Discussiones Mathematicae
Graph Theory 37 (2017) 13–28
doi:10.7151/dmgt.1916

A FINITE CHARACTERIZATION AND RECOGNITION OF INTERSECTION GRAPHS OF HYPERGRAPHS WITH RANK AT MOST 3 AND MULTIPLICITY AT MOST 2 IN THE CLASS OF THRESHOLD GRAPHS

Yury Metelsky

Department of Mathematical Cybernetics
Mechanics and Mathematics Faculty
Belarusian State University
Nezavisimosti Ave. 4, 220030 Minsk, Belarus
e-mail: metelsky@bsu.by

Kseniya Schemeleva

Université Lumière Lyon 2
Laboratoire d'Economie des Transports
14 avenue Berthelot, 69363 Lyon Cedex 07, France
e-mail: kseniya.schemeleva@let.ish-lyon.cnrs.fr

and

Frank Werner

Institute of Mathematical Optimization
Otto-von-Guericke-University of Magdeburg
Universitätsplatz 2, 39106 Magdeburg, Germany
e-mail: frank.werner@mathematik.uni-magdeburg.de

Abstract

We characterize the class L_3^2 of intersection graphs of hypergraphs with rank at most 3 and multiplicity at most 2 by means of a finite list of forbidden induced subgraphs in the class of threshold graphs. We also give an $O(n)$-time algorithm for the recognition of graphs from L_3^2 in the class of threshold graphs, where n is the number of vertices of a tested graph.

Keywords: intersection graph, hypergraph rank, hypergraph multiplicity, forbidden induced subgraph, threshold graph.

2010 Mathematics Subject Classification: 05C62, 05C75, 05C70, 05C65, 05C85.

1. Introduction

In this paper, we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively; $N(v) = N_G(v)$ is the neighborhood of a vertex v in G and $\deg(v)$ is the degree of v; the subgraph of G induced by a set $X \subseteq V(G)$ is denoted by $G(X)$. A vertex v of a graph G is called *dominating* if $N(v) \cup \{v\} = V(G)$.

The *intersection graph* $L(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is defined as follows:

(1) the vertices of $L(\mathcal{H})$ are in a bijective correspondence with the edges of $\mathcal{H}$;
(2) two vertices are adjacent in $L(\mathcal{H})$ if and only if the corresponding edges have a non-empty intersection.

The *rank* of a hypergraph $\mathcal{H}$ is the maximum size of its edges. The *multiplicity of a pair of vertices* u, v of $\mathcal{H}$ is the number of edges in $\mathcal{H}$ containing both u and v; the *multiplicity* $m(\mathcal{H})$ of $\mathcal{H}$ is the maximum multiplicity among all pairs of vertices in $\mathcal{H}$ (see for example [15]).

Denote by L_r^m the class of intersection graphs of hypergraphs with rank at most r and multiplicity at most m. So, we refer to L_r^∞ as the class of intersection graphs of hypergraphs with rank at most r. The class L_r^m, where $r \geq 1$, $m \geq 1$ or $m = \infty$, is hereditary (i.e., every induced subgraph of a graph in L_r^m is also in L_r^m). Therefore, it can be characterized by means of a list (finite or not) of forbidden induced subgraphs.

A non-trivial characterization of the class L_r^m is known only for $r \leq 2$. These are:

- Beineke's finite characterization of the class L_2^1 of line graphs (i.e., intersection graphs of simple graphs) [1];
- a finite characterization of the class L_2^∞ of intersection graphs of multigraphs by Bermond and Meyer [2];
- a finite characterization of the class L_2^m by Tashkinov [22].

Such finite characterizations of the classes above imply that there exist polynomial algorithms for recognizing graphs from these classes. (For efficient algorithms for recognizing graphs from L_2^1 see, e.g., [4, 11, 17, 19].) It is also known that for any $r \geq 3$ and m, where $m \geq 1$ or $m = \infty$, there does not exist a finite characterization for the class L_r^m (see [6, 15, 16, 10]).

Poljak, Rödl and Turzik [18] proved that the problem of determining whether a graph belongs to L_r^∞ is NP-complete for an arbitrary r. Moreover, they proved that for every fixed $r \geq 4$, the analogous problem remains NP-complete. The question whether or not the class L_3^∞ can be recognized in polynomial time is still open, but recognizing intersection graphs of hypergraphs without multiple edges with rank at most 3 is NP-complete as well [18]. The following result generalizing one from [18] was obtained in [7]: For every fixed $m \geq 1$ and an arbitrary r, the problem of determining whether a graph belongs to L_r^m is NP-complete.

Hliněný and Kratochvíl [8] proved that for every fixed $r \geq 3$, the problem of determining whether a graph belongs to L_r^1 is NP-complete. The class L_3^1 was studied in different papers, and several graph classes were found, where the problem of recognizing graphs from the class is polynomially solvable or remains NP-complete ([7, 9, 14, 15, 16, 21]).

A graph G is called *split* [5] if there exists a partition of its vertex set $V(G) = A \cup B$ into a clique A and a stable set B (*bipartition* (A, B)). It was proved in [12] that for every fixed r, there exists a finite characterization of the graphs from L_r^1 in the class of split graphs. In [13] (see also [7]), this result was generalized to the class L_r^m for every fixed m.

A split graph with the bipartition (A, B) is called *threshold* [3] if the vertices from B can be numbered as $b_1, b_2, \ldots, b_k$ so that $N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_k)$. In [20], the problem of determining the Krausz dimension of a graph (the minimum r such that the graph belongs to the class L_r^1) was solved in the subclass of threshold graphs of the form $K_n - E(K_p)$.

In Section 2 of this paper, we give some preliminary facts (e.g., a so-called Krausz type characterization of the class L_3^2 in terms of clique coverings), prove some technical lemmas and formulate Theorem 2 that gives a finite characterization of the class L_3^2 (consisting of 15 graphs) in the class of threshold graphs. In Sections 3 and 4, we prove the necessity and sufficiency of Theorem 2, respectively. In Section 5 we give an $O(n)$-time algorithm for the recognition of graphs from L_3^2 in the class of threshold graphs, where n is the number of vertices of a tested graph.

2. Some Preliminaries and the Formulation of Theorem 2

A finite family $\mathscr{C} = (C_1, C_2, \ldots, C_q)$ of cliques of the graph G is called a *covering* of G if every vertex as well as every edge of G is contained in some C_i. The cliques C_i are the *clusters* of $\mathscr{C}$. For a vertex $v \in V(G)$, denote by $\mathscr{C}(v)$ the subfamily of all clusters of $\mathscr{C}$ that contain v. A covering $\mathscr{C}$ of the graph G is called an (r, m)-*covering* if any vertex of G belongs to at most r clusters of $\mathscr{C}$, and any two clusters of $\mathscr{C}$ have at most m vertices in common.

Theorem 1 [7, 13]**.** *A graph G belongs to the class L_3^2 if and only if there exists a* $(3, 2)$*-covering of G.*

A clique of a graph G is called *maximal* if it is not contained in some other clique of G.

Let a threshold graph with the bipartition (A, B) be given, where $B = \{b_1, b_2, \ldots, b_k\}$ and $N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_k)$. We denote such a graph by $G(p, q_1, q_2, \ldots, q_k)$ if $|A| = p$ and $\deg(b_i) = q_i$ for any $i = 1, 2, \ldots, k$. Without loss of generality (W.l.o.g.), we assume below that any threshold graph

$G(p, q_1, q_2, \ldots, q_k)$ with the bipartition (A, B) satisfies the conditions $A = \{a_1, a_2, \ldots, a_p\}$, $B = \{b_1, b_2, \ldots, b_k\}$, $p > q_1$ and $N(b_i) = \{a_1, a_2, \ldots, a_{q_i}\}$ for any $i = 1, 2, \ldots, k$ (see Figure 1).

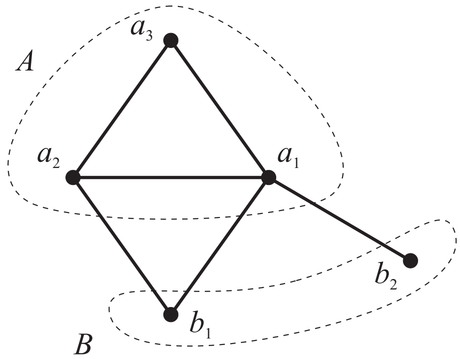

Figure 1. The graph $G(3, 2, 1)$ and its bipartition (A, B).

In this paper, we characterize the class L_3^2 by means of a finite list of forbidden induced subgraphs in the class of threshold graphs:

Theorem 2. *A threshold graph H belongs to the class L_3^2 if and only if it contains none of the graphs $K_{1,4}$, $G(12, 7)$, $G(11, 10)$, $G(10, 9, 5)$, $G(10, 9, 7)$, $G(10, 9, 9)$, $G(10, 7, k)$, $k = 1, 2, \ldots, 7$, $G(9, 8, 1)$, $G(9, 8, 2)$ as induced subgraphs.*

Now we formulate some technical statements that will be used for proving Theorem 2.

A $(3, 2)$-covering $\mathscr{C} = (C_1, C_2, \ldots, C_t)$ of a complete graph G is called a *decomposition* $(3, 2)$*-covering* if $C_i \neq V(G)$ for any $i = 1, 2, \ldots, t$.

Lemma 3. *Let $\mathscr{C} = (C_1, C_2, \ldots, C_t)$ be a decomposition $(3, 2)$-covering of a complete graph G. Then the following statements hold:*

(i) $|C_i| \leq 6$ *for any* $i = 1, 2, \ldots, t$.

(ii) *If* $C_i \setminus C_j \neq \emptyset$ *for some* $i, j \in \{1, 2, \ldots, t\}$, *then* $|C_j \setminus C_i| \leq 4$.

(iii) *If* $(C_i \cap C_j) \setminus C_k \neq \emptyset$ *for some different* $i, j, k \in \{1, 2, \ldots, t\}$, *then* $|C_k \setminus (C_i \cup C_j)| \leq 2$.

Proof. (i) Let, to the contrary, $C_i = \{a_1, a_2, \ldots, a_7, \ldots\}$ for some $i \in \{1, 2, \ldots, t\}$. Consider a vertex $v \in V(G) \setminus C_i$. By the definition of a $(3, 2)$-covering, each cluster of $\mathscr{C}$ contains at most two edges of va_s, $s = 1, 2, \ldots, 7$. Hence, the edges va_s, $s = 1, 2, \ldots, 7$, are covered by at least four clusters of $\mathscr{C}$, and, therefore, the vertex v is contained in at least four clusters of $\mathscr{C}$, which is a contradiction to the definition of $\mathscr{C}$.

(ii) Assume, to the contrary, that for a vertex $v \in V(G)$, we have $v \in C_i \backslash C_j$ and $C_j \backslash C_i = \{a_1, a_2, a_3, a_4, a_5, \ldots\}$. By the definition of a $(3,2)$-covering, the edges va_s, $s = 1, 2, \ldots, 5$, are covered by at least three clusters of $\mathscr{C}$, different from C_i. So, taking into account the cluster C_i, the vertex v is contained in at least four clusters of $\mathscr{C}$, which is a contradiction to the definition of $\mathscr{C}$.

(iii) Let, instead, $v \in (C_i \cap C_j) \setminus C_k \neq \emptyset$ and $C_k \setminus (C_i \cup C_j) = \{a_1, a_2, a_3, \ldots\}$. By the definition of a $(3,2)$-covering, the edges va_1, va_2, va_3 are covered by at least two clusters of $\mathscr{C}$, different from C_i and C_j. So, together with the clusters C_i, C_j, the vertex v is contained in at least four clusters of $\mathscr{C}$, which is a contradiction. ■

Lemma 4. *Let* $\mathscr{C} = (C_1, C_2, \ldots, C_t)$ *be a decomposition* $(3,2)$*-covering of a complete graph* G*. Then the following statements hold:*

(i) *If* G *has order* 11*, then it contains no cluster of size at most* 2*.*

(ii) *If* G *has order* 12*, then it contains no cluster of size at most* 3*.*

Proof. (i) Let $V(G) = \{a_1, a_2, \ldots, a_{11}\}$, $C_1 \in \mathscr{C}(a_1)$ and $|C_1| \leq 2$. W.l.o.g., assume that $\{a_3, a_4, \ldots, a_{11}\} \subseteq V(G) \setminus C_1$. By the definition of $\mathscr{C}$, there exists a cluster $C_2 \in \mathscr{C}(a_1)$ of size at least 6 among the clusters covering some of the nine edges a_1a_i, $i = 3, 4, \ldots, 11$. By Lemma 3(i),(ii), $|C_2| = 6$ and $C_1 \subseteq C_2$. Hence, $|V(G) \setminus (C_1 \cup C_2)| = 5$ and there exists a cluster $C_3 \in \mathscr{C}(a_1) \setminus \{C_1, C_2\}$ of size at least 6 containing the set $V(G) \setminus (C_1 \cup C_2)$. By Lemma 3(i), $C_3 = \{a_1\} \cup (V(G) \setminus (C_1 \cup C_2))$. We have $|C_2| = |C_3| = 6$ and $|C_2 \cap C_3| = 1$, which is a contradiction to Lemma 3(ii).

The statement (ii) of the lemma follows immediately from the statement (i). ■

3. Proof of Necessity of Theorem 2

By heredity of the class L_3^2, one has to show that none of the graphs $K_{1,4}$, $G(12,7)$, $G(11,10)$, $G(10,9,5)$, $G(10,9,7)$, $G(10,9,9)$, $G(10,7,k)$, $k = 1, 2, \ldots, 7$, $G(9,8,1)$ and $G(9,8,2)$ belongs to this class. Obviously, there exists no $(3,2)$-covering for the star $K_{1,4}$. Therefore, $K_{1,4} \notin L_3^2$ by Theorem 1.

Furthermore, let G be one of the graphs $G(12,7)$, $G(11,10)$, $G(10,9,5)$, $G(10,9,7)$, $G(10,9,9)$, $G(10,7,k)$, $k = 1, 2, \ldots, 7$, $G(9,8,1)$, $G(9,8,2)$ with the bipartition (A, B). Suppose, to the contrary, that there exists a $(3,2)$-covering $\mathscr{D} = (D_1, D_2, \ldots, D_t)$ of G.

W.l.o.g., we will assume that no cluster of $\mathscr{D}$ is contained in some other cluster of $\mathscr{D}$. By Theorem 1, it can be easily seen that $D_i \neq A$ for any $i = 1, 2, \ldots, t$, since $\deg(b_1) \geq 7$.

Put $\mathscr{C} = (C_1, C_2, \ldots, C_t)$, where $C_i = D_i \cap A$, $i = 1, 2, \ldots, t$. Then $\mathscr{C}$ is a decomposition $(3,2)$-covering of the subgraph $G(A)$, since $N(b_i) \neq A$ for each

$b_i \in B$. A cluster $C \in \mathscr{C}$ is called b_i-*reduced* with $b_i \in B$, if $C \cup \{b_i\} \in \mathscr{D}$. A cluster $C \in \mathscr{C}$ is called simply *reduced* if it is b_i-reduced for some $b_i \in B$. By Lemma 3(i), $\mathscr{C}$ contains two or three b_1-reduced clusters, since $\deg(b_1) \geq 7$.

Lemma 5. *The following statements hold:*

(i) *If $C_1, C_2 \in \mathscr{C}$ are two different b_i-reduced clusters with $b_i \in B$, then $|C_1 \cap C_2| \leq 1$.*

(ii) *If $C_1, C_2 \in \mathscr{C}$ are two different b_i-reduced clusters with $b_i \in B$, then $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$.*

(iii) *If $C_1, C_2, C_3 \in \mathscr{C}$ are three different reduced clusters, then $C_1 \cap C_2 \cap C_3 = \emptyset$.*

Proof. (i) The validity of the statement follows immediately from the definition of $\mathscr{C}$.

(ii) The statement follows from the above assumption that no cluster of $\mathscr{D}$ is contained in some other cluster of $\mathscr{D}$.

(iii) If, to the contrary, $a \in C_1 \cap C_2 \cap C_3$, then the edge aa_p is not covered by a cluster from $\mathscr{C}(a) = \{C_1, C_2, C_3\}$, which is a contradiction to the definition of $\mathscr{C}$. ■

We consider the following separate cases and come to a contradiction in each of them.

(1) $G = G(12, 7)$.

(a) Assume that there exist exactly two b_1-reduced clusters $C_1, C_2 \in \mathscr{C}$. By Lemma 4(ii), $|C_1| \geq 4$ and $|C_2| \geq 4$. Hence, by Lemma 5(i) and the equality $|C_1 \cup C_2| = 7$, we obtain $|C_1| = |C_2| = 4$ and $|C_1 \cap C_2| = 1$. W.l.o.g., assume that $C_1 \cap C_2 = \{a_1\}$. Consider the cluster $C_3 \in \mathscr{C}(a_1) \setminus \{C_1, C_2\}$. Then $\{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\} \subseteq C_3$. By Lemma 3(i), $C_3 = \{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ (see Figure 2). We have $|C_3 \setminus C_1| = 5$, which is a contradiction to Lemma 3(ii).

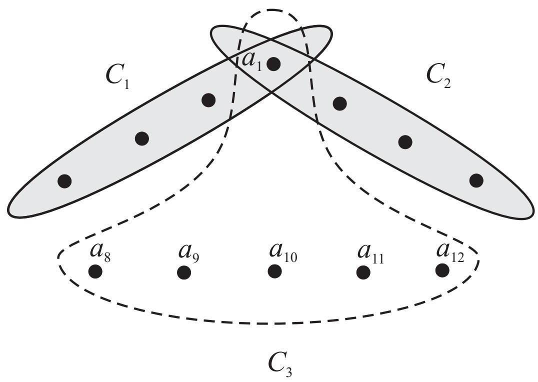

Figure 2. The clusters C_1, C_2 and C_3 of the covering $\mathscr{C}$ in the case (1).

(b) Suppose that there exist exactly three b_1-reduced clusters $C_1, C_2, C_3 \in \mathscr{C}$. Taking into account Lemmas 5(i) and 4(ii), we obtain that $|C_1 \cup C_2| \geq 7$ and, therefore, $|C_1 \cup C_2 \cup C_3| \geq 9 > 7 = \deg(b_1)$, which is a contradiction.

(2) $G = G(11, 10)$.

(a) Assume that there exist exactly two b_1-reduced clusters $C_1, C_2 \in \mathscr{C}$. By Lemma 5(i), $|C_1 \cap C_2| \leq 1$. By Lemmas 5(ii) and 3(ii), $|C_1 \setminus C_2| \leq 4$ and $|C_2 \setminus C_1| \leq 4$. Therefore, $\deg(b_1) = |C_1 \cup C_2| \leq 9$, which is a contradiction.

(b) Let $\mathscr{C}$ contain three b_1-reduced clusters C_1, C_2 and C_3.

First, we suppose that C_1, C_2 and C_3 are pairwise disjoint. By Lemmas 3(ii) and 4(i), we have $3 \leq |C_i| \leq 4$ for any $i = 1, 2, 3$. W.l.o.g., assume that $C_1 = \{a_1, a_2, a_3\}$, $C_2 = \{a_4, a_5, a_6\}$, $C_3 = \{a_7, a_8, a_9, a_{10}\}$. By the definition of $\mathscr{C}$ and Lemma 3(i), we have $|\mathscr{C}(a_1)| = 3$, since $|A \setminus C_1| = 8$.

Let C_4 and C_5 be two clusters in $\mathscr{C}(a_1) \setminus \{C_1\}$. Each of the clusters C_4 and C_5 has at least one common vertex with any of the clusters C_2, C_3. If, for example, $C_4 \cap C_2 = \emptyset$, then $a_1 \in (C_1 \cap C_4) \setminus C_2$ and $|C_2 \setminus (C_1 \cup C_4)| = |C_2| = 3$, which is a contradiction to Lemma 3(iii). Since $C_3 \subseteq C_4 \cup C_5$ by the definition of $\mathscr{C}$ and $|C_3| = 4$, then each of the clusters C_4 and C_5 has exactly two common vertices with the cluster C_3.

The inequalities $|C_4| \geq 5$ and $|C_5| \geq 5$ hold. Otherwise, let, for example, $|C_4| \leq 4$. Then $|C_5| \geq 6$, since $|C_4 \cup C_5| \geq 9$. Hence, by Lemma 3(i), $|C_5| = 6$. Therefore, $C_4 \cap C_5 = \{a_1\}$ and $|C_5 \setminus C_4| = 5$, which is a contradiction to Lemma 3(ii).

W.l.o.g., assume that $\{a_4, a_7, a_8\} \subseteq C_4$, $\{a_6, a_9, a_{10}, a_{11}\} \subseteq C_5$. Since $|C_5 \setminus C_1| \leq 4$ by Lemma 3(ii), then $a_5 \notin C_5$. Hence, $a_5 \in C_4$. We have $a_5 \in (C_2 \cap C_4) \setminus C_5$. By Lemma 3(iii), $|C_5 \setminus (C_2 \cup C_4)| \leq 2$. Then $a_{11} \in C_4$ and, by Lemma 3(i), $C_4 = \{a_1, a_4, a_5, a_7, a_8, a_{11}\}$ (see Figure 3). Therefore, $|C_4 \setminus C_1| = 5$, which is a contradiction to Lemma 3(ii).

Now, w.l.o.g., assume that $a_1 \in C_1 \cap C_2$. By Lemma 5(i), $C_1 \cap C_2 = \{a_1\}$. By Lemmas 5(ii) and 3(ii), $|C_1| \leq 5$ and $|C_2| \leq 5$. Each of the clusters C_1, C_2 has size at least 4. If not, then $a_1 \in (C_1 \cap C_2) \setminus C_3$ by Lemma 5(iii), and $|C_3 \setminus (C_1 \cup C_2)| \geq 10 - (3 + 5 - 1) = 3$, which is a contradiction to Lemma 3(iii).

Furthermore, assume that at least one of the clusters C_1, C_2, say C_1, has size 5. Then $|C_1 \setminus C_3| \leq 4$ by Lemmas 5(ii) and 3(ii), and so $|C_1 \cap C_3| = 1$ by Lemma 5(i). Let $C_1 \cap C_3 = \{a_2\}$. Then $a_2 \in (C_1 \cap C_3) \setminus C_2$ by Lemma 5(iii). We obtain that $|C_2 \setminus (C_1 \cup C_3)| \leq 2$ by Lemma 3(iii). Therefore, $|C_2 \cap C_3| = 1$. Let $C_2 \cap C_3 = \{a_3\}$. We have $a_3 \in (C_2 \cap C_3) \setminus C_1$ and $|C_1 \setminus (C_2 \cup C_3)| = 3$, contradicting Lemma 3(iii).

Thus, $|C_1| = |C_2| = 4$. Let, w.l.o.g., $C_1 = \{a_1, a_2, a_3, a_4\}$, $C_2 = \{a_1, a_5, a_6, a_7\}$. Then $\{a_8, a_9, a_{10}\} \subseteq C_3$, since $\{a_1, a_2, \ldots, a_{10}\} = N(b_1)$. By Lemma 5(iii), $a_1 \in (C_1 \cap C_2) \setminus C_3$. However, then $|C_3 \setminus (C_1 \cup C_2)| = 3$, which is a contradiction to Lemma 3(iii).

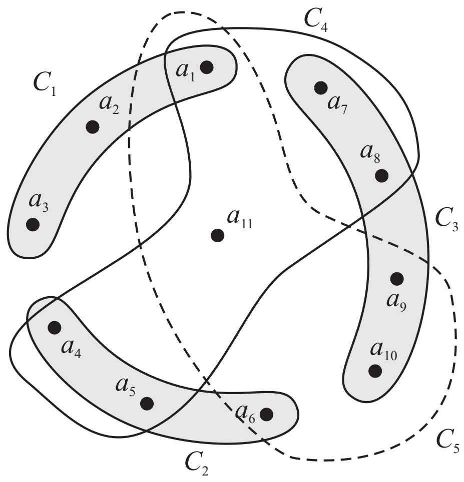

Figure 3. The clusters C_1, C_2, C_3, C_4 and C_5 of the covering $\mathscr{C}$ in the case (2).

(3) $G = G(10, 9, 5)$.

Each vertex a_i, where $i = 1, 2, \dots, 5$, belongs to one b_1- and one b_2-reduced clusters. Therefore, by Lemma 5(iii), each two of the b_2-reduced clusters have no common vertices. By Lemma 5(iii), if a vertex belongs to two of the b_1-reduced clusters, then this vertex belongs to the set $\{a_6, a_7, a_8, a_9\}$.

(a) Let $\mathscr{C}$ contain exactly two b_1-reduced clusters C_1, C_2. Since $|C_1 \cup C_2| = 9$, we get $|C_1 \cap C_2| = 1$ and $|C_1| = |C_2| = 5$ by Lemmas 5(i),(ii) and 3(ii). Let, w.l.o.g., $C_1 \cap C_2 = \{a_9\}$. By the definition of $\mathscr{C}$, any vertex a_i, where $i = 1, 2, \dots, 8$, belongs to exactly two clusters from $\mathscr{C}(a_i) \setminus \{C_1, C_2\}$. Moreover, it is easy to obtain that, for any vertex a_i, where $i = 1, 2, \dots, 8$, each cluster $C \in \mathscr{C}(a_i) \setminus \{C_1, C_2\}$ satisfies the equalities $|C \cap (C_1 \setminus C_2)| = 2$ and $|C \cap (C_2 \setminus C_1)| = 2$. Since every b_2-reduced cluster is a subset of $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ and belongs to $\mathscr{C}(a_i) \setminus \{C_1, C_2\}$, it has size 4, which is a contradiction.

(b) Let $\mathscr{C}$ contain three pairwise non-intersecting b_1-reduced clusters C_1, C_2 and C_3. By Lemma 3(ii), $|C_i| \le 4$ for every $i = 1, 2, 3$.

(b1) First, suppose that $|C_1| = 1$, $|C_2| = 4$ and $|C_3| = 4$. Put $C_1 = \{a_1\}$. Consider the clusters $C_4, C_5 \in \mathscr{C}(a_1) \setminus \{C_1\}$. By the definition of $\mathscr{C}$, $|C_i \cap C_j| = 2$ for any $i = 2, 3$ and $j = 4, 5$. In particular, $(C_4 \cap C_5) \cap (C_2 \cup C_3) = \emptyset$. Since $(C_2 \cap C_4) \setminus C_5 \neq \emptyset$, then $|C_5 \setminus (C_2 \cup C_4)| \le 2$ by Lemma 3(iii). Similarly, $|C_4 \setminus (C_2 \cup C_5)| \le 2$. Therefore, $a_{10} \in C_4 \cap C_5$. We obtain that there does not exist a b_2-reduced cluster in $\mathscr{C}(a_1)$, which is a contradiction.

Now, let $C_1 \subset \{a_6, a_7, a_8, a_9\}$. W.l.o.g., put $C_1 = \{a_9\}$. Note that each b_2-reduced cluster C in $\mathscr{C}$ has size at most 4. If not (i.e., $|C| = \deg(b_2) = 5$), then the inclusion $C \subseteq C_2 \cup C_3$ implies that $|C \cap C_2| \ge 3$ or $|C \cap C_3| \ge 3$, which is a contradiction to the definition of $\mathscr{C}$. Let C_4 be a b_2-reduced cluster in $\mathscr{C}$ with size at most 2. Let $a_1 \in C_4 \cap C_2$. Consider the cluster $C_5 \in \mathscr{C}(a_1) \setminus \{C_2, C_4\}$. By the definition of $\mathscr{C}$, we have $C_3 \setminus C_4 \subseteq C_5$. Since $|C_4| \leq 2$ and $C_4 \cap C_2 \neq \emptyset$, we have $|C_3 \setminus C_4| \geq 3$. Therefore, $|C_3 \cap C_5| \geq 3$, which is a contradiction.

(b2) Suppose that $|C_1| = 2$, $|C_2| = 3$ and $|C_3| = 4$. Let $a \in C_1$, where $a \in \{a_1, a_2, \ldots, a_9\}$. Consider the clusters $C_4, C_5 \in \mathscr{C}(a) \setminus \{C_1\}$. By the definition of $\mathscr{C}$, $1 \leq |C_i \cap C_2| \leq 2$ and $|C_i \cap C_3| = 2$ for any $i = 4, 5$. Moreover, at least one of the clusters C_4, C_5, say C_5, has exactly two common vertices with C_2. Clearly, $(C_4 \cap C_5) \cap C_3 = \emptyset$ and $|(C_4 \cap C_5) \cap C_2| \leq 1$. If $a_{10} \in C_5$, then $|C_5| = 6$ by Lemma 3(i). We have $C_1 \setminus C_5 \neq \emptyset$ and $|C_5 \setminus C_1| = 5 > 4$, which is a contradiction to Lemma 3(ii). Therefore, $a_{10} \in C_4 \setminus C_5$. By Lemma 3(i), at least one vertex a' of the set $C_5 \cap C_2$ does not belong to C_4. We obtain that $a' \in (C_2 \cap C_5) \setminus C_4$ and $|C_4 \setminus (C_2 \cup C_5)| \geq 3$, which is a contradiction to Lemma 3(iii).

(b3) Let $|C_1| = |C_2| = |C_3| = 3$. Assume that there exists a b_2-reduced cluster in $\mathscr{C}$ with size at most 2. Therefore, this cluster does not intersect with some of the clusters C_1, C_2 and C_3, which is a contradiction to the definition of $\mathscr{C}$.

Now, let $C_4 = N(b_2)$ be the only b_2-reduced cluster in $\mathscr{C}$. W.l.o.g., assume that $C_1 = \{a_1, a_6, a_7\}$, $C_2 = \{a_2, a_3, a_8\}$ and $C_3 = \{a_4, a_5, a_9\}$. Consider the clusters $C' \in \mathscr{C}(a_2) \setminus \{C_2, C_4\}$ and $C'' \in \mathscr{C}(a_3) \setminus \{C_2, C_4\}$. By the definition of $\mathscr{C}$, we have $a_6, a_7, a_9, a_{10} \in C' \cap C''$. Therefore, $C' = C''$. Put $C_5 = C'$. Then $C_3 \setminus C_5 \neq \emptyset$ and $|C_5 \setminus C_3| = 5 > 4$, which is a contradiction to Lemma 3(ii).

(c) Let $\mathscr{C}$ contain three b_1-reduced clusters C_1, C_2, C_3 and $C_1 \cap C_2 \neq \emptyset$. W.l.o.g., assume that $|C_1| \geq |C_2|$. By Lemma 5(iii), we obtain that $(C_1 \cap C_2) \setminus C_3 \neq \emptyset$. It follows from Lemma 3(iii) that $|C_3 \setminus (C_1 \cup C_2)| \leq 2$. Hence, $|C_1 \cup C_2| \geq 7$. Then $|C_1| \geq 4$. Moreover, by Lemmas 5(ii) and 3(ii), we have $|C_1| \leq 5$.

(c1) Let $|C_1| = 5$. Then $C_1 \cap C_3 \neq \emptyset$ by Lemmas 5(ii) and 3(ii). Furthermore, $C_2 \cap C_3 = \emptyset$ by Lemmas 5(iii) and 3(iii). Since $(C_1 \cap C_3) \setminus C_2 \neq \emptyset$ and, by Lemma 3(iii), $|C_2 \setminus (C_1 \cup C_3)| \leq 2$, we have $|C_1| = 5$, $|C_2| = 3$ and $|C_3| = 3$. Recall that $C_1 \cap C_2, C_1 \cap C_3 \subseteq \{a_6, a_7, a_8, a_9\}$. W.l.o.g., assume that $C_1 \cap C_2 = \{a_8\}$, $C_1 \cap C_3 = \{a_9\}$. Consider the clusters $C_4 \in \mathscr{C}(a_8) \setminus \{C_1, C_2\}$ and $C_5 \in \mathscr{C}(a_9) \setminus \{C_1, C_3\}$. By the definition of $\mathscr{C}$, we have $|C_4 \cap (C_1 \setminus \{a_8, a_9\})| \leq 1$ and $|C_5 \cap (C_1 \setminus \{a_8, a_9\})| \leq 1$. Note that $C_4 \cap (C_2 \setminus \{a_8\}) = \emptyset$. If, to the contrary, $a \in C_4 \cap (C_2 \setminus \{a_8\})$, then $\mathscr{C}(a) = \{C_2, C_4, C_5\}$ and some vertex of the set $C_1 \setminus \{a_8, a_9\}$ does not belong to the set $C_2 \cup C_4 \cup C_5$, contradicting the definition of $\mathscr{C}$. Analogously, $C_5 \cap (C_3 \setminus \{a_9\}) = \emptyset$. At least one of the clusters C_2, C_3, say C_3, contains a vertex $a' \in \{a_1, a_2, \ldots, a_5\}$, since $|\{a_1, a_2, \ldots, a_5\} \cap C_1| \leq 3$. Let a'' be another vertex in the set $C_3 \setminus \{a_9\}$. Consider the clusters $C' \in \mathscr{C}(a') \setminus \{C_3, C_4\}$ and $C'' \in \mathscr{C}(a'') \setminus \{C_3, C_4\}$. Each of them contains the set $(C_2 \setminus \{a_8\}) \cup (C_1 \setminus (C_3 \cup C_4))$ of size at least 4. Therefore, $C' = C'' = C_6$ is a cluster of $\mathscr{C}$ of size at least 6. By Lemma 3(i), $|C_6| = 6$ (see Figure 4). Since $a' \in \{a_1, a_2, \ldots, a_5\}$, then C_6 is a b_2-reduced cluster in $\mathscr{C}$, which is a contradiction.

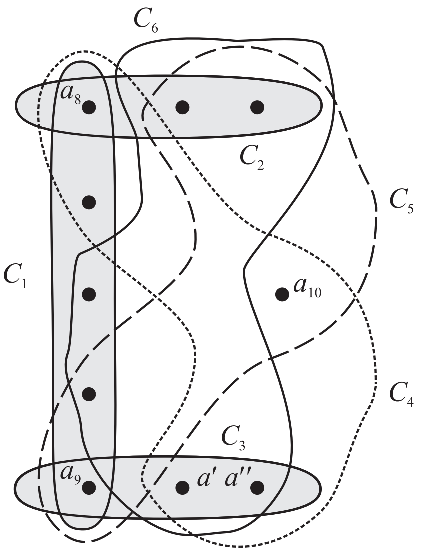

Figure 4. The clusters C_1, C_2, C_3, C_4, C_5 and C_6 of the covering $\mathscr{C}$ in the case (3).

(c2) Now, let $|C_1| = 4$. Then, taking into consideration the inequalities $|C_1 \cup C_2| \geq 7$ and $|C_1| \geq |C_2|$, we have $|C_2| = 4$.

Let C_3 intersect with C_1 or C_2. Then, by Lemma 3(iii), C_3 intersects with both C_1 and C_2. By Lemma 5(i),(iii), we can assume, w.l.o.g., that $C_1 = \{a_1, a_2, a_7, a_8\}$, $C_2 = \{a_3, a_4, a_7, a_9\}$ and $C_3 = \{a_5, a_6, a_8, a_9\}$. Consider the cluster $C_4 \in \mathscr{C}(a_7) \setminus \{C_1, C_2\}$. By the definition of $\mathscr{C}$, we have $a_5, a_6, a_{10} \in C_4$. Initially, let $C_4 = \{a_5, a_6, a_7, a_{10}\}$. Consider the cluster $C_5 \in \mathscr{C}(a_5) \setminus \{C_3, C_4\}$. By the definition of $\mathscr{C}$, we have $a_1, a_2, a_3, a_4 \in C_5$. Both clusters C_3, C_4 are not b_2-reduced since each of them contains at least one of the vertices $a_6, a_7, a_8, a_9, a_{10}$. Hence C_5 is a b_2-reduced cluster. It follows from the inclusion $N(b_2) \subseteq C_5$ that $C_5 = N(b_2) = \{a_1, a_2, \ldots, a_5\}$. Consider the cluster $C_6 \in \mathscr{C}(a_6) \setminus \{C_3, C_4\}$. By the definition of $\mathscr{C}$, we have $a_1, a_2, a_3, a_4 \in C_6$. Thus $C_6 \neq C_5$ and $|C_6 \cap C_5| \geq 4 > 2$, which is a contradiction to the definition of $\mathscr{C}$. If the cluster C_4 has a non-empty intersection with the set $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$, for example $a_1 \in C_4$, then at least one of the vertices a_3, a_4 also belongs to C_4. Otherwise, by the definition of $\mathscr{C}$, the cluster $C_5 \in \mathscr{C}(a_1) \setminus \{C_1, C_4\}$ contains the vertices a_3, a_4 and a_9. We obtain that $C_5 \neq C_2$ and $|C_5 \cap C_2| \geq 3 > 2$, which is a contradiction. Let $a_3 \in C_4$ and $C_5 \in \mathscr{C}(a_1) \setminus \{C_1, C_4\}$. Then $a_4, a_9 \in C_5$. We obtain that none of the clusters $C_1, C_4, C_5 \in \mathscr{C}(a_1)$ is b_2-reduced, which is a contradiction.

Assume that the cluster C_3 does not intersect with C_1 and C_2. Then $|C_3| = 2$. One of the vertices a_6, a_7, a_8 and a_9, say a_9, belongs to $C_1 \cap C_2$. Consider the cluster $C_4 \in \mathscr{C}(a_9) \setminus \{C_1, C_2\}$. Clearly, $C_3 \cup \{a_{10}\} \subseteq C_4$. We show that $|C_4 \cap$

$(C_1 \setminus C_2)| = 1$ and $|C_4 \cap (C_2 \setminus C_1)| = 1$. Indeed, if C_4 has no common vertices with one of the sets $C_1 \setminus C_2$ or $C_2 \setminus C_1$, say with $C_1 \setminus C_2$, then $(C_3 \cap C_4) \setminus C_1 \neq \emptyset$ and $|C_1 \setminus (C_3 \cup C_4)| = 3$, contradicting Lemma 3(iii). Let $C_3 = \{a', a''\}$. Consider the clusters $C' \in \mathscr{C}(a') \setminus \{C_3, C_4\}$ and $C'' \in \mathscr{C}(a'') \setminus \{C_3, C_4\}$. We have $(C_1 \setminus C_4) \cup (C_2 \setminus C_4) \subseteq C' \cap C''$. Since $|(C_1 \setminus C_4) \cup (C_2 \setminus C_4)| = 4$, we obtain that $C' = C''$ by the definition of $\mathscr{C}$. Denote the cluster C' by C_5. It can be easily obtained by the definition of $\mathscr{C}$ that there are two clusters C_6 and C_7 in $\mathscr{C}$ such that $((C_1 \setminus C_2) \cap C_4) \cup (C_2 \cap C_5) \cup \{a_{10}\} \subseteq C_6$ and $((C_2 \setminus C_1) \cap C_4) \cup (C_1 \cap C_5) \cup \{a_{10}\} \subseteq C_7$. Each vertex from the set $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ belongs to exactly three of the non-b_2-reduced clusters $C_1, C_2, C_4, C_5, C_6, C_7$. Clearly, at least three of the vertices $a_1, a_2, \ldots, a_5$ belong to the set $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$, which is a contradiction.

(4) We can come to a contradiction for each of the graphs $G = G(10, 9, 9)$ and $G = G(10, 9, 7)$ analogously to the graph $G = G(10, 9, 5)$.

(5) $G = G(10, 7, k)$, $k = 1, 2, \ldots, 7$.

(a) First, assume that $4 \leq k \leq 7$. For any $i = 1, 2, 3, 4$, denote by C_{i1} and C_{i2}, respectively, b_1- and b_2-reduced clusters from $\mathscr{C}(a_i)$. Consider the cluster $C_{i3} \in \mathscr{C}(a_i) \setminus \{C_{i1}, C_{i2}\}$. Since $C_{i1}, C_{i2} \subseteq \{a_1, a_2, \ldots, a_7\}$, we have $\{a_8, a_9, a_{10}\} \subseteq C_{i3}$ for any $i = 1, 2, 3, 4$. By the definition of $\mathscr{C}$, we obtain $C_{13} = C_{23} = C_{33} = C_{43}$ and $\{a_1, a_2, a_3, a_4, a_8, a_9, a_{10}\} \subseteq C_{i3}$ for any $i = 1, 2, 3, 4$, which is a contradiction to Lemma 3(i).

(b) Put $k = 1$. Let C_1 and C_2, respectively, be b_1- and b_2-reduced clusters from $\mathscr{C}(a_1)$. Then $C_1 \subseteq \{a_1, a_2, \ldots, a_7\}$, $C_2 = \{a_1\}$. By Lemma 3(i), $|C_1| \leq 6$. Consider the cluster $C_3 \in \mathscr{C}(a_1) \setminus \{C_1, C_2\}$. The equality $C_1 \cup C_2 \cup C_3 = C_1 \cup C_3 = A$ implies that $|C_1| \geq 5$ by Lemma 3(i).

W.l.o.g., assume that $C_1 = \{a_1, a_2, \ldots, a_5\}$. Then $C_3 = \{a_1, a_6, a_7, \ldots, a_{10}\}$ by Lemma 3(i). We obtain $C_1 \setminus C_3 \neq \emptyset$ and $|C_3 \setminus C_1| = 5$, contradicting Lemma 3(ii). Now, w.l.o.g. put $C_1 = \{a_1, a_2, \ldots, a_6\}$. Then $\{a_1, a_7, a_8, a_9, a_{10}\} \subseteq C_3$. By Lemma 3(ii), $|C_1 \setminus C_3| \leq 4$. Therefore, one of the vertices $a_2, a_3, \ldots, a_6$, say a_2, belongs to C_3. By Lemma 3(i), $C_3 = \{a_1, a_2, a_7, a_8, a_9, a_{10}\}$. Let C_4 be a b_1-reduced cluster from $\mathscr{C}(a_7)$. We get $C_3 \neq C_4$, since $C_3 \not\subseteq N(b_1)$. By Lemma 5(i), $|C_4 \cap C_1| \leq 1$. We obtain that $a_7 \in (C_3 \cap C_4) \setminus C_1$ and $|C_1 \setminus (C_3 \cup C_4)| \geq 3$, which is a contradiction to Lemma 3(iii).

(c) Put $k = 2$. Let C_1 and C_2, respectively, be b_1- and b_2-reduced clusters from $\mathscr{C}(a_1)$. Taking into account the case (b), we can assume that $C_2 = \{a_1, a_2\}$. Then we can proceed analogously to the case (b).

(d) Finally, we assume that $k = 3$. For any $i = 1, 2, 3$, denote by C_{i1} and C_{i2}, respectively, b_1- and b_2-reduced clusters from $\mathscr{C}(a_i)$. Taking into account the cases (b) and (c), we can assume that $C_{12} = \{a_1, a_2, a_3\}$. Consider the cluster $C_{i3} \in \mathscr{C}(a_i) \setminus \{C_{i1}, C_{i2}\}$. Since $C_{i1}, C_{i2} \subseteq \{a_1, a_2, \ldots, a_7\}$, we have $\{a_8, a_9, a_{10}\} \subseteq C_{i3}$ for any $i = 1, 2, 3$. By the definition of $\mathscr{C}$, $C_{13} = C_{23} = C_{33}$

and $\{a_1, a_2, a_3, a_8, a_9, a_{10}\} \subseteq C_{i3}$ for any $i = 1, 2, 3$. By Lemma 3(i), $C_{i3} = \{a_1, a_2, a_3, a_8, a_9, a_{10}\}$. We obtain that $C_{12} \neq C_{13}$ and $|C_{12} \cap C_{13}| = 3$, which is a contradiction to the definition of $\mathscr{C}$.

(6) $G = G(9, 8, 1)$.

(a) Assume that there exist exactly two b_1-reduced clusters $C_1, C_2 \in \mathscr{C}$. Clearly, $\mathscr{C}$ contains a unique b_2-reduced cluster $C_3 = \{a_1\}$. If $C_1 \cap C_2 = \emptyset$, then $|C_1| = |C_2| = 4$ by Lemma 3(ii). W.l.o.g., assume that $a_1 \in C_1$. Thus, $a_1 \in (C_1 \cap C_3) \setminus C_2$ and $|C_2 \setminus (C_1 \cup C_3)| = 4 > 2$, which is a contradiction to Lemma 3(iii).

Let $C_1 \cap C_2 \neq \emptyset$. It follows from Lemma 5(i) that $|C_1 \cap C_2| = 1$. Then $C_1 \cap C_2 \neq \{a_1\}$ by Lemma 5(iii). Let $C_1 \cap C_2 = \{a_2\}$ and $a_1 \in C_1$. Since $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$, we have $|C_1| \leq 5$ and $|C_2| \leq 5$ by Lemma 3(ii). The equality $|C_1 \cup C_2| = 8$ implies $|C_1| \geq 4$ and $|C_2| \geq 4$. We have $a_1 \in (C_1 \cap C_3) \setminus C_2$ and $|C_2 \setminus (C_1 \cup C_3)| \geq 3$, which is a contradiction to Lemma 3(iii).

(b) Now, let C_1, C_2, C_3 and $C_4 = \{a_1\}$, respectively, be three b_1- and a unique b_2-reduced clusters in $\mathscr{C}$. W.l.o.g., assume that $a_1 \in C_1$. By Lemma 5(iii), $C_1 \cap C_i \neq \{a_1\}$ for any $i = 2, 3$.

Furthermore, we have $|C_1| \geq 5$. Otherwise, $|A \setminus C_1| \geq 5$ and, by the definition of $\mathscr{C}$, there exists a cluster $C_5 \in \mathscr{C}(a_1) \setminus \{C_1, C_4\}$ such that $(A \setminus C_1) \cup \{a_1\} \subseteq C_5$. By Lemma 3(i), it follows that $|A \setminus C_1| = 5$, i.e., $|C_1| = 4$. We have $C_1 \setminus C_5 \neq \emptyset$ and $|C_5 \setminus C_1| \geq 5$, which is a contradiction to Lemma 3(ii). Therefore, by the same lemma, $C_1 \cap C_2 \neq \emptyset$ and $C_1 \cap C_3 \neq \emptyset$. By Lemmas 5(ii) and 3(ii), we have $|C_1 \setminus C_2| \leq 4$ and, consequently, $|C_1| = 5$.

The equality $C_2 \cap C_3 = \emptyset$ holds. Otherwise, by Lemma 5(i) and (iii), we have $(C_2 \cap C_3) \setminus C_1 \neq \emptyset$ and $|C_1 \setminus (C_2 \cup C_3)| = 3$, which is a contradiction to Lemma 3(iii).

Let $C_5 \in \mathscr{C}(a_1) \setminus \{C_1, C_4\}$. Since $A \setminus (C_1 \cup C_4) \subset C_5$, we have $|C_5| \geq 5$. Since $|C_1 \cap C_i| = 1$ for any $i = 2, 3$, $C_2 \cap C_3 = \emptyset$ and $|(C_2 \cup C_3) \setminus C_1| = 3$, one of the clusters C_2, C_3, say C_2, has size 2. So, we have $(C_2 \cap C_5) \setminus C_1 \neq \emptyset$ and $|C_1 \setminus (C_2 \cup C_5)| \geq 3$ both in the case $|C_5| = 6$ (since $C_2 \subseteq C_5$ by Lemma 3(ii)) and in the case $|C_5| = 5$, which is a contradiction to Lemma 3(iii).

(7) We can come to a contradiction for the graph $G = G(9, 8, 2)$ analogously to the graph $G = G(9, 8, 1)$.

4. PROOF OF SUFFICIENCY OF THEOREM 2

Let a threshold graph $H = G(p, q_1, q_2, \ldots, q_k)$ with the bipartition (A, B) not contain any of the graphs $K_{1,4}$, $G(12, 7)$, $G(11, 10)$, $G(10, 9, 9)$, $G(10, 9, 7)$, $G(10, 9, 5)$, $G(10, 7, k)$, $k = 1, 2, \ldots, 7$, $G(9, 8, 2)$, $G(9, 8, 1)$ as an induced subgraph. By Theorem 1, we have to show that there exists a $(3, 2)$-covering of H.

W.l.o.g., assume that H is a connected non-complete graph. Therefore, H

has a dominating vertex by the definition of H. Furthermore, $|B| \leq 2$, since H does not contain $K_{1,4}$ as an induced subgraph. Thus, we have $H = G(p, q_1)$ or $H = G(p, q_1, q_2)$.

First, we suppose that $|A| = p \geq 14$. Then $q_1 \leq 6$, since H does not contain any of the graphs $G(11, 10)$ and $G(12, 7)$ as an induced subgraph. For any vertex $b \in B$, partition the set $N(b)$ into $n_b \leq 3$ pairwise disjoint cliques C_i^b each having size at most 2. Obviously, the list of cliques $(C_i^b \cup \{b\} : b \in B,\ i = 1, \dots, n_b)$ together with the clique A gives a desired $(3, 2)$-covering of H.

If $|A| \leq 7$, then $q_1 \leq 6$ by the maximality of the clique A. Therefore, a desired $(3, 2)$-covering of H can be constructed as above.

Now, let $8 \leq |A| \leq 13$. Taking into account the above considerations, we can assume that $q_1 \geq 7$.

Let $H = G(p, q_1)$. Since H does not contain any of the graphs $G(12, 7)$ and $G(11, 10)$ as an induced subgraph, it is isomorphic to one of the graphs $G(13, 9)$, $G(12, 9)$, $G(12, 8)$, $G(11, 9)$, $G(11, 8)$, $G(11, 7)$, $G(10, 9)$, $G(10, 8)$, $G(10, 7)$, $G(9, 8)$, $G(9, 7)$, $G(8, 7)$. Clearly, the set of cliques

$$\begin{aligned}\mathscr{C} = \{&\{a_1, a_2, a_3, a_4, a_5, b_1\}, \{a_1, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_{10}, a_{11}, a_{12}, a_{13}\},\\ &\{a_2, a_3, a_6, a_7, a_{10}, a_{11}\}, \{a_2, a_3, a_8, a_9, a_{12}, a_{13}\}, \{a_4, a_5, a_6, a_7, a_{12}, a_{13}\},\\ &\{a_4, a_5, a_8, a_9, a_{10}, a_{11}\}\}\end{aligned}$$

of the graph $G(13, 9)$ is one of its $(3, 2)$-coverings. Each of the graphs $G(12, 9)$, $G(12, 8)$, $G(11, 9)$, $G(11, 8)$, $G(11, 7)$, $G(10, 9)$, $G(10, 8)$, $G(10, 7)$, $G(9, 8)$, $G(9, 7)$ and $G(8, 7)$ is an induced subgraph of $G(13, 9)$. Therefore, a desired $(3, 2)$-covering for each of these graphs can be obtained from the covering $\mathscr{C}$.

Now, let $H = G(p, q_1, q_2)$. Since H does not contain any of the graphs $G(12, 7)$, $G(11, 10)$, $G(10, 9, 9)$, $G(10, 9, 7)$, $G(10, 9, 5)$, $G(10, 7, k)$, $k = 1, 2, \dots, 7$, $G(9, 8, 2)$ and $G(9, 8, 1)$ as an induced subgraph, it is isomorphic to one of the graphs $G(11, 9, 8)$, $G(11, 9, 6)$, $G(11, 9, 4)$, $G(10, 9, 8)$, $G(10, 9, 6)$, $G(10, 9, 4)$, $G(10, 8, 8)$, $G(10, 8, 7)$, $G(10, 8, 6)$, $G(10, 8, 5)$, $G(10, 8, 4)$, $G(10, 8, 3)$, $G(9, 8, 8)$, $G(9, 8, 7)$, $G(9, 8, 6)$, $G(9, 8, 5)$, $G(9, 8, 4)$, $G(9, 8, 3)$, $G(9, 7, 7)$, $G(9, 7, 6)$, $G(9, 7, 5)$, $G(9, 7, 4)$, $G(9, 7, 3)$, $G(9, 7, 2)$, $G(9, 7, 1)$, $G(8, 7, 7)$, $G(8, 7, 6)$, $G(8, 7, 5)$, $G(8, 7, 4)$, $G(8, 7, 3)$, $G(8, 7, 2)$, $G(8, 7, 1)$. Some of the desired $(3, 2)$-coverings $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$ for the graphs $G(11, 9, 8)$, $G(11, 9, 6)$, $G(11, 9, 4)$, $G(9, 7, 1)$, respectively, are given below:

$$\begin{aligned}\mathscr{C}_1 = \{&\{a_1, a_2, a_3, a_4, a_9, b_1\}, \{a_5, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_2, a_7, a_8, b_2\},\\ &\{a_3, a_4, a_5, a_6, b_2\}, \{a_1, a_2, a_5, a_6, a_{10}, a_{11}\}, \{a_3, a_4, a_7, a_8, a_{10}, a_{11}\},\\ &\{a_9, a_{10}, a_{11}\}\},\\ \mathscr{C}_2 = \{&\{a_1, a_2, a_7, a_9, b_1\}, \{a_3, a_4, a_7, a_8, b_1\}, \{a_5, a_6, a_8, a_9, b_1\},\\ &\{a_1, a_2, a_3, a_4, a_5, a_6, b_2\}, \{a_5, a_6, a_7, a_{10}, a_{11}\}, \{a_1, a_2, a_8, a_{10}, a_{11}\},\\ &\{a_3, a_4, a_9, a_{10}, a_{11}\}\},\end{aligned}$$

$$\begin{aligned}\mathscr{C}_3 = \{&\{a_1, a_2, a_3, a_4, a_9, b_1\}, \{a_5, a_6, a_7, a_8, a_9, b_1\}, \{a_1, a_2, a_7, a_8, b_2\},\\ &\{a_3, a_4, a_5, a_6\}, \{a_1, a_2, a_5, a_6, a_{10}, a_{11}\}, \{a_3, a_4, a_7, a_8, a_{10}, a_{11}\},\\ &\{a_9, a_{10}, a_{11}\}\},\\ \mathscr{C}_4 = \{&\{a_1, a_2, a_3, a_4, a_5, b_1\}, \{a_5, a_6, a_7, b_1\}, \{a_1, b_2\}, \{a_1, a_2, a_6, a_7, a_8, a_9\},\\ &\{a_3, a_4, a_6, a_7\}, \{a_3, a_4, a_8, a_9\}, \{a_5, a_6, a_7\}, \{a_5, a_8, a_9\}\}.\end{aligned}$$

Each of the remaining graphs $G(10, 9, 8)$, $G(10, 9, 6)$, $G(10, 9, 4)$, $G(10, 8, 8)$, $G(10, 8, 7)$, $G(10, 8, 6)$, $G(10, 8, 5)$, $G(10, 8, 4)$, $G(10, 8, 3)$, $G(9, 8, 8)$, $G(9, 8, 7)$, $G(9, 8, 6)$, $G(9, 8, 5)$, $G(9, 8, 4)$, $G(9, 8, 3)$, $G(9, 7, 7)$, $G(9, 7, 6)$, $G(9, 7, 5)$, $G(9, 7, 4)$, $G(9, 7, 3)$, $G(9, 7, 2)$, $G(8, 7, 7)$, $G(8, 7, 6)$, $G(8, 7, 5)$, $G(8, 7, 4)$, $G(8, 7, 3)$, $G(8, 7, 2)$, $G(8, 7, 1)$ is an induced subgraph for some of the graphs $G(11, 9, 8)$, $G(11, 9, 6)$, $G(11, 9, 4)$, $G(9, 7, 1)$. Therefore, a desired $(3, 2)$-covering for each of the remaining graphs can be obtained from one of the coverings $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$.

5. Recognition Algorithm

The proof of sufficiency of Theorem 2 implies the following linear algorithm for recognizing graphs from L_3^2 in the class of threshold graphs.

Algorithm

Input: a connected threshold graph H with bipartition (A, B), where A is a maximal clique in H.

Output: 1 if $H \in L_3^2$, and 0 otherwise.

begin
    if $B = \emptyset$, i.e., the graph H is complete,
        return 1;
    if $|B| \geq 3$
        return 0;
    if $\deg(b) \leq 6$ for every $b \in B$
        return 1;
    if $|A| \geq 14$
        return 0;
    if H contains some of the graphs $G(12, 7)$, $G(11, 10)$, $G(10, 9, 9)$, $G(10, 9, 7)$, $G(10, 9, 5)$, $G(10, 7, k)$, $k = 1, 2, \ldots, 7$, $G(9, 8, 2)$, $G(9, 8, 1)$ as an induced subgraph
        return 0;
    return 1;
end.

The complexity of the algorithm in lines 1–9 is at most $O(n)$, where $n = |V(H)|$. Since the order of the graph H in line 10 is at most 13, this line takes $O(1)$ time.

So, the total complexity of the recognition algorithm is $O(n)$.

Acknowledgements

This work is partially supported by BRFFR (Projects F11OB-064 and F13MLD-012), and by DAAD.

References

[1] L.W. Beineke, *Derived graphs and digraphs*, in: Beitrage zur Graphentheorie, H. Sachs, H.-J. Voss, H.-J. Walter (Ed(s)), (Leipzig, Teubner, 1968) 17–33.

[2] J.C. Bermond and J.C. Meyer, *Graphs representatif des arêtes d'un multigraphe*, J. Math. Pures Appl. **52** (1973) 299–308.

[3] V. Chvátal and P.L. Hammer, Set-Packing and Threshold Graphs (Comp. Sci. Dept. Univ. of Waterloo, Ontario, 1973).

[4] D.G. Degiorgi and K. Simon, *A dynamic algorithm for line graph recognition*, Lect. Notes in Comput. Sci. **1017** (1995) 37–48.
doi:10.1007/3-540-60618-1_64

[5] S. Földes and P.L. Hammer, *Split graphs having Dilworth number two*, Canad. J. Math. **29** (1977) 666–672.
doi:10.4153/CJM-1977-069-1

[6] M.L. Gardner, *Forbidden configurations in intersection graphs of r-graphs*, Discrete Math. **31** (1980) 85–88.
doi:10.1016/0012-365X(80)90175-2

[7] O.V. Glebova, Yu.M. Metelsky and P.V. Skums, *Krausz dimension and its generalizations in special graph classes*, Discrete Math. Theor. Comput. Sci. **15** (2013) 107–120.

[8] P. Hliněný and J. Kratochvíl, *Computational complexity of the Krausz dimension of graphs*, Lect. Notes in Comput. Sci. **1335** (1997) 214–228.
doi:10.1007/BFb0024500

[9] M.S. Jacobson, A.E. Kezdy and J. Lehel, *Recognizing intersection graphs of linear uniform hypergraphs*, Graphs Combin. **4** (1997) 359–367.
doi:10.1007/BF03353014

[10] A.G. Levin and R.I. Tyshkevich, *Edge Hypergraphs*, Diskret. Mat. **5** (1993) 112–129, in Russian.

[11] P.G.H. Lehot, *An optimal algorithm to detect a line graph and output its root graph*, J. Assoc. Comput. Mach. **21** (1974) 569–575.
doi:10.1145/321850.321853

[12] Yu. Metelsky, *Split intersection graphs of hypergraphs with bounded rank*, Vestsi Nats. Akad. Navuk Belarusi. Ser. Fiz.-Mat. Navuk **3** (1997) 117–122, in Russian.

[13] Yu. Metelsky and K.N. Shchamialiova, *Finite characterizability of intersection graphs of hypergraphs with bounded rank and multiplicity in the class of split graphs*, Vestn. Beloruss. Gos. Univ. Ser. 1 Fiz. Mat. Inform. **1** (2008) 102–105, in Russian.

[14] Yu. Metelsky and R. Tyshkevich, *Line graphs of linear* 3*-uniform hypergraphs*, J. Graph Theory **25** (1997) 243–251.
doi:10.1002/(SICI)1097-0118(199708)25:4⟨243::AID-JGT1⟩3.0.CO;2-K

[15] R.N. Naik, S.B. Rao, S.S. Shrikhande and N.M. Singhi, *Intersection graphs of k-uniform linear hypergraphs*, Ann. Discrete Math. **6** (1980) 275–279.
doi:10.1016/S0167-5060(08)70711-8

[16] R.N. Naik, S.B. Rao, S.S. Shrikhande and N.M. Singhi, *Intersection graphs of k-uniform linear hypergraphs*, European J. Combin. **3** (1982) 159–172.
doi:10.1016/S0195-6698(82)80029-2

[17] J. Naor and M.B. Novick, *An efficient reconstruction of a graph from its line graph in parallel*, J. Algorithms **11** (1990) 132–143.
doi:10.1016/0196-6774(90)90034-C

[18] S. Poljak, V. Rödl and D. Turzik, *Complexity of representation of graphs by set systems*, Discrete Appl. Math. **3** (1981) 301–312.
doi:10.1016/0166-218X(81)90007-X

[19] N.D. Roussopoulos, *A max*$\{m,n\}$ *algorithm for determining the graph H from its line graph G*, Inform. Process. Lett. **2** (1973) 108–112.
doi:10.1016/0020-0190(73)90029-X

[20] P.V. Skums, *Krausz decomposition in special classes of split graphs*, Vestn. Beloruss. Gos. Univ. Ser. 1 Fiz. Mat. Inform. **3** (2005) 96–100, in Russian.

[21] P.V. Skums, S.V. Suzdal and R.I. Tyshkevich, *Edge intersection graphs of linear* 3*-uniform hypergraphs*, Discrete Math. **309** (2009) 3500–3517.
doi:10.1016/j.disc.2007.12.082

[22] V.A. Tashkinov, *Characterization of intersection graphs of p-graphs*, in: Proc. 5th all-USSR Conference on Problems of Theoretical Cybernetics (Novosibirsk, 1980) 135–137, in Russian.

Received 26 February 2015
Revised 29 January 2016
Accepted 29 January 2016