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SHARP UPPER BOUNDS ON THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF STRONGLY CONNECTED DIGRAPHS[1]

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Abstract

Let $G = (V(G), E(G))$ be a simple strongly connected digraph and $q(G)$ be the signless Laplacian spectral radius of G. For any vertex $v_i \in V(G)$, let d_i^+ denote the outdegree of v_i, m_i^+ denote the average 2-outdegree of v_i, and N_i^+ denote the set of out-neighbors of v_i. In this paper, we prove that:

(1) $q(G) = d_1^+ + d_2^+$, $(d_1^+ \neq d_2^+)$ if and only if G is a star digraph $\overleftrightarrow{K}_{1,n-1}$, where d_1^+, d_2^+ are the maximum and the second maximum outdegree, respectively ($\overleftrightarrow{K}_{1,n-1}$ is the digraph on n vertices obtained from a star graph $K_{1,n-1}$ by replacing each edge with a pair of oppositely directed arcs).

(2) $q(G) \leq \max\left\{\frac{1}{2}\left(d_i^+ + \sqrt{{d_i^+}^2 + 8d_i^+ m_i^+}\right) : v_i \in V(G)\right\}$ with equality if and only if G is a regular digraph.

(3) $q(G) \leq \max\left\{\frac{1}{2}\left(d_i^+ + \sqrt{{d_i^+}^2 + \frac{4}{d_i^+}\sum_{v_j \in N_i^+} d_j^+(d_j^+ + m_j^+)}\right) : v_i \in V(G)\right\}$. Moreover, the equality holds if and only if G is a regular digraph or a bipartite semiregular digraph.

(4) $q(G) \leq \max\left\{\frac{1}{2}\left(d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}\right) : (v_j, v_i) \in E(G)\right\}$. If the equality holds, then G is a regular digraph or $G \in \Omega$, where Ω is a class of digraphs defined in this paper.

Keywords: digraph, signless Laplacian spectral radius.

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1. Introduction

Let G be a finite simple digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. Two vertices are called adjacent if they are connected by an arc. For $e = (v_i, v_j) \in E(G)$, v_i is called to be adjacent to v_j by an out-arc and v_j is called to be adjacent to v_i by an in-arc. For any vertex $v_i \in V(G)$, $N_i^+ = N_{v_i}^+(G) = \{v_j : (v_i, v_j) \in E(G)\}$ and $N_i^- = N_{v_i}^-(G) = \{v_j : (v_j, v_i) \in E(G)\}$ are called the sets of out-neighbors and in-neighbors of v_i, respectively. Let $d_i^+ = |N_i^+|$ denote the outdegree of the vertex v_i and $d_i^- = |N_i^-|$ denote the indegree of the vertex v_i in the digraph G. The maximum vertex outdegree is denoted by Δ^+ and the minimum outdegree by δ^+. If $\delta^+ = \Delta^+$, then G is a regular digraph. Let $t_i^+ = \sum_{v_j \in N_i^+} d_j^+$ be the 2-outdegree of the vertex v_i, $m_i^+ = \frac{t_i^+}{d_i^+}$ be the average 2-outdegree of the vertex v_i. A digraph is simple if it has no loops and multiarcs. A digraph is strongly connected if for every pair of vertices $v_i, v_j \in V(G)$, there exists a directed path from v_i to v_j. A digraph is a bipartite semiregular digraph if it is a strongly connected digraph whose vertex set can be partitioned into two subsets S and T, such that each arc has one end in S and one end in T, all vertices in S have the same outdegree, and all vertices in T have the same outdegree.

For a digraph G, we assume that the vertices are ordered such that $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Let $A(G) = (a_{ij})$ denote the adjacency matrix of G, where a_{ij} is equal to the number of arcs (v_i, v_j). Let $D(G) = \mathrm{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix with outdegrees of the vertices of G and $Q(G) = D(G) + A(G)$ the signless Laplacian matrix of G. The spectral radius of $Q(G)$, i.e., the largest modulus of the eigenvalues of $Q(G)$, is called the signless Laplacian spectral radius of G, denoted by $q(G)$. It follows from Perron Frobenius Theorem that $q(G)$ is an eigenvalue of $Q(G)$, and there is a positive unit eigenvector corresponding to $q(G)$ when G is a strongly connected digraph. Therefore, throughout this paper, we only consider finite simple strongly connected digraphs.

There are a lot of results on the (signless) Laplacian spectral radius of an undirected graph, see [3–7, 9, 13–15, 17] and so on. Recently, some lower or upper bounds for the spectral radius of a digraph are given in [2, 8, 16], and some lower or upper bounds for the signless Laplacian spectral radius of a digraph can be found in [1, 10].

In 2014, Lang and Wang [12] presented the following bounds for the signless Laplacian spectral radius of a digraph.

$$(1) \qquad q(G) \leq \max\left\{ \frac{d_i^+ + \sqrt{{d_i^+}^2 + 4m_i^+(d_j^+ + m_j^+)}}{2} : (v_i, v_j) \in E(G) \right\}.$$

$$(2) \qquad q(G) \le \max\left\{ d_i^+ + \frac{d_i^+\left(m_i^+ + \sqrt{m_i^+}\right)}{d_i^+ + \sqrt{d_i^+}} : v_i \in V(G) \right\}.$$

$$(3) \quad q(G) \le \min_{1\le i\le n}\left\{ \frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2} \right\}.$$

In this paper, we investigate the signless Laplacian spectral radius of a strongly connected digraph G. We obtain some upper bounds for $q(G)$, and we also characterize the digraphs which achieve the upper bounds for the signless Laplacian spectral radius $q(G)$ of a strongly connected digraph G. Finally, we give an example to compare those upper bounds.

2. Lemmas and Results

Lemma 1 ([11])**.** *Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix with spectral radius $\rho(M)$ and let $R_i = R_i(M)$ be the i-th row sum of M, i.e., $R_i(M) = \sum_{j=1}^{n} m_{ij}$ $(1 \le i \le n)$. Then*

$$(4) \qquad \min\{R_i(M) : 1 \le i \le n\} \le \rho(M) \le \max\{R_i(M) : 1 \le i \le n\}.$$

Moreover, if M is irreducible, then any equality holds in (4) *if and only if $R_1 = R_2 = \cdots = R_n$.*

Lemma 2 ([11])**.** *Let M be an irreducible nonnegative matrix. Then $\rho(M)$ is an eigenvalue of M and there is a positive vector X such that $MX = \rho(M)X$.*

Lemma 3. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $d_1^+ + m_1^+ = d_2^+ + m_2^+ = \cdots = d_n^+ + m_n^+$ holds if and only if G is a regular digraph or a bipartite semiregular digraph.*

Proof. If G is a regular digraph or a bipartite semiregular digraph, then $d_1^+ + m_1^+ = d_2^+ + m_2^+ = \cdots = d_n^+ + m_n^+$ holds.

Conversely, suppose that $d_1^+ + m_1^+ = d_2^+ + m_2^+ = \cdots = d_n^+ + m_n^+$. Assume that G is not regular. We will show that G is a bipartite semiregular digraph.

As G is strongly connected and not regular , then G contains an arc (u, v) such that $d_u^+ = \delta^+ < d_v^+$, where $d_v^+ = \max\{d_w^+ : w \in N_u^+\}$. If the vertices in N_u^+ have different outdegrees, then

$$d_u^+ + m_u^+ < \delta^+ + d_v^+ \le d_v^+ + m_v^+,$$

which yields a contradiction. So all out-neighbors of u have the same outdegree d_v^+. Consider now the vertex v. If one of its out-neighbors has outdegree greater than δ^+, then $d_v^+ + m_v^+ > d_v^+ + \delta^+ = m_u^+ + d_u^+$, also a contradiction. So all out-neighbors of v have the same outdegree δ^+.

Repeating the above discussion on an out-neighbor of v with outdegree δ^+ and so on, as G is strongly connected, we get that G has only two distinct outdegrees δ^+, d_v^+, and each arc joins two vertices with outdegrees δ^+, d_v^+ respectively. We have a bipartition of the vertex set $V(G) = S \cup T$, where S (respectively, T) consists of vertices with outdegree δ^+ (respectively, d_v^+). Any two vertices in S or T are not adjacent by an arc, as each arc joins two vertices with distinct outdegrees. So G is bipartite semiregular. ■

Lemma 4. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be a positive eigenvector corresponding to the eigenvalue $q(G)$ of $Q(G)$. If x_i is the largest eigencomponent, then the outdegree of v_i is greater than or equal to $\frac{q(G)}{2}$.*

Proof. Since $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector corresponding to the eigenvalue $q(G)$ of $Q(G)$, we get from $Q(G)\mathbf{X} = q(G)\mathbf{X}$ that

$$q(G)x_i = d_i^+ x_i + \sum_{(v_i,v_k)\in E(G)} x_k,$$

i.e.,

$$q(G) - d_i^+ = \sum_{(v_i,v_k)\in E(G)} \frac{x_k}{x_i} \quad (\text{as } x_i \text{ is the largest, } x_i \neq 0).$$

Thus

$$q(G) - d_i^+ \leq d_i^+, \quad \text{and hence,} \quad d_i^+ \geq \frac{q(G)}{2}.$$ ■

Corollary 5. *If $q(G) = d_1^+ + d_2^+$, then the vertex corresponding to the largest eigencomponent is the largest outdegree vertex.*

Proof. From Lemma 4 we get $d_i^+ \geq \frac{d_1^+ + d_2^+}{2}$. Then we deduce $d_i^+ = d_1^+$. ■

Lemma 6. *If $q(G) = d_1^+ + d_2^+$ $(d_1^+ \neq d_2^+)$, then*

(i) *the vertices respectively corresponding to the second largest and the largest eigencomponents are adjacent by an out-arc, where the latter vertex is the head and the former is the tail,*

(ii) *the second largest eigencomponent is greater than or equal to $\frac{d_2^+}{d_1^+} x_i$, where x_i is the largest eigencomponent.*

Proof. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be a positive eigenvector corresponding to the eigenvalue $q(G)$ of $Q(G)$, x_i and x_j be the largest and the second largest eigencomponents, respectively. Note that $Q(G)\mathbf{X} = q(G)\mathbf{X}$.

(i) Assume to the contrary that $(v_j, v_i) \notin E(G)$. We have

$$q(G)x_j = d_j^+ x_j + \sum_{(v_j,v_k)\in E(G)} x_k, \quad \text{i.e., } \ (q(G) - d_j^+)x_j \leq d_j^+ x_j,$$

hence

$$d_j^+ \geq \frac{q(G)}{2} = \frac{d_1^+ + d_2^+}{2}, \quad \text{thus } \ d_j^+ = d_1^+,$$

which is a contradiction because $d_i^+ = d_1^+$ and $d_1^+ \neq d_2^+$.

(ii) We have

$$q(G)x_i = d_i^+ x_i + \sum_{(v_i,v_k)\in E(G)} x_k.$$

By Corollary 5,

$$(q(G) - d_1^+)x_i \leq d_1^+ x_j, \quad \text{hence } \ x_j \geq \frac{d_2^+}{d_1^+} x_i.$$

■

Theorem 7. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and $d_1^+ \neq d_2^+$. Then $q(G) = d_1^+ + d_2^+$ if and only if G is a star digraph $\overleftrightarrow{K}_{1,n-1}$, where $\overleftrightarrow{K}_{1,n-1}$ is the digraph on n vertices obtained from a star graph $K_{1,n-1}$ by replacing each edge with a pair of oppositely directed arcs.*

Proof. If G is a star digraph $\overleftrightarrow{K}_{1,n-1}$, then $q(G) = n - 1 + 1 = d_1^+ + d_2^+$.

Conversely, let $q(G) = d_1^+ + d_2^+$. We will show that G is a star digraph $\overleftrightarrow{K}_{1,n-1}$.

Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be a positive eigenvector corresponding to the eigenvalue $q(G)$ of $Q(G)$, x_1 and x_j be the largest and the second largest eigencomponent, respectively. We have

$$q(G)x_j = d_j^+ x_j + \sum_{(v_j,v_k)\in E(G)} x_k.$$

Hence

$$(q(G) - d_j^+)x_j = \sum_{(v_j,v_k)\in E(G)} x_k \leq x_1 + (d_j^+ - 1)x_j \quad \text{(by Lemma 6(i))},$$

$$d_1^+ \leq (d_1^+ + d_2^+ - d_j^+) \leq \frac{x_1}{x_j} + (d_j^+ - 1) \leq \frac{d_1^+}{d_2^+} + (d_2^+ - 1) \ \text{ (by Lemma 6(ii))},$$

and thus

$$(d_1^+ - d_2^+)(d_2^+ - 1) \leq 0.$$

It follows that $d_2^+ = 1$, since G is a strongly connected digraph and $d_1^+ \neq d_2^+$.

Furthermore,

$$q(G)x_j = d_j^+ x_j + x_1 \quad \text{(by Lemma 6(i))},$$

and

$$d_1^+ x_j = x_1 = \sum_{(v_1,v_k)\in E(G)} x_k, \quad \text{since } (d_1^+ + d_2^+)x_1 = d_1^+ x_1 + \sum_{(v_1,v_k)\in E(G)} x_k.$$

Since x_j is the second largest eigencomponent, we have

$$x_k = x_j \text{ for all } (v_1, v_k) \in E(G).$$

Because x_j is the second largest eigencomponent, by Lemma 6(i), we have $(v_k, v_1) \in E(G)$ for all $(v_1, v_k) \in E(G)$.

By the previous discussion, we know that G contains a star digraph centered at v_1. If $d_1^+ \neq n-1$, v_1 must have an out-neighbor, say u, which is adjacent to a vertex outside the star digraph by an out-arc, as G is strongly connected. Then $d_u^+ \geq 2$, which yields a contradiction.

Therefore, $d_1^+ = n-1$. Since $(v_k, v_1) \in E(G)$ for all $(v_1, v_k) \in E(G)$, so $d_1^- = n-1$. Then G is a star digraph $\overleftrightarrow{K}_{1,n-1}$. ■

Remark 8. From [1], we have that $q(G) \leq \max\{d_i^+ + d_j^+ : (v_i, v_j) \in E(G)\} \leq d_1^+ + d_2^+$. But the extremal digraph which achieve the upper bound was not determined. Here we characterize this extreme digraph.

Let R^+ denote the set of real positive numbers. We have the following theorem.

Theorem 9. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then*

$$q(G) \leq \max\left\{\frac{d_i^+ + \sqrt{{d_i^+}^2 + \frac{4c_ic_i'}{b_i}}}{2} : v_i \in V(G)\right\}, \tag{5}$$

where $b_i \in R^+$, $b_i' = \frac{1}{b_i}\sum_{(v_i,v_j)\in E(G)} b_j$, $c_i = b_i(d_i^+ + b_i')$, $c_i' = \frac{\sum_{(v_i,v_j)\in E(G)} c_j}{c_i}$. Moreover, the equality holds if and only if $d_1^+ + b_1' = d_2^+ + b_2' = \cdots = d_n^+ + b_n'$.

Proof. From Lemma 2, there exists an positive eigenvector $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ corresponding to the eigenvalue $q(G)$ of $B^{-1}Q(G)B$, where $B = \mathrm{diag}(b_1, b_2, \ldots,$

b_n). We assume that one eigencomponent x_i is equal to 1 and the other eigencomponents are less than or equal to 1, that is, $x_i = 1$ and $0 < x_k \leq 1$, for all $1 \leq k \leq n$. From

$$(B^{-1}Q(G)B)\mathbf{X} = q(G)\mathbf{X},$$

we have

$$q(G)x_i = d_i^+ x_i + \sum_{(v_i,v_j)\in E(G)} \frac{b_j}{b_i} x_j.$$

That is

$$q(G) = d_i^+ + \sum_{(v_i,v_j)\in E(G)} \frac{b_j}{b_i} x_j. \tag{6}$$

And

$$q(G)x_j = d_j^+ x_j + \sum_{(v_j,v_k)\in E(G)} \frac{b_k}{b_j} x_k. \tag{7}$$

Multiplying both sides of (6) by $q(G)$, then replacing $q(G)x_j$ with (7), we get

$$\begin{aligned}
q(G)^2 &= d_i^+ q(G) + \sum_{(v_i,v_j)\in E(G)} \frac{b_j}{b_i}(d_j^+ x_j + \sum_{(v_j,v_k)\in E(G)} \frac{b_k}{b_j} x_k) \\
&= d_i^+ q(G) + \sum_{(v_i,v_j)\in E(G)} \frac{b_j d_j^+}{b_i} x_j + \frac{1}{b_i} \sum_{(v_i,v_j)\in E(G)} \sum_{(v_j,v_k)\in E(G)} b_k x_k \\
&\leq d_i^+ q(G) + \sum_{(v_i,v_j)\in E(G)} \frac{b_j d_j^+}{b_i} + \frac{1}{b_i} \sum_{(v_i,v_j)\in E(G)} \sum_{(v_j,v_k)\in E(G)} b_k \qquad (8) \\
&= d_i^+ q(G) + \sum_{(v_i,v_j)\in E(G)} \frac{b_j d_j^+}{b_i} + \sum_{(v_i,v_j)\in E(G)} \frac{1}{b_i} b_j b_j', \text{ as } x_j, x_k \leq 1 \\
&= d_i^+ q(G) + \sum_{(v_i,v_j)\in E(G)} \frac{1}{b_i} b_j (d_j^+ + b_j') = d_i^+ q(G) + \frac{1}{b_i} c_i c_i'.
\end{aligned}$$

From the above the bound (5) follows.

Now suppose that the equality holds in (5). Then all inequalities in the above argument must be equalities. From equality in (8), we get $x_j = 1$ for all j such that $(v_i, v_j) \in E(G)$ and $x_k = 1$ for all k such that $(v_i, v_j) \in E(G)$ and $(v_j, v_k) \in E(G)$. Since G is a strongly connected digraph, from this one can easily show that $x_j = 1$ for all $v_j \in V(G)$. Thus we have $d_1^+ + b_1' = d_2^+ + b_2' = \cdots = d_n^+ + b_n'$. Conversely, if $d_1^+ + b_1' = d_2^+ + b_2' = \cdots = d_n^+ + b_n'$, then $B^{-1}Q(G)B$ has the same row sum $d_i^+ + b_i'$, which is the spectral radius of $B^{-1}Q(G)B$ and $Q(G)$ by Lemma 1. Thus the equality holds. ■

Corollary 10. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then*

$$q(G) \leq \max\left\{\frac{d_i^+ + \sqrt{{d_i^+}^2 + 8d_i^+ m_i^+}}{2} : v_i \in V(G)\right\}. \tag{9}$$

Moreover, the equality holds if and only if G is a regular digraph.

Proof. Taking $b_i = 1$ in (5), the result follows. ■

Corollary 11. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then*

$$q(G) \leq \max\left\{\frac{d_i^+ + \sqrt{{d_i^+}^2 + \frac{4}{d_i^+}\sum\limits_{v_j \in N_i^+} d_j^+(d_j^+ + m_j^+)}}{2} : v_i \in V(G)\right\}. \tag{10}$$

Moreover, the equality holds if and only if G is a regular digraph or a bipartite semiregular digraph.

Proof. Taking $b_i = d_i^+$ in (5), we get the result. And the equality holds if and only if $d_i^+ + m_i^+$ is a constant . Then by Lemma 3, the equality holds if and only if G is a regular digraph or a bipartite semiregular digraph. ■

Let Ω be the class of digraphs $P = (V(P), E(P))$ such that P is a strongly connected digraph with $V(P) = \{1\} \cup V_1$, $d_1^+ = \Delta^+$, $d_1^- = n - 1$, $V_1 = \{k \in N_1^- : d_k^+ = \delta^+\}$ and $\Delta^+ > \delta^+$.

The spectral radius of the signless Laplacian matrix of $P \in \Omega$ is given by

$$q(P) = \frac{\Delta^+ + 2\delta^+ - 1 + \sqrt{(\Delta^+ - 2\delta^+ + 1)^2 + 4\Delta^+}}{2}.$$

Now we give another upper bound on the spectral radius of the signless Laplacian matrix of digraphs.

Theorem 12. *Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, arc set $E(G)$, the maximum vertex outdegree Δ^+, the second maximum outdegree Δ_2^+ and the minimum outdegree δ^+. Then $q(G)$ is less than or equal to*

$$\max\left\{\frac{d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}}{2} : (v_j, v_i) \in E(G)\right\}. \tag{11}$$

If the equality in (11) holds, then G is a regular digraph or $G \in \Omega$.

Proof. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be a positive eigenvector of $Q(G)$ corresponding to the eigenvalue $q(G)$. We assume that one eigencomponent x_i is equal to 1 and the other eigencomponents are less than or equal to 1, that is, $x_i = 1$ and $0 < x_k \leq 1$, for all k. Also let $x_j = \max_{k:k\neq i} x_k$. As previously,

$$Q(G)\mathbf{X} = q(G)\mathbf{X}. \tag{12}$$

From the i-th equation of (12), we have

$$q(G)x_i = d_i^+ x_i + \sum_{(v_i,v_k)\in E(G)} x_k, \text{ i.e., } q(G) \leq d_i^+ + d_i^+ x_j.$$

Therefore

$$0 < \sum_{(v_i,v_k)\in E(G)} x_k = q(G) - d_i^+ \leq d_i^+ x_j. \tag{13}$$

From the j-th equation of (12), we have

$$q(G)x_j = d_j^+ x_j + \sum_{(v_j,v_k)\in E(G)} x_k,$$

thus

$$q(G)x_j \leq d_j^+ x_j + 1 + (d_j^+ - 1)x_j,$$

and hence

$$(q(G) - 2d_j^+ + 1)x_j \leq 1. \tag{14}$$

From (13) and (14), we get

$$(q(G) - d_i^+)(q(G) - 2d_j^+ + 1) \leq d_i^+,$$

hence

$$q(G)^2 - (d_i^+ + 2d_j^+ - 1)q(G) + 2d_i^+(d_j^+ - 1) \leq 0,$$

thus

$$q(G) \leq \frac{1}{2}(d_i^+ + 2d_j^+ - 1 + \sqrt{(d_i^+ - 2d_j^+ + 1)^2 + 4d_i^+}).$$

The first part of the proof is done.

Now suppose that equality holds in (11). Then all inequalities in the above argument must be equalities. In particular, from (13) we get

$$x_k = x_j \text{ for all } v_k, (v_i, v_k) \in E(G).$$

By (14), we get that there exists an out-neighbor w of v_j with $x_w = 1 = x_i$, and for any other out-neighbor v_k of v_j, $x_k = x_j$.

Let $V_1 = \{v_k : v_k \neq v_i, x_k = x_j\}$. If $V_1 \neq V(G) \setminus \{v_i\}$, then V_1 is a proper subset of $V(G) \setminus \{v_i\}$, that is, $V_1 \subset V(G) \setminus \{v_i\}$. Hence there exist vertices $v_p \in V_1$, $v_q \notin V_1$, $v_q \neq v_i$ such that $(v_p, v_q) \in E(G)$, as G is strongly connected. Thus we have $x_q < x_j$ as x_j is the second maximum eigencomponent. For $v_p \in V(G)$, from above, we must have $x_q = x_j$, a contradiction. Thus $V_1 = V(G) \setminus \{v_i\}$.

If $x_j = 1 = x_i$, then all eigencomponent of $\mathbf{X}$ are 1's. Thus

$$q(G) = 2d_i^+, i = 1, 2, \dots, n.$$

Hence G is a regular digraph.

Otherwise, $x_j < 1$. By the above observation, all vertices with eigencomponent x_j have the vertex v_i as an out-neighbor, which implies that $d_i^- = n-1$. In this case, let $V_1 = V(G) \setminus \{v_i\}$. For any two vertices v_j and v_k in V_1, we have

$$q(G)x_j = d_j^+ x_j + \sum_{(v_j, v_r) \in E(G)} x_r = d_j^+ x_j + 1 + (d_j^+ - 1)x_j,$$

and

$$q(G)x_k = d_k^+ x_k + \sum_{(v_k, v_r) \in E(G)} x_r,$$

i.e., $q(G)x_j = d_k^+ x_j + 1 + (d_k^+ - 1)x_j$, as for any vertex v_k in $V(G) \setminus \{v_i\}, x_k = x_j$. Therefore $d_j^+ = d_k^+$. If $d_j^+ \geq d_i^+$, then

$$q(G)x_j = (2d_j^+ - 1)x_j + 1 > (2d_j^+ - 1)x_j + x_j = 2d_j^+ x_j \geq 2d_i^+ x_j,$$

so $q(G) > 2d_i^+$, but $q(G) = d_i^+ + d_i^+ x_j < 2d_i^+$, a contradiction. Therefore $d_j^+ < d_i^+$. Thus $d_i^+ = \Delta^+$, $d_i^- = n-1$, $V_1 = \{v_k, d_k^+ = \delta^+ : v_k \in N_i^-\}$, $\Delta^+ > \delta^+$. Hence $G \in \Omega$. In addition, note that each digraph in Ω has n vertices. ∎

3. Example

Let G_1, G_2 and G_3 be the digraphs of orders 4, 5 and 5, respectively, as shown in the following figure.

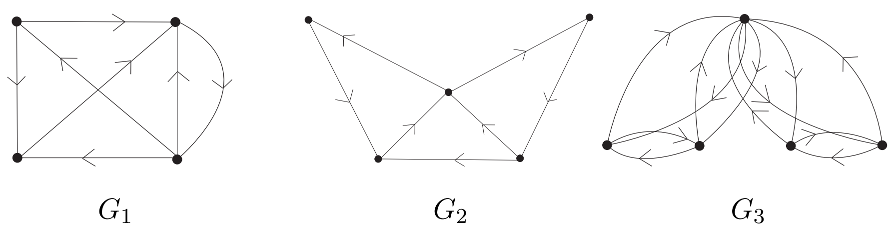

Table 1. Values of the various bounds for the three digraphs G_1, G_2 and G_3.

	$q(G)$	(1)	(2)	(3)	(9)	(10)	(11)
G_1	3.7693	4.2538	4.5774	4.8284	4.7016	4.0000	5.5616
G_2	3.0000	3.3452	3.5961	3.5616	3.6456	3.3452	4.0000
G_3	5.5616	5.7417	6.2761	5.5616	6.4721	5.7417	5.5616

Remark 13. Obviously, from Table 1, the bound (10) is the best in all known upper bounds for G_1, and the bound (1) is the second-best bound for G_1. Bounds (1) and (10) are the best for G_2, and the bound (3) is the second-best bound for G_2. However, the bounds (3) and (11) are the best for G_3, and the bounds (1) and (10) are the second-best bounds for G_3. In general, these bounds are incomparable.

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References

[1] Ş.B. Bozkurt and D. Bozkurt, *On the signless Laplacian spectral radius of digraphs*, Ars Combin. **108** (2013) 193–200.

[2] B.K. Bulter and P.H. Siegel, *Sharp bounds on the spectral radius of nonnegative matrices and digraphs*, Linear Algebra Appl. **439** (2013) 1468–1478. doi:10.1016/j.laa.2013.04.029

[3] Y.Q. Chen and L.G. Wang, *Sharp bounds for the largest eigenvalue of the signless Laplacian of a graph*, Linear Algebra Appl. **433** (2010) 908–913. doi:10.1016/j.laa.2010.04.026

[4] S.Y. Cui, C.X. Tian and J.J. Guo, *A sharp upper bound on the signless Laplacian spectral radius of graphs*, Linear Algebra Appl. **439** (2013) 2442–2447. doi:10.1016/j.laa.2013.06.015

[5] K.Ch. Das, *A characterization on graphs which achieve the upper bound for the largest Laplacian eigenvalue of graphs*, Linear Algebra Appl. **376** (2004) 173–186. doi:10.1016/j.laa.2003.06.009

[6] K.Ch. Das, *The Laplacian spectrum of a graph*, Comput. Math. Appl. **48** (2004) 715–724. doi:10.1016/j.camwa.2004.05.005

[7] K.Ch. Das, *Maximizing the sum of the squares of the degrees of a graph*, Discrete Math. **285** (2004) 57–66. doi:10.1016/j.disc.2004.04.007

[8] E. Gudiño and J. Rada, *A lower bound for the spectral radius of a digraph*, Linear Algebra Appl. **433** (2010) 233–240.
doi:10.1016/j.laa.2010.02.012

[9] A.D. Güngör, K.C. Das and A.S. Çevik, *Sharp upper bounds on the spectral radius of the signless Laplacian matrix of a graph*, Appl. Math. Comput. **219** (2013) 5025–5032.
doi:10.1016/j.amc.2012.11.039

[10] W.X. Hong and L.H. You, *Spectral radius and signless Laplacian spectral radius of strongly connected digraphs*, Linear Algebra Appl. **457** (2014) 93–113.
doi:10.1016/j.laa.2014.05.007

[11] R.A. Horn and C.R. Johnson, Matrix Analysis (Cambridge University Press, New York, 1985).
doi:10.1017/CBO9780511810817

[12] W.W. Lang and L.G. Wang, *Sharp bounds for the signless Laplacian spectral radius of digraphs*, Appl. Math. Comput. **238** (2014) 43–49.
doi:10.1016/j.amc.2014.04.001

[13] M.H. Liu, B.L. Liu and B. Cheng, *Ordering* (*signless*) *Laplacian spectral radii with maximum degrees of graphs*, Discrete Math. **338** (2015) 159–163.
doi:10.1016/j.disc.2014.09.007

[14] C.S. Oliveira, L.S. de Lima, N.M.M. de Abreu and P. Hansen, *Bounds on the index of the signless Laplacian of a graph*, Discrete Appl. Math. **158** (2010) 355–360.
doi:10.1016/j.dam.2009.06.023

[15] Y.L. Pan, *Sharp upper bounds for the Laplacian graph eigenvalues*, Linear Algebra Appl. **355** (2002) 287–295.
doi:10.1016/S0024-3795(02)00353-1

[16] G.H. Xu, K.F. Fang and J. Shen, *Bounds on the spectral radii of digraphs in terms of walks*, Appl. Math. Comput. **219** (2012) 3721–3728.
doi:10.1016/j.amc.2012.09.073

[17] G.L. Yu, Y.R. Wu and J.L. Shu, *Sharp bounds on the signless Laplacian spectral radii of graphs*, Linear Algebra Appl. **434** (2011) 683–687.
doi:10.1016/j.laa.2010.09.029

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