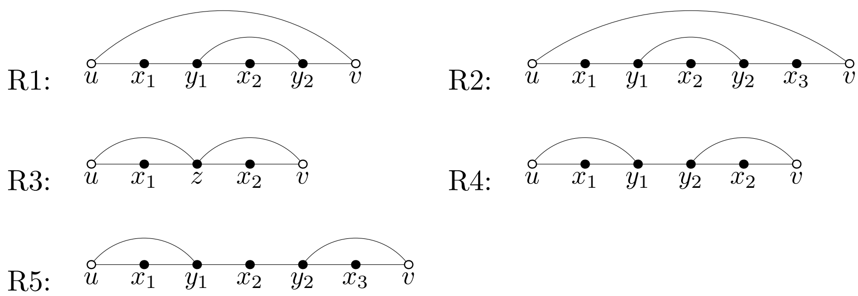

Figure 2. Reducible configurations.

Proof. Let H be the weak dual (that is, an induced subgraph of the dual graph whose vertices correspond to the bounded faces of the primal graph) of G. Fleischner *et al.* [16] showed that the weak-dual of any outerplane graph is a forest.

Observe that the forest H contains a 2-vertex adjacent with exactly one leaf or it contains a k-vertex v, $k \geq 2$, adjacent with two leaves, say v_1, v_2, such that v_1vv_2 is a facial path in H. In the former case one of the configurations R1, R2 is presented in G. In the latter case one of the configurations R3, R4, R5 can be found in G. □

Next we will show that all configurations of Claim 4 are reducible.

Case 1. First suppose that the configuration R1 appears in G. Let $G' = G \setminus \{x_1, y_1, x_2, y_2\}$. The graph G' has a facial (P_3, P_4)-WORM coloring c with colors a, b which can be extended to a facial (P_3, P_4)-WORM coloring of G in the following way: if $c(u) = c(v) = a$, then put $c(y_1) = c(x_2) = a$ and $c(x_1) = c(y_2) = b$; if $c(u) = a$ and $c(v) = b$, then put $c(y_1) = c(y_2) = a$ and $c(x_1) = c(x_2) = b$.

Case 2. Now suppose that the configuration R2 appears in G. Let $G' = G \setminus \{x_1, y_1, x_2, y_2, x_3\}$. A facial (P_3, P_4)-WORM coloring c of G' can be extended to a facial (P_3, P_4)-WORM coloring of G in the following way: if $c(u) = c(v) = a$, then set $c(y_1) = c(y_2) = a$ and $c(x_1) = c(x_2) = c(x_3) = b$; if $c(u) = a$ and $c(v) = b$, then set $c(x_2) = c(x_3) = a$ and $c(x_1) = c(y_1) = c(y_2) = b$.

Case 3. If the configuration R3 appears in G, then $G' = G \setminus \{x_1, x_2\}$ and the extension is the following: if $c(u) = c(v) = a$, then $c(x_1) = c(x_2) = b$; if $c(u) = a$ and $c(v) = b$, then $c(x_1) = b$ and $c(x_2) = a$.

Case 4. If the configuration R4 appears in G, then we delete the vertices x_1, y_1, y_2, x_2 from G and add a new edge uv if it is not already present in G. The extension is the following: if $c(u) = c(v) = a$, then $c(y_1) = a$ and $c(x_1) = c(y_2) = c(x_2) = b$; if $c(u) = a$ and $c(v) = b$, then $c(x_1) = c(y_1) = b$ and $c(y_2) = c(x_2) = a$.

Case 5. If the configuration R5 appears in G, then we delete the vertices x_1, y_1, x_2, y_2, x_3 from G and add a new edge uv if it is not already present in G. The extension is the following: if $c(u) = c(v) = a$, then $c(x_2) = a$ and $c(x_1) = c(y_1) = c(y_2) = c(x_3) = b$; if $c(u) = a$ and $c(v) = b$, then $c(x_1) = c(y_1) = c(x_2) = b$ and $c(y_2) = c(x_3) = a$. ■

Theorem 14. *Let G be a 3-connected plane graph. Then $W^-_{3,5}(G) \leq 2$, i.e., G admits a 2-coloring without a monochromatic facial 5-path.*

Proof. It is well known that any connected plane graph has an even number of odd-gonal faces. Let this number be $2n$. Since G is 3-connected, its dual G^* is also 3-connected. By a theorem of Barnette [3], G^* contains a spanning tree of maximum degree at most three. Let S be a minimal subtree of this spanning tree that contains all odd-vertices of G^*. We color the vertices of S with black and white in the following way: Let all vertices of S that correspond to odd-vertices of G^* be black and all other vertices of S be white. Notice that every leaf of S is black.

Let Q be a closed shortest walk containing all edges of S starting in a leaf x_1^* of S. We label the black vertices $x_1^*, y_1^*, x_2^*, y_2^*, \dots, x_n^*, y_n^*$ by the order of their first occurrence on Q. We define P_i as the segment of Q from x_i^* to y_i^*, see Figure 3 for illustration.

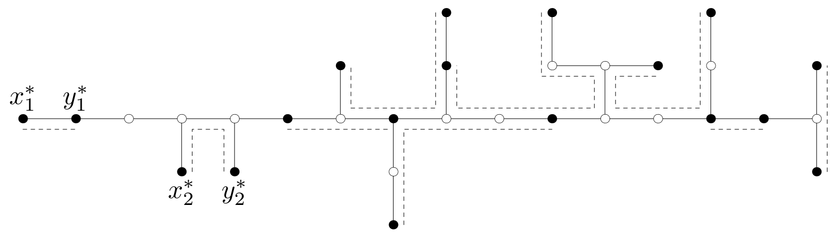

Figure 3. A minimal subtree S.

Notice that no leaf of S can appear as an internal vertex of some P_j.

Any edge e^* of G^* corresponds to an edge e of G. If the edge e^* appears on t paths from $\{P_1, \dots, P_n\}$ (notice that any edge appears on at most two such paths), then we subdivide the corresponding edge e of G by t vertices of degree two. If we do this for every edge e of G, we obtain a plane graph H.

Let $\deg_{P_i}(x^*)$ denote the degree of a vertex x^* of G^* on the path P_i (i.e., $\deg_{P_i}(x^*) = 1$ if x^* is an endvertex of P_i, $\deg_{P_i}(x^*) = 2$ if x^* is an internal vertex of P_i and $\deg_{P_i}(x^*) = 0$ if x^* does not lie on P_i). Put

$$\sigma(x^*) = \sum_{i=1}^{n} \deg_{P_i}(x^*).$$

Then for every odd-vertex v^* of G^* we have $\sigma(v^*) \in \{1, 3, 5\}$ and for every even-vertex v^* of G^* it holds $\sigma(v^*) \in \{0, 2, 4, 6\}$.

Let $\tilde{f}$ denote the face of H that corresponds to the face f of G. Then for every face $\tilde{f}$ of H we have

$$\deg_H(\tilde{f}) = \deg_G(f) + \sigma(f^*),$$

where f^* is the vertex of G^* corresponding to the face f of G. Because the vertex f^* is odd in G^* if and only if the corresponding face f is an odd-gonal face, the graph H is bipartite and so it has a proper coloring c with two colors. As the maximum degree of S is at most three, we subdivided at most three edges on the boundary of every face of G. Therefore, the coloring c restricted to the vertices of G is a facial (P_3, P_5)-WORM one. ■

Using the same approach as in the proof of Theorem 14 one can prove the following.

Theorem 15. *Let G^* be the dual of a connected plane graph G. Let $S \subseteq G^*$ be a tree containing all odd vertices of G^*. If the maximum degree of S is d, then*

$$W^-_{3,d+2}(G) \leq 2.$$

We strongly believe that the following is true.

Conjecture 16. *Let G be a connected plane graph . Then $W^-_{3,4}(G) \leq 2$.*

4. Concluding Remarks

1. Since any proper coloring of P_n with two colors is automatically a (P_k, P_ℓ)-WORM coloring, we have $W^-_{P_k,P_\ell}(P_n) \leq 2$ for $k, n \geq 3$, $\ell \geq 2$.

2. It is easy to determine the formula for $W^+_{F,H}(G)$ when F, H, and G are paths. Note that in this case the "general" WORM and facial WORM colorings coincide.

(P_k, P_k)-WORM colorings of P_n were studied in [20]. The authors there determined the exact value of $W^+_{P_k,P_k}(P_n)$ for $k \geq 3$. The idea of their proof also works for (P_k, P_ℓ)-WORM colorings, $k, \ell \geq 3$.

Theorem 17. *If $k, \ell, n \geq 3$, then $W^+_{P_k,P_\ell}(P_n) = n - \left\lceil \frac{n-k+1}{k-1} \right\rceil$.*

In case when the coloring is proper we get the following.

Theorem 18. *If $k, n \geq 3$, then $W^+_{P_k,P_2}(P_n) = n - \left\lceil \frac{n-k+1}{k-2} \right\rceil$.*

Proof. Let the path P_n be $v_1 v_2 \cdots v_n$. First we prove the lower bound.

The case $k = 3$ is trivial. If $k = 4$, then color the vertices v_{2i}, $i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, with the same color and the other vertices with different colors. The total number of colors is $n - \left\lfloor \frac{n}{2} \right\rfloor + 1$, as claimed. Let $k \geq 5$. Color the vertices $v_{i(k-2)}$ and $v_{i(k-2)+2}$ with color i for $i = 1, 2, \ldots, \left\lfloor \frac{n}{k-2} \right\rfloor$, and then color the remaining vertices with different colors. This proper coloring uses $n - \left\lceil \frac{n-k+1}{k-2} \right\rceil$ colors. Moreover, any path of length k contains two vertices of the same color.

To prove the upper bound we use induction on n for fixed k. For $n \leq 2k - 3$ the claim holds. By the induction hypothesis, the number of colors used on the first $n - (k - 2)$ vertices of P_n is at most

$$n - (k-2) - \left\lceil \frac{n-(k-2)-k+1}{k-2} \right\rceil = n - \left\lceil \frac{n-k+1}{k-2} \right\rceil - (k-3).$$

The last $k-2$ vertices of P_n use at most $k-3$ colors other than those used on the first $n-(k-2)$ vertices, otherwise the last k vertices of P_n form a rainbow path. Therefore, the total number of colors used is at most $n - \left\lceil \frac{n-k+1}{k-2} \right\rceil$. ■

3. As a by-product of the proof of Theorem 14 we have the following.

Lemma 19. *In a* 3*-connected plane graph, one can subdivide at most three edges of each face such that the resultant graph is bipartite.*

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