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ALMOST SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

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Abstract

It is known that self-complementary 3-uniform hypergraphs on n vertices exist if and only if n is congruent to 0, 1 or 2 modulo 4. In this paper we define an almost self-complementary 3-uniform hypergraph on n vertices and prove that it exists if and only if n is congruent to 3 modulo 4. The structure of corresponding complementing permutation is also analyzed. Further, we prove that there does not exist a regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4, and it is proved that there exist a quasi regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4.

Keywords: uniform hypergraph, self-complementary hypergraph, almost complete 3-uniform hypergraph, almost self-complementary hypergraph, quasi regular hypergraph.

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1. Introduction

The study of self-complementary graphs was initiated by Sachs and Ringel, independently. Ringel [11] and Sachs [12] have both proved the results concerning order and cycle structure of a complementing permutation of self-complementary graphs. Das [3] introduced the concept of almost self-complementary graphs which is similar to graphs self-complementary in $K_n - e$ introduced by Clapham [1]. They proved similar results on order and cycle structure of complementing permutation of almost self-complementary graphs. Kocay [9] extended the results of self-complementary graphs to self-complementary 3-uniform hypergraphs. He has analysed the cycle structure of complementing permutation of self-complementary 3-uniform hypergraphs. Szymański and Wojda [13] have characterized n and k for which there exist k-uniform self-complementary hypergraphs and gave the structure of corresponding self-complementing permutations. Gosselin [5] has characterized all n and k for which there exists a regular k-uniform self-complementary hypergraph of order n.

Potočnik and Šajana [10] raised the following question strengthening Hartman's conjecture [2, 6] about existence of large sets of (not necessarily isomorphic) designs.

Question [10]. *Is it true that for every triple of integers $t < k < n$ such that $\binom{n-i}{k-i}$ is even for all $i = 0, \ldots, t$, there exists a self-complementary t-subset-regular k-uniform hypergraph of order n?*

The answer to the above question is affirmative for $k = 2$ and $t = 1$ (see [12]). The answer was proved affirmative also for the case $k = 3$ and $t = 1$ (see [10]). And in [8] it is shown that the answer to the question above is affirmative for the remaining case of 3-uniform hypergraph, namely for the case $k = 3$, $t = 2$.

It is clear that if the number of triples in the complete design (K_n^3) is odd, then there does not exist a self-complementary t-subset-regular 3-uniform hypergraph of order n. In this case one may modify the problem by "Does there exist a partition of $K_n^3 - e$ into two isomorphic t-subset-regular 3-uniform hypergraphs of order n?" Das and Rosa [4] proved that there exists a partition of Steiner triple system (STS) into two isomorphic 3-uniform hypergraphs of order n, if $n \equiv 3$ or 7 (mod 12). In this paper we prove that there does not exist a partition of $K_n^3 - e$ into two isomorphic 1-subset-regular 3-uniform hypergraphs of order n, if $n \equiv 3 \pmod 4$ and some partial answers to the above question are given.

In Section 2, we define almost self-complementary 3-uniform hypergraph on n vertices and further prove that such a hypergraph exists if and only if n is congruent to 3 modulo 4.

In Section 3, the structure of a complementing permutation of such an almost self-complementary 3-uniform hypergraph is analyzed.

In Section 4, we prove that there does not exist a regular almost self complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4. Further, we prove that there exists a quasi regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4.

2. Necessary and Sufficient Condition for Existence of Almost Self-Complementary 3-Uniform Hypergraph

Suppose H is a 3-uniform hypergraph with vertex set V and edge set E. A partition of $E = \bigcup_{i=1}^{s} E_i$ is called a factorization of H and the 3-uniform hypergraph $H_i(V, E_i)$ is called a factor of H for $i = 1, 2, \ldots, s$. A factorization in which all factors are isomorphic is called an isomorphic factorization.

A factor in a factorization of complete 3-uniform hypergraph K_n^3 with only two isomorphic factors is nothing but a self complementary 3-uniform hypergraph. A partitioning of the edge set of K_n^3 into two isomorphic factors is not possible when K_n^3 has an odd number of edges, i.e., when n is congruent to 3 modulo 4. However, after deleting some odd number of edges from K_n^3 the remaining 3-uniform hypergraph may be partitioned into two isomorphic factors.

In this paper we delete one edge from K_n^3 and define an almost self-complementary 3-uniform hypergraph. We always denote by e the edge deleted from K_n^3, call it the missing edge and the corresponding vertices of e the special vertices.

Definition. The hypergraph $\tilde{K_n^3} = K_n^3 - e$ is called an almost complete 3-uniform hypergraph.

Definition. A 3-uniform hypergraph H with n vertices is almost self-complementary if it is isomorphic with its complement $\bar{H}$ with respect to $\tilde{K_n^3}$.

This means that a 3-uniform hypergraph H with n vertices is almost self-complementary if $\tilde{K_n^3}$ can be decomposed into two isomorphic factors with H as one factor.

Since $\tilde{K_n^3}$ has $\binom{n}{3} - 1$ edges, such a factorization is possible only if this number is divisible by 2. Thus it is necessary that $n \equiv 3 \pmod 4$. We can compare this with the fact that isomorphic factorizations of K_n^3, into 2 factors, i.e., self-complementary 3-uniform hypergraphs, exist only if $n \equiv 0, 1$ or $2 \pmod 4$. Almost self-complementary 3-uniform hypergraphs in a sense fill the gap where self-complementary 3-uniform hypergraphs do not exist.

Following theorem gives a necessary and sufficient condition on the order of an almost self-complementary 3-uniform hypergraph.

Theorem 1. *There exists an almost self-complementary 3-uniform hypergraph on n vertices if and only if $n \equiv 3 \pmod 4$.*

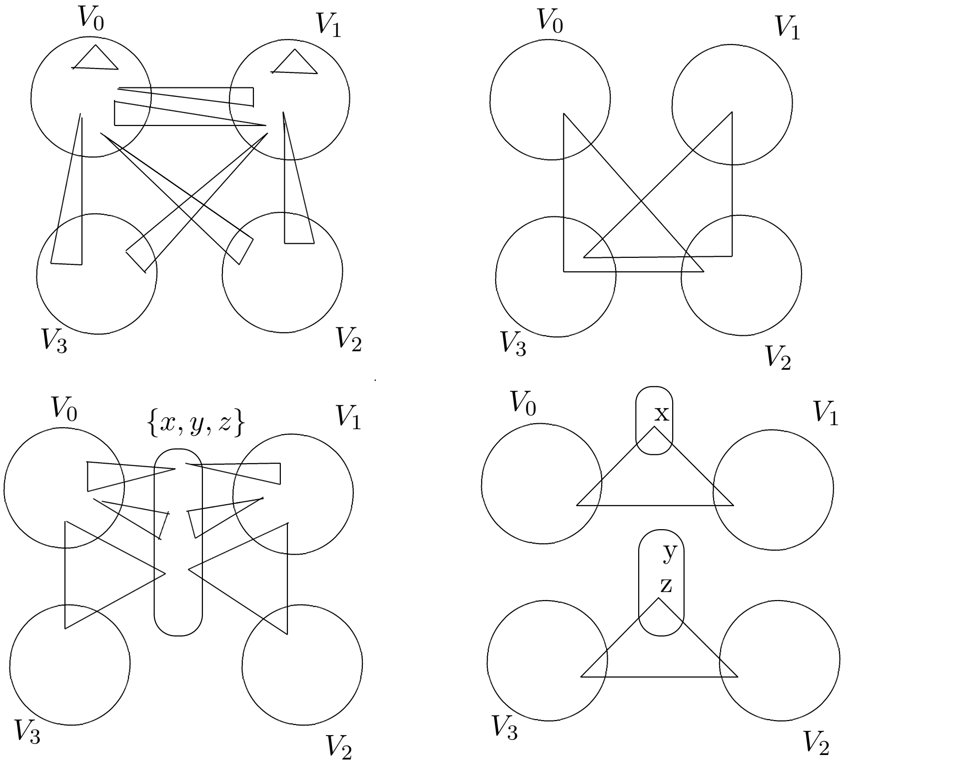

Figure 1. The types of triples making up the edge set of an almost self-complementary 3-uniform hypergraph on $n = 4m + 3$ vertices.

Proof. Necessity is obvious from the above discussions. To prove sufficiency, we construct a 3-uniform hypergraph which is self-complementary in $\tilde{K}_n^3$ on n vertices with $n \equiv 3 \pmod 4$. Denote the missing edge by $e = \{x, y, z\}$.

Let m be a positive integer such that $n = 4m + 3$ and $V = V_0 \cup V_1 \cup V_2 \cup V_3 \cup \{x, y, z\}$, where $V_i = \{v_j^i : j \in \mathbb{Z}_m\}$ for all $i \in \mathbb{Z}_4$.

For pairwise distinct $i, i', i'' \in \mathbb{Z}_4$ we consider the following partition of edges of $\tilde{K}_n^3$.

$$E_i = V_i^{(3)} = \text{all 3-subsets of } V_i,$$
$$E_{i,i'} = \{\{v_{j_1}^i, v_{j_2}^i, v_{j'}^{i'}\} : j_1, j_2, j' \in \mathbb{Z}_m, j_1 \neq j_2\},$$
$$E_{i,i',i''} = \{\{v_j^i, v_{j'}^{i'}, v_{j''}^{i''}\} : j, j', j'' \in \mathbb{Z}_m\},$$
$$E_i^x = \{\{x, v_{j_1}^i, v_{j_2}^i\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\},$$
$$E_i^y = \{\{y, v_{j_1}^i, v_{j_2}^i\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\},$$
$$E_i^z = \{\{z, v_{j_1}^i, v_{j_2}^i\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\},$$
$$E_{i,i'}^x = \{\{x, v_j^i, v_{j'}^{i'}\} : j, j' \in \mathbb{Z}_m\},$$
$$E_{i,i'}^y = \{\{y, v_j^i, v_{j'}^{i'}\} : j, j' \in \mathbb{Z}_m\},$$
$$E_{i,i'}^z = \{\{z, v_j^i, v_{j'}^{i'}\} : j, j' \in \mathbb{Z}_m\},$$

$$E_i^{(x,y)} = \{\{x, y, v_j^i\} : j \in \mathbb{Z}_m\},$$
$$E_i^{(x,z)} = \{\{x, z, v_j^i\} : j \in \mathbb{Z}_m\},$$
$$E_i^{(y,z)} = \{\{y, z, v_j^i\} : j \in \mathbb{Z}_m\}.$$

Let

$$E = \bigcup_{k=0,1} \left(E_k \cup E_{2,k} \cup E_{3,k} \cup E_{k,2,3} \cup E_k^x \cup E_k^y \cup E_k^z \cup E_k^{(x,y)} \cup E_k^{(x,z)} \cup E_k^{(y,z)} \right)$$
$$\cup\, E_{0,1} \cup E_{1,0} \cup E_{0,1}^x \cup E_{1,2}^x \cup E_{0,3}^x \cup \bigcup_{k=1,2,3} \left(E_{k,k+1}^y \cup E_{k,k+1}^z \right).$$

Let H be the 3-uniform hypergraph with vertex set V and edge set E. Figure 1 gives a diagrammatic construction of H.

To prove that H is almost self-complementary, we define a permutation $\phi : V \to V$ by $\phi(x) = x, \phi(y) = y, \phi(z) = z$, $\phi(v_j^0) = v_j^3$, $\phi(v_j^3) = v_j^1$, $\phi(v_j^1) = v_j^2$, and $\phi(v_j^2) = v_j^0$, for all $j \in \mathbb{Z}_m$. It is checked easily that ϕ is a complementing permutation of H and therefore H is almost self-complementary. ■

3. THE COMPLEMENTING PERMUTATION

It is known (see [9]) that for a 3-uniform self-complementary (s.c.) hypergraph, if τ is a complementing permutation of the vertices that maps H onto its complement $\bar{H}$, then

(i) every cycle of τ has even length, or
(ii) τ has 1 or 2 fixed points, and the length of all other cycles is a multiple of 4.

We prove similar results for the complementing permutation of an almost self-complementary 3-uniform hypergraph.

Given an almost self-complementary 3-uniform hypergraph H, let the edges of H be coloured red and the remaining edges of $\tilde{K_n^3}$ be coloured green. Since the 2 factors are isomorphic, there is a permutation τ of the vertices of $\tilde{K_n^3}$ that induces a mapping of the red edges onto the green edges. We consider τ as a permutation of the vertices of K_n^3, and denote by τ' the corresponding mapping induced on the set of edges of K_n^3. Thus τ' maps each red edge onto a green edge. However, the mapping τ' need not necessarily map each green edge onto a red edge. This would be so if τ' mapped e onto itself, but it may be that τ' maps e onto a red edge and some green edge onto e. Such a τ (which, for definiteness we shall always assume induces a mapping from red to green) will (as for s.c. 3-uniform hypergraphs) be called a complementing permutation. It will be useful to consider the cycles of the induced mapping τ'.

The following remarks regarding the cycles of induced mapping τ' will be used to prove a number of results about the structure of complementing permutation τ.

Remark 2. A cycle of τ' that does not include e must be of even length, consisting of edges alternately red and green.

Remark 3. The cycle of τ' that includes e has odd length, consisting of e followed by red and green edges alternately. Further, this length equals 1 when τ' maps e onto itself.

Lemma 4. *If the special vertices x, y, z occur in different cycles of τ, then they must be three fixed points $(x)(y)(z)$.*

Proof. Suppose that x occurs in a cycle of length L_1, y occurs in a cycle of length L_2 and z occurs in a cycle of length L_3.

Consider the cycle of τ' including e. The number of edges in this cycle is the least common multiple of L_1, L_2 and L_3. By Remark 3 this number must be odd. If $L_1 \geq 3$ then any triple $\{i, j, k\}$ of this cycle gives rise to a sequence of triples $\{i, j, k\}, \tau\{i, j, k\}, \tau^2\{i, j, k\}, \ldots$ etc. These must be alternately edges of H and $\bar{H}$. This is possible only if L_1 is even. Hence $L_1 = 1$. Similarly $L_2 = L_3 = 1$. ■

Lemma 5. *If all special vertices x, y, z occur in the same cycle C of τ, then C has length* 3.

Proof. Suppose all the special vertices x, y, z occur in a cycle C of length L. Consideration of the cycle of τ' including e shows that L must be odd.

If $L > 3$, one finds that there is another cycle of τ' not including e, of odd length, which is a contradiction to Remark 2. Thus $L = 3$. ■

Lemma 6. *If any two of the special vertices, say x, y, occur in the same cycle C_1 of τ, then C_1 has length $4h + 2, h \geq 0$, with $\tau^{2h+1}(x) = y$ and special vertex z fixed.*

Proof. Suppose special vertices x, y occur in the same cycle C_1 of length L_1. Let the remaining special vertex z occur in a cycle C_2 of length L_2. Clearly $L_1 \geq 2$ and $L_2 \geq 1$. Since $L_1 \geq 2$, it must be even as argued in Lemma 4. If $L_2 > 1$ we get a contradiction to Remark 3. Hence $L_2 = 1$.

Let $\tau^m(x) = y$, therefore $\tau^{L_1 - m}(y) = x$. Consider the sequence of triples $\{x, y, z\}$, $\tau\{x, y, z\}, \ldots$. This cycle has length either L_1 or m if $m = \frac{L_1}{2}$ and it must be odd. Since L_1 is even the only possibility is that the length is m and m is odd. Hence $L_1 = 2m = 4h + 2, h \geq 0$. ■

Lemma 7. *The cycles of τ that do not include the special vertices are of length multiple of* 4.

Proof. If a cycle $(u_1u_2\cdots u_L)$ does not involve the special vertices, then by Remark 2 L is even. We get the following cases depending on occurence of special vertices in τ.

Case (i) Special vertices x, y, z are fixed points.

Case (ii) Special vertices x, y, z occur in the cycle of length 3, say $C' = (x, y, z)$.

Case (iii) Two special vertices say x, y occur in the same cycle and z is fixed.

In all these cases consider the cycle of τ' including the edge $\{u_1, u_{(\frac{L}{2}+1)}, z\}$ which is of length $\frac{L}{2}$. From Remark 2 we find that $\frac{L}{2}$ must be even. Thus L must be a multiple of 4. ■

Complete description of complementing permutation of almost self complementary 3-uniform hypergraph is given below.

Theorem 8. *Let τ be a complementing permutation of an almost self-complementary 3-uniform hypergraph on $n \geq 3$ vertices and $e = \{x, y, z\}$ be the deleted edge. Then $n \equiv 3 \pmod 4$. Further*

(a) *τ consists of 3 fixed special vertices and all other cycles of length multiple of 4, or*

(b) *τ consists of a cycle of length $4h + 2$, $h \geq 0$, including two special vertices x and y with $\tau^{2h+1}(x) = y$, one fixed special vertex z and all other cycles of length multiple of 4, or*

(c) *τ consists of the cycle (x, y, z) and all other cycles are of length multiple of 4.*

4. REGULAR AND QUASI REGULAR ALMOST SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPH

It is known that (see [10]) a regular self-complementary 3-uniform hypergraph on n vertices exists if and only if $n \geq 5$ and n is congruent to 1 or 2 modulo 4. In the next theorem we prove that there does not exist a regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4.

Theorem 9. *There does not exist a regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4.*

Proof. Suppose there exists a regular almost self-complementary 3-uniform hypergraph, say H of regular degree r. Then the total number of edges in H is $\frac{1}{2}(\binom{n}{3} - 1) = \frac{n(n-1)(n-2)-6}{12}$. Since H is regular we get $rn = 3\times$ number of edges in H, i.e., $rn = \frac{3(n(n-1)(n-2)-6)}{12}$. Hence $r = \frac{n(n-1)(n-2)-6}{4n}$ which is not an integer for any n congruent to 3 modulo 4, a contradiction. Hence, there does not exist a regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4. ∎

A hypergraph H is said to be quasi regular if the degree of every vertex is either r or $r-1$ for some positive integer r. In [7], it is proved that there exists a quasi regular self-complementary 3-uniform hypergraph on n vertices if and only if n is congruent to 0 modulo 4. In the following theorem we prove that there exists a quasi regular almost self-complementary 3-uniform hypergraph on n vertices where n is congruent to 3 modulo 4.

Theorem 10. *There exists a quasi regular almost self complementary* 3*-uniform hypergraph on* n *vertices where* n *is congruent to* 3 *modulo* 4*.*

Proof. H constructed in Theorem 1 is already shown to be almost self-complementary 3-uniform hypergraph. We show that H is quasi regular. Considering the same notation as in proof of Theorem 1, take any vertex v_j^i.

Case (i) If $i \in \{0,1\}$ then, for fixed $i', i'' \in \mathbb{Z}_4$ distinct from i, the vertex v_j^i lies in $\binom{m-1}{2}$ triples of E_i, $3\binom{m}{2}$ triples of $E_{i',i}$, $(m-1)m$ triples of $E_{i,i'}$, m^2 triples of $E_{i,i',i''}$, $(m-1)$ triples of each E_i^x, E_i^y, E_i^z, $4m$ triples of $E_{i,i'}^x$, $E_{i,i'}^y$, $E_{i,i'}^z$ and 1 triple of each $E_i^{(x,y)}$, $E_i^{(x,z)}$, $E_i^{(y,z)}$. Hence, for every vertex v_j^i in H with $i \in \{0,1\}$, we have

$$\deg(v_j^i) = \binom{m-1}{2} + 3\binom{m}{2} + m(m-1) + m^2 + 3(m-1) + 4m + 3 = 4m^2 + 3m + 1.$$

Case (ii) If $i \in \{2,3\}$ then the vertex v_j^i lies in $2(m-1)m$ triples of $E_{i,i'}$, $2m^2$ triples of $E_{i,i',i''}$, $5m$ triples of $E_{i,i'}^x$, $E_{i,i'}^y$, $E_{i,i'}^z$. Hence, for every vertex v_j^i in H with $i \in \{2,3\}$, we obtain

$$\deg(v_j^i) = 2(m-1)m + 2m^2 + 5m = 4m^2 + 3m.$$

Case (iii) x lies in $2\binom{m}{2}$ triples of E_i^x, m triples of each $E_i^{(x,y)}$, $E_i^{(x,z)}$ and $3m^2$ triples of $E_{i,i'}^x$. Hence

$$\deg(x) = 2\binom{m}{2} + 4m + 3m^2 = 4m^2 + 3m.$$

y lies in $2\binom{m}{2}$ triples of E_i^y, $4m$ triples of $E_i^{(x,y)}$, $E_i^{(y,z)}$ and $3m^2$ triples of $E_{i,i'}^y$. Hence

$$\deg(y) = 2\binom{m}{2} + 4m + 3m^2 = 4m^2 + 3m.$$

Similarly,

$$\deg(z) = 2\binom{m}{2} + 4m + 3m^2 = 4m^2 + 3m.$$

Hence, H is quasi regular with degrees $r = 4m^2+3m+1$ and $r-1 = 4m^2+3m$. ∎

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