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RAINBOW CONNECTION NUMBER OF GRAPHS WITH DIAMETER 3

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Abstract

A path in an edge-colored graph G is *rainbow* if no two edges of the path are colored the same. The *rainbow connection number* $rc(G)$ of G is the smallest integer k for which there exists a k-edge-coloring of G such that every pair of distinct vertices of G is connected by a rainbow path. Let $f(d)$ denote the minimum number such that $rc(G) \leq f(d)$ for each bridgeless graph G with diameter d. In this paper, we shall show that $7 \leq f(3) \leq 9$.

Keywords: edge-coloring, rainbow path, rainbow connection number, diameter.

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1. Introduction

All graphs in this paper are undirected, finite, and simple. We refer to book [2] for notation and terminology not described here. A path $u_0u_1 \cdots u_k$ is called a P_{uv} *path*, where $u = u_0$ and $u_k = v$. The *distance* between two vertices x and y in G, denoted by $d(x, y)$, is the number of edges of a shortest path between them. The *eccentricity* of a vertex x, denoted by $\mathrm{ecc}(x)$, is $\max_{y \in V(G)} d(x, y)$. The *radius* and *diameter* of G, denoted by $\mathrm{rad}(G)$ and $\mathrm{diam}(G)$, are $\min_{x \in V(G)} \mathrm{ecc}(x)$ and $\max_{x \in V(G)} \mathrm{ecc}(x)$, respectively. A vertex u is a *center* if $\mathrm{ecc}(u) = \mathrm{rad}(G)$.

A path in an edge-colored graph G, where adjacent edges may have the same color, is *rainbow* if no two edges of the path are colored the same. An edge-coloring of a graph G is a *rainbow-connected edge-coloring* if every pair of distinct vertices of G is connected by a rainbow path. The *rainbow connection number* $rc(G)$ of G is the minimum integer k for which there exists a *rainbow-connected k-edge-coloring* of G. It is easy to see that $\operatorname{diam}(G) \leq rc(G)$ for any connected graph G.

The rainbow connection number was introduced by Chartrand, Johns, McKeon, and Zhang in [4]. It has application in transferring information of high security in multicomputer networks. We refer the readers to [3, 8] for details.

Chakraborty, Fischer, Matsliah, and Yuster [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G, deciding if $rc(G) = 2$ is NP-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius and diameter, etc. [1, 5, 6, 7].

Let $f(d)$ denote the minimum number such that each bridgeless graph G with diameter d has a rainbow-connected $f(d)$-edge-coloring. It is easy to check that $f(1) = 1$. In [7], we showed that $f(2) = 5$. In this paper, we shall show that $7 \leq f(3) \leq 9$.

The following theorem will be used in this paper.

Theorem 1 [5]**.** *For every bridgeless graph G,*

$$rc(G) \leq \sum_{i=1}^{\operatorname{rad}(G)} \min\{2i+1, \eta(G)\} \leq \operatorname{rad}(G)\eta(G),$$

where $\eta(G)$ is the smallest integer such that every edge of G is contained in a cycle of length at most $\eta(G)$.

In this paper, we investigate the upper bound on the rainbow connection number of bridgeless graphs with diameter 3, and obtain the following result.

Theorem 2. *For every bridgeless graph G with diameter 3, $rc(G) \leq 9$.*

If each edge of a bridgeless graph G with diameter 3 belongs to a triangle, then $rc(G) \leq 9$ by Theorem 1. Thus, we suppose that there exists an edge e such that e does not belong to any triangle in G.

This paper is organized as follows. In Section 2, we partition $V(G)$, and present a partial edge-coloring of G under this partition. In Section 3, we further partition $V(G)$ and give a complete edge-coloring of G under this partition. In Section 4, we prove that the edge-coloring in Section 3 is a rainbow-connected 9-edge-coloring of G, and give a class of bridgeless graphs with diameter 3 and rainbow connection number at least 7.

2. A Partial Edge-Coloring

Let G be a graph. For any integer $k \geq 1$, the *k-step open neighborhood* $N^k(X)$ is $\{y \in V(G) : d(X, y) = k\}$. We simply write $N(X)$ for $N^1(X)$ and $N^k(x)$ for $N^k(\{x\})$. Similarly, the *k-step closed neighborhood* $N^k[X]$ is $\{y \in V(G) : d(X, y) \leq k\}$. We simply write $N[X]$ for $N^1[X]$ and $N^k[x]$ for $N^k[\{x\}]$.

Let c be an edge-coloring of G, and let P be a rainbow path in G. We use $c(P)$ to denote the set of colors used on P, that is, $c(P) = \{c(e)$: the edge e belongs to $P\}$. If $c(P) \subseteq \{k_1, k_2, \ldots, k_r\}$, then P is a *$\{k_1, k_2, \ldots, k_r\}$-rainbow path.* In particular, an edge e is a *k-color edge* if $c(e) = k$. We use $x_0 \overset{c_1}{\sim} x_1 \overset{c_2}{\sim} \cdots \overset{c_k}{\sim} x_k$ to denote a rainbow path $x_0 x_1 \cdots x_k$ with $c(x_{i-1}x_i) = c_i$ for each $1 \leq i \leq k$. Let $X_1, X_2, \ldots, X_{k-1}$ be pairwise disjoint vertex subsets of G. The notation $x_0 \overset{c_1}{\sim} X_1 \overset{c_2}{\sim} \cdots \overset{c_{k-1}}{\sim} X_{k-1} \overset{c_k}{\sim} x_k$ means that there exists a rainbow path $x_0 \overset{c_1}{\sim} x_1 \overset{c_2}{\sim} \cdots \overset{c_k}{\sim} x_k$, where $x_i \in X_i$ for $1 \leq i \leq k-1$.

Recall that e is an edge not belonging to any triangle in G. Let u and v be the ends of e.

Since e does not belong to any triangle, for the open neighborhood, $N(\{u, v\})$, of $\{u, v\}$ in G, we can divide it as follows:

$$A = N(u) \setminus \{v\},$$
$$B = N(v) \setminus \{u\}.$$

See Figure 1 for details.

For the 2-step open neighborhood, $N^2(\{u, v\})$, of $\{u, v\}$ in G, we can divide it as follows:

$$X = \{x \in N(A) \setminus N(B) : x \notin A \cup B \cup \{u, v\}\},$$
$$Y = \{x \in N(B) \setminus N(A) : x \notin A \cup B \cup \{u, v\}\},$$
$$Z = \{x \in N(A) \cap N(B) : x \notin A \cup B \cup \{u, v\}\}.$$

See Figure 1 for details. It is easy to see that $x \in X$ if and only if $x \notin N[\{u, v\}]$, $d(x, u) = 2$ and $d(x, v) = 3$; $y \in Y$ if and only if $y \notin N[\{u, v\}]$, $d(y, u) = 3$ and $d(y, v) = 2$; $z \in Z$ if and only if $z \notin N[\{u, v\}]$, $d(x, u) = 2$ and $d(x, v) = 2$.

Note that for $x \in N^3(\{u, v\})$, we have $d(x, u) = d(x, v) = 3$, since $\operatorname{diam}(G) = 3$, that is, $N(x) \cap N(A) \neq \emptyset$ and $N(x) \cap N(B) \neq \emptyset$.

For the 3-step open neighborhood, $N^3(\{u, v\})$, of $\{u, v\}$ in G, we can partition $N^3\{u, v\}$ based on the distribution of the neighbors of x as follows:

$$W = \{x \in N^3(\{u, v\}) : N(x) \cap X \neq \emptyset \text{ and } N(x) \cap Y \neq \emptyset\},$$
$$I = \{x \in N^3(\{u, v\}) \setminus W : N(x) \cap X \neq \emptyset \text{ and } N(x) \cap Z \neq \emptyset\},$$

$$K = \{x \in N^3(\{u,v\}) \setminus (W \cup I) : N(x) \cap Y \neq \emptyset \text{ and } N(x) \cap Z \neq \emptyset\},$$
$$J = \{x \in N^3(\{u,v\}) \setminus (W \cup I \cup K) : N(x) \cap Z \neq \emptyset\}.$$

See Figure 1 for details. It is easy to see that $N^3(\{u,v\}) = I \cup J \cup K \cup W$.

At this point, we further partition A and B as follows:

$$\begin{aligned}
A_1 &= \{x \in A : N(x) \cap (B \cup X \cup Z) \neq \emptyset\},\\
A_2 &= \{x \in A \setminus A_1 : N(x) \cap (A \setminus A_1) \neq \emptyset\},\\
A_3 &= A \setminus (A_1 \cup A_2),\\
B_1 &= \{x \in B : N(x) \cap (A \cup Y \cup Z) \neq \emptyset\},\\
B_2 &= \{x \in B \setminus B_1 : N(x) \cap (B \setminus B_1) \neq \emptyset\},\\
B_3 &= B \setminus (B_1 \cup B_2).
\end{aligned}$$

That is, A_1 consists of vertices which have neighbors outside $A \cup \{u\}$, A_2 consists of vertices which do not have neighbors outside A (apart from u) but have neighbors in $A \setminus A_1$, and A_3 consists of vertices which have neighbors only in A_1 (apart from u). It is clear that for each $x \in A_2$, there exists a vertex $x' \in A_2$ such that $xx'u$ is a triangle. Similar results also hold for B_1, B_2 and B_3.

Note that there may exist edges between between A_1 and A_2, but it does not matter for our proof.

Meanwhile, we partition X and Y as follows:

$$\begin{aligned}
X_1 &= \{x \in X : N(x) \cap (Y \cup Z \cup I \cup W) \neq \emptyset\},\\
X_2 &= \{x \in X \setminus X_1 : N(x) \cap (X \setminus X_1) \neq \emptyset\},\\
X_3 &= \{x \in X \setminus (X_1 \cup X_2) : N(x) \subseteq A\},\\
X_4 &= X \setminus (X_1 \cup X_2 \cup X_3),\\
Y_1 &= \{y \in Y : N(y) \cap (X \cup Z \cup K \cup W) \neq \emptyset\},\\
Y_2 &= \{y \in Y \setminus Y_1 : N(y) \cap (Y \setminus Y_1) \neq \emptyset\},\\
Y_3 &= \{y \in Y \setminus (Y_1 \cup Y_2) : N(y) \subseteq B\},\\
Y_4 &= Y \setminus (Y_1 \cup Y_2 \cup Y_3).
\end{aligned}$$

That is, X_1 consists of vertices which have neighbors outside X (apart from A_1), X_2 consists of vertices which do not have neighbors outside X (apart from A_1) but have neighbors in $X \setminus X_1$, X_3 consists of vertices which have neighbors only in A_1, and X_4 consists of vertices which have neighbors only in X_1 (apart from A_1). Similar results also hold for Y_1, Y_2, Y_3 and Y_4.

By the definitions of sets A_1, A_2 and A_3, we know that $N(X_3) \subseteq A_1$ and $N(Y_3) \subseteq B_1$. Thus $X_3 = \{x \in X \setminus (X_1 \cup X_2) : N(x) \subseteq A_1\}$ and $Y_3 = \{y \in Y \setminus (Y_1 \cup Y_2) : N(y) \subseteq B_1\}$.

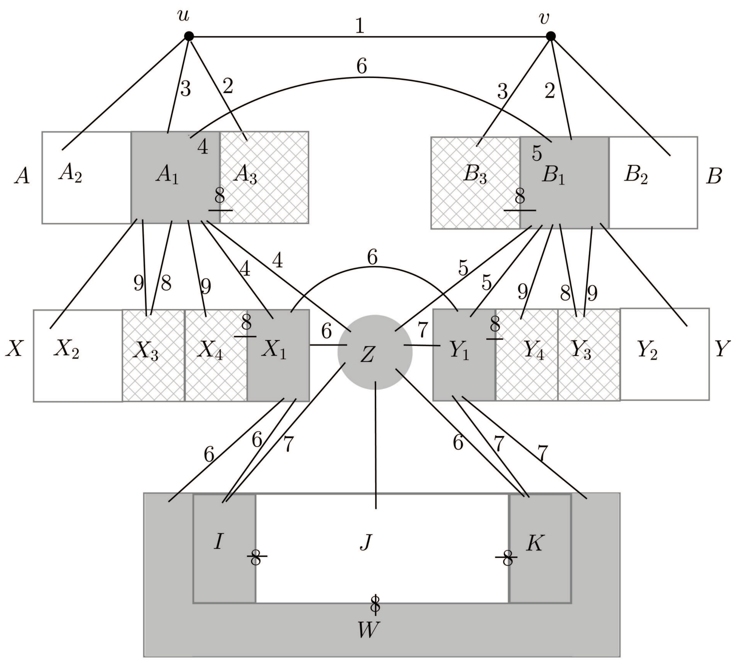

Figure 1. A partial edge-coloring of G.

We denote the above set partition by $\mathcal{P}$. The following observation holds for $\mathcal{P}$ since G is bridgeless.

Lemma 3. (1) *For* $x \in A_3$, $N(x) \cap A_1 \neq \emptyset$.
(2) *For* $x \in B_3$, $N(x) \cap B_1 \neq \emptyset$.
(3) *For* $x \in X_4$, $N(x) \cap X_1 \neq \emptyset$.
(4) *For* $x \in Y_4$, $N(x) \cap Y_1 \neq \emptyset$.

We give a partial 9-edge-coloring of G as follows:

$$c(e) = \begin{cases} 1, & \text{if } e = uv; \\ 2, & \text{if } e \in E[u, A_3] \cup E[v, B_1]; \\ 3, & \text{if } e \in E[u, A_1] \cup E[v, B_3]; \\ 4, & \text{if } e \in E[A_1, X_1 \cup Z] \cup E(G[A_1]); \\ 5, & \text{if } e \in E[B_1, Y_1 \cup Z] \cup E(G[B_1]); \\ 6, & \text{if } e \in E[A_1, B_1] \cup E[Z, K] \cup E[X_1, Z \cup I \cup W \cup Y_1]; \\ 7, & \text{if } e \in E[Z, I] \cup E[Y_1, K \cup W \cup Z]; \\ 8, & \text{if } e \in E[A_1, A_3] \cup E[B_1, B_3] \cup E[X_1, X_4] \\ & \qquad \cup E[Y_1, Y_4] \cup E[J, I \cup K \cup W]; \\ 9, & \text{if } e \in E[A_1, X_4] \cup E[B_1, Y_4]. \end{cases}$$

See Figure 1 for details.

For each $x \in X_3$, $N(x) \subseteq A_1$ by the above set partition. Since G is a bridgeless graph, $|N(x)| \geq 2$. Thus, we can color one edge incident to x by 8, and color the others incident to x by 9. Similarly, for each vertex $y \in Y_3$, we can color edges incident to y by colors 8 and 9.

Lemma 4. (1) *For $x \in X_1$, there exists an $x \overset{6}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{6}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{6}{\sim} I \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{6}{\sim} W \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring.*

(2) *For $y \in Y_1$, there exists a $y \overset{6}{\sim} X_1 \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{7}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{7}{\sim} W \overset{6}{\sim} X_1 \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{7}{\sim} K \overset{6}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path under the above partial edge-coloring.*

Proof. We only show (1) since the proofs are similar. For any $x \in X_1$, by the definition of set X_1, we know that x has a neighbor, say x', in $Y \cup Z \cup I \cup W$.

If $x' \in Y$, then $x' \in Y_1$ by the definition of set Y_1. Thus $xx'x''v$ is an $x \overset{6}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where x'' is a neighbor of x' in B_1.

If $x' \in Z$, then $xx'x''v$ is an $x \overset{6}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where x'' is a neighbor of x' in B_1.

If $x' \in I$, then $xx'x''x'''v$ is an $x \overset{6}{\sim} I \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where x'' is a neighbor of x' in Z and x''' is a neighbor of x'' in B_1.

Otherwise, $x' \in W$, and then $xx'x''x'''v$ is an $x \overset{6}{\sim} W \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring, where x'' is a neighbor of x' in Y_1 and x''' is a neighbor of x'' in B_1. ■

Lemma 5. (1) *For $x \in A_1$, there exists an $x \overset{6}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} I \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, or $x \overset{4}{\sim} X_1 \overset{6}{\sim} W \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring.*

(2) *For $y \in B_1$, there exists a $y \overset{6}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{6}{\sim} X_1 \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{7}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{7}{\sim} W \overset{6}{\sim} X_1 \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path, or $y \overset{5}{\sim} Y_1 \overset{7}{\sim} K \overset{6}{\sim} Z \overset{4}{\sim} A_1 \overset{3}{\sim} u$-rainbow path under the above partial edge-coloring.*

Proof. We only show (1) since the proofs are similar. For any $x \in A_1$, by the definition of set A_1, we know that x has a neighbor, say, x', in $B_1 \cup Z \cup X_1$.

If $x' \in B_1$, then $xx'v$ is an $x \overset{6}{\sim} B_1 \overset{2}{\sim} v$-rainbow path under the above partial edge-coloring.

If $x' \in Z$, then $xx'x''v$ is an $x \overset{4}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v$-rainbow path, where x'' is a neighbor of x' in B_1.

Otherwise, $x' \in X_1$. By Lemma 4, there exists a desired rainbow path. ■

Lemma 6. (1) *For $x \in Z$, there exists an $x \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u \overset{3}{\sim} A_1 \overset{4}{\sim} x$-rainbow cycle under the above partial edge-coloring.*

(2) *For $x \in I$, there exists an $x \overset{7}{\sim} Z \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u \overset{3}{\sim} A_1 \overset{4}{\sim} X_1 \overset{6}{\sim} x$-rainbow cycle under the above partial edge-coloring.*

(3) *For $x \in K$, there exists an $x \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u \overset{3}{\sim} A_1 \overset{4}{\sim} Z \overset{6}{\sim} x$-rainbow cycle under the above partial edge-coloring.*

(4) *For $x \in W$, there exists an $x \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u \overset{3}{\sim} A_1 \overset{4}{\sim} X_1 \overset{6}{\sim} x$-rainbow cycle under the above partial edge-coloring.*

Proof. We only show (4) since (1), (2) and (3) can be proved similarly. For any $x \in W$, by the definition of set W, the vertex x has a neighbor $v_1 \in X_1$ and a neighbor $v_2 \in Y_1$. Moreover, by the definitions of sets X_1 and Y_1, the vertex v_1 has a neighbor $v_3 \in A_1$, and the vertex v_2 has a neighbor $v_4 \in B_1$. Thus $x \overset{7}{\sim} v_2 \overset{5}{\sim} v_4 \overset{2}{\sim} v \overset{1}{\sim} u \overset{3}{\sim} v_3 \overset{4}{\sim} v_1 \overset{6}{\sim} x$ is a rainbow cycle, that is, there exists an $x \overset{7}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u \overset{3}{\sim} A_1 \overset{4}{\sim} X_1 \overset{6}{\sim} x$-rainbow cycle under the above partial edge-coloring. ■

Lemma 7. *For any two vertices $x, y \in V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J)$, there exists a rainbow path joining x and y under the above partial edge-coloring.*

Proof. Let x and y be any two vertices in $V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J)$. It is easy to see that there exists a rainbow path between u (respectively v) and another vertex $w \in V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J)$ in the partial edge-color graph G. Thus suppose that $\{u, v\} \cap \{x, y\} = \emptyset$.

Case 1. $x, y \in A_1 \cup B_1 \cup X_1 \cup Y_1 \cup Z \cup I \cup K \cup W$. By Lemmas 4, 5 and 6, we can pick a special rainbow path P_1 between x and v and a special rainbow path P_2 between y and v such that $c(P_1) \cap c(P_2) = \emptyset$. Thus we can obtain a rainbow path joining x and y by combining the paths P_1 and P_2.

Case 2. Exactly one of x and y belongs to $A_1 \cup B_1 \cup X_1 \cup Y_1 \cup Z \cup I \cup K \cup W$. Without loss of generality, say $x \in A_1 \cup B_1 \cup X_1 \cup Y_1 \cup Z \cup I \cup K \cup W$ and $y \in A_3 \cup B_3 \cup X_3 \cup X_4 \cup Y_3 \cup Y_4$. We only check the case $y \in A_3 \cup X_3 \cup X_4$ since the case $y \in B_3 \cup Y_3 \cup Y_4$ can be checked similarly.

For $y \in A_3 \cup X_3 \cup X_4$, there exists a $y \overset{9\,(\mathrm{or}\,8)}{\sim} A_1 \overset{3}{\sim} u \overset{1}{\sim} v$-rainbow path P_1 joining y and v. Moreover, there exists a $\{2, 4, 5, 6, 7\}$-rainbow path P_2 joining x and v. Thus a rainbow path joining x and y can be obtained from P_1 and P_2.

Case 3. $x \in A_3 \cup X_3 \cup X_4$ and $y \in B_3 \cup Y_3 \cup Y_4$.

Subcase 3.1. $x \in A_3$. There exist an $x \overset{2}{\sim} u$-rainbow path P_1 and an $x \overset{8}{\sim} A_1 \overset{3}{\sim} u$-rainbow path P_2 by Figure 1 and Lemma 3.

If $y \in Y_3 \cup Y_4$, then there exists a $y \overset{9}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u$-rainbow path P_3. Thus a rainbow path joining x and y can be obtained from P_2 and P_3.

If $y \in B_3$, then there exists a $y \overset{3}{\sim} v \overset{1}{\sim} u$-rainbow path P_4. Thus a rainbow path joining x and y can be obtained from P_1 and P_4.

Subcase 3.2. $x \in X_3 \cup X_4$. There exists an $x \overset{9}{\sim} A_1 \overset{3}{\sim} u$-rainbow path P_1 by Figure 1. Moreover, there exists a $y \overset{8}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u$-rainbow path P_2 if $y \in B_3 \cup Y_3$, or there exists a $y \overset{8}{\sim} Y_1 \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u$-rainbow path P_2 if $y \in Y_4$. Thus a rainbow path joining x and y can be obtained from P_1 and P_2.

Case 4. $x, y \in A_3 \cup X_3 \cup X_4$ or $x, y \in B_3 \cup Y_3 \cup Y_4$. We only check the case $x, y \in A_3 \cup X_3 \cup X_4$ since the case $x, y \in B_3 \cup Y_3 \cup Y_4$ can be checked similarly.

Subcase 4.1. $x \in A_3$ or $y \in A_3$. Without loss of generality, say $x \in A_3$. Then there exists a $x \overset{2}{\sim} u \overset{3}{\sim} A_1 \overset{8(\text{or } 9)}{\sim} y$-rainbow path connecting x and y.

Subcase 4.2. At least one of x and y belongs to X_3. Without loss of generality, assume that $x \in X_3$. Let x' and y' be neighbors of x and y in A_1 such that $c(xx') = 8$ and $c(yy') = 9$. By Lemma 5, there exists a $\{2, 4, 5, 6, 7\}$-rainbow path P joining y' and v. Thus $yy'Pvux'x$ is a rainbow path connecting x and y.

Subcase 4.3. Both x and y belong to X_4. Let x' be a neighbor of x in A_1, and let y' be a neighbor of y in X_1. By Lemma 4, there exists a $\{2, 5, 6, 7\}$-rainbow path P joining y' and v. Thus $yy'Pvux'x$ is a rainbow path connecting x and y. ■

3. A Complete Edge-Coloring

To complete our edge-coloring, we further partition J as follows:

$$
\begin{aligned}
J_0 &= \{x \in J : x \text{ is not an isolated vertex in } G[J]\},\\
J_1 &= \{x \in J \setminus J_0 : x \text{ has at least a neighbor in } K\},\\
J_2 &= \{x \in J \setminus (J_0 \cup J_1) : x \text{ has at least a neighbor in } W\},\\
J_3 &= \{x \in J \setminus (J_0 \cup J_1 \cup J_2) : x \text{ has at least a neighbor in } I\},\\
J_4 &= J \setminus (J_0 \cup J_1 \cup J_2 \cup J_3).
\end{aligned}
$$

Now we further color the edges of G as follows: color the edges in $E[Z, J_1 \cup J_2 \cup J_3]$ by color 7; for any $x \in J_4$, color one in $E[x, Z]$ by 8, color the others in $E[x, Z]$ by 9 (there exists at least one such edge since G is bridgeless).

To color the remaining edges, we need the following lemma.

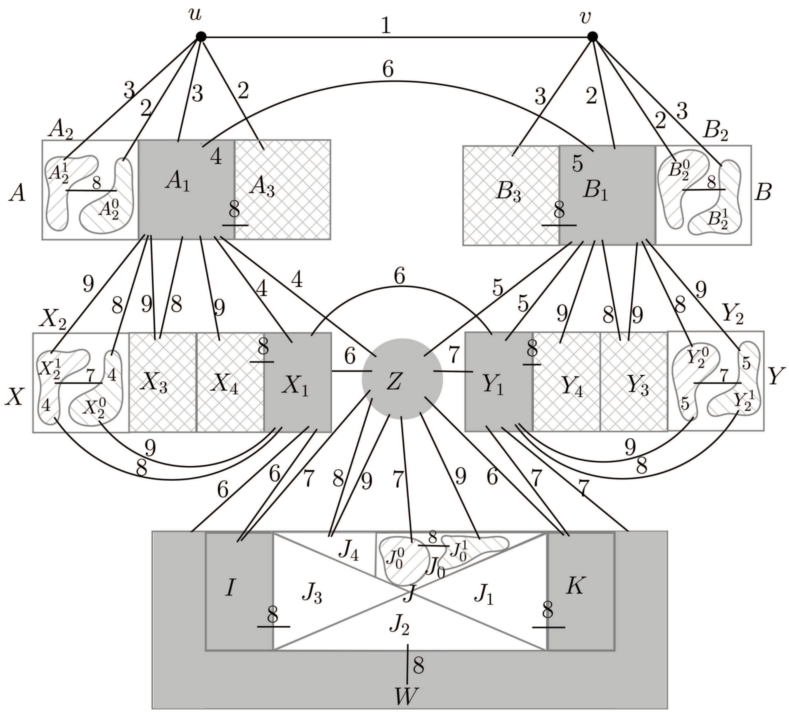

Figure 2. A complete edge-coloring of G (we omit the line between Z and J_1, the line between Z and J_2, and the line between Z and J_3).

Lemma 8. *Let S and T be two disjoint vertex sets of a graph G such that $S \subseteq N(T)$. If the induced subgraph $G[S]$ has no trivial components, then there is an $\{\alpha, \beta, \gamma\}$-edge-coloring of $G[S] \cup E[S,T]$ such that there exist two rainbow paths P_1 and P_2 joining s and T for every $s \in S$. Furthermore, if P_1 has color $\{\alpha\}$, then P_2 has colors $\{\beta, \gamma\}$; if P_1 has color $\{\beta\}$, then P_2 has colors $\{\alpha, \gamma\}$.*

Proof. Let F be a maximal spanning forest of $G[S]$, and let (X, Y) be any of bipartitions defined by this forest F. We give a 3-edge-coloring $c : E(G[S]) \cup E[S,T] \to \{\alpha, \beta, \gamma\}$ of G by defining

$$c(e) = \begin{cases} \alpha, & \text{if } \ e \in E[T, X]; \\ \beta, & \text{if } \ e \in E[T, Y]; \\ \gamma, & \text{otherwise.} \end{cases}$$

Clearly, for the edge-coloring above, there exist two rainbow paths P_1 and P_2 joining s and T for every $s \in S$. Furthermore, if P_1 has color $\{\alpha\}$, then P_2 has colors $\{\beta, \gamma\}$; if P_2 has color $\{\beta\}$, then P_2 has colors $\{\alpha, \gamma\}$. ■

Remark. The edge-coloring in Lemma 8 is called an $\langle \alpha, \beta, \gamma\rangle$-*edge-coloring* for T and $X \cup Y$. Let $T_{A_2}, T_{B_2}, T_{X_2}, T_{Y_2}$ and T_{J_0} be maximal spanning forests of $G[A_2], G[B_2]$, $G[X_2], G[Y_2]$ and $G[J_0]$, respectively. Clearly, the forests have no isolated vertex. Let A_2^0 and A_2^1, B_2^0 and B_2^1, X_2^0 and X_2^1, Y_2^0 and Y_2^1, and J_0^0 and J_0^1 be bipartitions of $T_{A_2}, T_{B_2}, T_{X_2}, T_{Y_2}$ and T_{J_0}. Now we give a $\langle 2,3,8\rangle$-edge-coloring for u and $A_2^0 \cup A_2^1$, a $\langle 2,3,8\rangle$-edge-coloring for v and $B_2^0 \cup B_2^1$, an $\langle 8,9,7\rangle$-edge-coloring for A_1 and $X_2^0 \cup X_2^1$, an $\langle 8,9,7\rangle$-edge-coloring for B_1 and $Y_2^0 \cup Y_2^1$, a $\langle 7,9,8\rangle$-edge-coloring for Z and $J_0^0 \cup J_0^1$ as shown in Figure 2.

Furthermore, we color the edges in subgraphs $G[A_1]$, $G[X_2^0]$ and $G[X_2^1]$ by 4, the edges in subgraphs $G[B_1]$, $G[Y_2^0]$ and $G[Y_2^1]$ by 5, the edges in $E[X_1, X_2^1]$ and $E[Y_1, Y_2^1]$ by 8, and the edges in $E[X_1, X_2^0]$ and $E[Y_1, Y_2^0]$ by 9.

For the remaining edges, we can color them arbitrarily. Up to now, we give the graph G a complete edge-coloring. Let $\mathcal{P}$ be our final vertex set partition and let c be our final edge-coloring.

Lemma 9. *For any two vertices $x \in A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J$ and $y \in V(G) \setminus (A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J)$, there exists a rainbow path under the above partial edge-coloring.*

Proof. We consider the following three cases.

Case 1. $x \in A_2 \cup B_2$. We only consider the case $x \in A_2$ since the case $x \in B_2$ can be checked similarly.

Subcase 1.1. $x \in A_2^0$. By observing Figure 2, there exist an $x \overset{2}{\sim} u$-rainbow path P_1 joining x and u, or an $x \overset{8}{\sim} A_2^1 \overset{3}{\sim} u$-rainbow path P_2 joining x and u.

If $y \in A_3$, then $P_2 y$ is a rainbow path joining x and y.

If $y \in B_3$, then $P_1 v y$ is a rainbow path joining x and y.

If $y \in B_1 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z \cup I \cup K \cup W$, then there exists a $\{1,2,5,6,7,9\}$-rainbow path Q_1 joining u and y. Thus a rainbow path joining x and y can be obtained by combining P_2 and Q_1.

If $y \in A_1 \cup X_1 \cup X_3 \cup X_4$, then there exists a $\{3,4,9\}$-rainbow path Q_2 joining u and y. Thus a rainbow path joining x and y can be obtained by combining P_1 and Q_2.

Subcase 1.2. $x \in A_2^1$. By observing Figure 2, there exist an $x \overset{3}{\sim} u$-rainbow path P_1 joining x and u, or an $x \overset{8}{\sim} A_2^0 \overset{2}{\sim} u$-rainbow path P_2 joining x and u.

If $y \in A_3$, then $P_1 y$ is a rainbow path joining x and y.

If $y \in B_3$, then $P_2 v y$ is a rainbow path joining x and y.

If $y \in B_1 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z \cup I \cup K \cup W$, then there exists a $\{1,2,5,6,7,9\}$-rainbow path Q_1 joining u and y. Thus a rainbow path joining x and y can be obtained by combining P_1 and Q_1.

If $y \in A_1 \cup X_1 \cup X_3 \cup X_4$, then there exists a $\{3,4,9\}$-rainbow path Q_2 joining u and y. Thus a rainbow path joining x and y can be obtained by combining P_2 and Q_2.

Case 2. $x \in X_2 \cup Y_2$. We only consider the case $x \in X_2$ since the case $x \in Y_2$ can be checked similarly.

Subcase 2.1. $x \in X_2^0$. By observing Figure 2, there exists an $x \overset{8}{\sim} A_1 \overset{3}{\sim} u$-rainbow path P_1 joining x and u.

If $y \in A_3$, then $P_1 y$ is a rainbow path joining x and y.

If $y \in B_3$, then $x \overset{7}{\sim} X_2^1 \overset{9}{\sim} A_1 \overset{3}{\sim} u \overset{1}{\sim} v \overset{2}{\sim} B_1 \overset{8}{\sim} y$ is a rainbow path joining x and y.

If $y \in B_1 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z \cup I \cup K \cup W$, then there exists a $\{1, 2, 5, 6, 7, 9\}$-rainbow path Q_1 joining v and y. Thus a rainbow path joining x and y can be obtained by combining P_1 and Q_1.

If $y \in A_1 \cup X_1$, then there exists a $\{2, 4, 5, 6, 7\}$-rainbow path Q_1 joining v and y by Lemmas 4 and 5. Thus a $\{1, 2, 3, 4, 5, 6, 7, 8\}$-rainbow path joining x and y can be obtained by combining P_1, Q_1 and edge uv.

If $y \in X_3 \cup X_4$, then y has a neighbor y' in A_1 such that $c(yy') = 9$. Note that there exists a $\{1, 2, 3, 4, 5, 6, 7, 8\}$-rainbow path P joining x and y' by the arguments of the above paragraph. Thus Py is a rainbow path joining x and y.

Subcase 2.2. $x \in X_2^1$. By observing Figure 2, there exist an $x \overset{9}{\sim} A_1 \overset{3}{\sim} u$-rainbow path P_1 joining x and u.

If $y \in A_3$, then $P_1 y$ is a rainbow path joining x and y.

If $y \in B_3$, then $P_1 v y' y$ is a rainbow path joining x and y, where y' is a neighbor of y in B_1.

If $y \in B_1 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z \cup I \cup K \cup W$, then there exists a $\{1, 2, 5, 6, 7, 8\}$-rainbow path Q_1 joining u and y. Thus a rainbow path joining x and y can be obtained by combining P_1 and Q_1.

If $y \in A_1 \cup X_1$, then there exists a $\{2, 4, 5, 6, 7\}$-rainbow path Q_1 joining v and y by Lemmas 4 and 5. Thus a $\{1, 2, 3, 4, 5, 6, 7, 9\}$-rainbow path joining x and y can be obtained by combining P_1, Q_1 and edge uv.

If $y \in X_3 \cup X_4$, then y has a neighbor y' in A_1 or X_1 such that $c(yy') = 8$. Note that there exists a $\{1, 2, 3, 4, 5, 6, 7, 9\}$-rainbow path P joining x and y' by the arguments of the above paragraph. Thus Py is a rainbow path joining x and y.

Case 3. $x \in J$. By observing Figure 2, there exists a $\{7, 9\}$-rainbow path P joining x and some vertex $z \in Z$. Furthermore, there exist a $z \overset{4}{\sim} A_1 \overset{3}{\sim} u \overset{1}{\sim} v$-rainbow path Q_1 joining z and v, and a $z \overset{5}{\sim} B_1 \overset{2}{\sim} v \overset{1}{\sim} u$-rainbow path Q_2 joining z and u. Thus a $\{1, 3, 4, 7, 9\}$-rainbow path Q_1' joining x and v can be obtained from P and Q_1, and a $\{1, 2, 5, 7, 9\}$-rainbow path Q_2' joining x and u can be obtained from P and Q_2.

If $y \in B_1 \cup B_3 \cup Y_1 \cup Y_3 \cup Y_4$, then there exists a $\{2, 5, 8\}$-rainbow path R_1 between v and y. Thus a rainbow path joining x and y can be obtained from Q_1' and R_1.

If $y \in A_1 \cup A_3 \cup X_1 \cup X_3 \cup X_4 \cup Z \cup I \cup K \cup W$, then there exists a $\{3,4,6,8\}$-rainbow path R_2 between u and y. Thus a rainbow path joining x and y can be obtained from Q_2' and R_2. ∎

4. 9-Rainbow-Connected Edge-Coloring

In this section, we check that the above 9-edge-coloring is rainbow-connected 9-edge-coloring. It suffices to check that for any two vertices $x, y \in A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J$, there exists a rainbow path under the above partial edge-coloring.

Lemma 10. *There exists a rainbow path joining any two vertices of X_2 under the edge-coloring c.*

Proof. Let x and y be any two vertices in X_2. We consider the following two cases.

Case 1. $x \in X_2^0$ and $y \in X_2^1$, or $x \in X_2^0$ and $y \in X_2^1$. Without loss of generality, assume that $x \in X_2^0$ and $y \in X_2^1$. Let x' and y' be neighbors of x and y in A_1, respectively. By Figure 2, we know that $c(xx') = 8$ and $c(yy') = 9$. By Lemma 5, there exists a $\{2,4,5,6,7\}$-rainbow path $P_{y',v}$. Thus, a $\{1,2,3,4,5,6,7,8,9\}$-rainbow path joining x and y is obtained from the edge yy', rainbow paths $P_{y',v}$ and $vux'x$.

Case 2. $x, y \in X_2^0$ or $x, y \in X_2^1$. We only check the case $x, y \in X_2^0$ since the case $x, y \in X_2^1$ can be checked similarly.

Subcase 2.1. $d(x, B_1) = 2$ or $d(y, B_1) = 2$. Without loss of generality, assume $d(x, B_1) = 2$. Let $x' \in N(x) \cap N(B_1)$. By the definition of the above set partition, we know $x' \in A_1$. So, $xx'x''vu$ is a $\{1,2,6,8\}$-rainbow path, where x'' is a neighbor of x' in B_1. By Figure 2 and Lemma 8, u and y are connected by a $\{3,7,9\}$-rainbow path P. Thus a $\{1,2,3,6,7,8,9\}$-rainbow path joining x and y can be obtained from rainbow paths $xx'x''vu$ and P.

Subcase 2.2. $d(x, B_1) = d(y, B_1) = 3$. Let $xx_1x_2x_3$ be a path joining x and some vertex $x_3 \in B_1$. By the set partition above, $x_1 \in A_1 \cup X_1 \cup X_2$.

Subsubcase 2.2.1. $x_1 \in A_1$. By the definition of $\mathcal{P}$, $x_2 \in A_1 \cup B_1 \cup Z$. So $xx_1x_2x_3$ is a $\{4,5,6\}$-rainbow path. Furthermore, $xx_1x_2x_3vu$ is $\{1,2,4,5,6,8\}$-rainbow. By Figure 2, there exists a $\{3,7,9\}$-rainbow path P joining u and y. Hence a rainbow path joining x and y can be obtained from rainbow paths $xx_1x_2x_3vu$ and P.

Subsubcase 2.2.2. $x_1 \in X_1$. By the definition of the above set partition, $x_2 \in A_1 \cup Z$. Thus $xx_1x_2x_3$ is a $\{4,5,6,9\}$-rainbow path. Thus $xx_1x_2x_3vuy'y$ is a $\{1,2,3,4,5,6,8,9\}$-rainbow path joining x and y, where y_1 is a neighbor of y in A_1.

Subsubcase 2.2.3. $x_1 \in X_2$. If $x_1 \in X_2^0$, then $c(xx_1) = 4$. Furthermore, $x_2 \in A_1$. Thus $xx_1x_2x_3vu$ is a $\{1,2,4,6,8\}$-rainbow path. By Figure 2, there exists a $\{3,7,9\}$-rainbow path P joining u and y. Hence a rainbow path joining x and y can be obtained from $xx_1x_2x_3vu$ and P.

If $x_1 \in X_2^1$, then $c(xx_1) = 7$. Furthermore, $x_2 \in A_1$. Thus $xx_1x_2x_3vu$ is a $\{1,2,4,6,7,9\}$-rainbow path. By Figure 2, there exists a $\{3,8\}$-rainbow path P joining u and y. Hence a rainbow path joining x and y can be obtained from $xx_1x_2x_3vu$ and P. ■

Similarly to Lemma 8, the following lemma holds.

Lemma 11. *There exists a rainbow path joining any two vertices of Y_2 under the edge-coloring above.*

Lemma 12. *For any two vertices $x, y \in A_2 \cup B_2 \cup X_2 \cup Y_2 \cup J$, there exists a rainbow path under the above partial edge-coloring.*

Proof. For $x, y \in X_2$ or $x, y \in Y_2$, there exists a rainbow path joining x and y by Lemmas 10 or 11. For the others, we can easily check them by Lemmas 4, 5, 6 and 8 in a similar way. ■

Combining Lemmas 7, 9 and 12, we have the following result.

Theorem 13. *Let G be a bridgeless graph with diameter* 3. *If there exists an edge e such that e does not belongs to any triangle in G, then $rc(G) \leq 9$.*

For a bridgeless graph G with diameter 3, if each edge belongs to a triangle in G, then $rc(G) \leq 9$ by Theorem 1. Combining this result with Theorem 13, we know that Theorem 2 holds.

We can give the following example of graphs with diameter 3 for which the rainbow connection number reaches 7.

Example 2. Let K_n be a complete graph with vertex set $\{v_1, \ldots, v_n\}$, where $n \geq 217$. For every v_i, we add a pendant path $\langle v_i, v_{i,1}, v_{i,2}, v_{i,3} \rangle$, denoted by P_i, and then we identify the vertex $v_{i,3}$ with a vertex v. The resulting graph is denoted by G. Clearly, $\text{diam}(G) = 3$. Let c be any 6-edge-coloring of G with colors $\{1, \ldots, 6\}$. Since $6^3 = 216$, at least two of them are colored the same. Without loss generality, say P_1 and P_2, that is, $c(v_1v_{1,1}) = c(v_2v_{2,1}), c(v_{1,1}v_{1,2}) = c(v_{2,1}v_{2,2})$ and $c(v_{1,2}v) = c(v_{2,2}v)$. By the structure of G, it is easy to see that there exists no rainbow path joining $v_{1,1}$ and $v_{2,1}$ in G under c. Thus $rc(G) \geq 7$.

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References

[1] M. Basavaraju, L.S. Chandran, D. Rajendraprasad and A. Ramaswamy, *Rainbow connection number and radius*, Graphs Combin. **30** (2014) 275–285. doi:10.1007/s00373-012-1267-7

[2] J.A. Bondy and U.S.R. Murty, Graph Theory (New York, Springer, 2008). doi:10.1007/978-1-84628-970-5

[3] S. Chakraborty, E. Fischer, A. Matsliah and R. Yuster, *Hardness and algorithms for rainbow connectivity*, J. Comb. Optim. **21** (2011) 330–347. doi:10.1007/s10878-009-9250-9

[4] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[5] X. Huang, H. Li, X. Li and Y. Sun, *Oriented diameter and rainbow connection number of a graph*, Discrete Math. Theor. Comput. Sci. **16** (2014) 51–60.

[6] X. Huang, X. Li, Y. Shi, J. Yue and Y. Zhao, *Rainbow connections for outerplanar graphs with diameter* 2 *and* 3, Appl. Math. Comput. **242** (2014) 277–280. doi:10.1016/j.amc.2014.05.066

[7] H. Li, X. Li, and S. Liu, *Rainbow connection of graphs with diameter* 2, Discrete Math. **312** (2012) 1453–1457. doi:10.1016/j.disc.2012.01.009

[8] X. Li and Y. Sun, Rainbow Connections of Graphs (New York, Springer, 2012). doi:10.1007/978-1-4614-3119-0

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