*Discussiones Mathematicae*
*Graph Theory 36 (2016) 959–976*
*doi:10.7151/dmgt.1913*

# BOUNDS FOR THE *b*-CHROMATIC NUMBER OF SUBGRAPHS AND EDGE-DELETED SUBGRAPHS

P. Francis[1] and S. Francis Raj

*Department of Mathematics*
*Pondicherry University*
*Puducherry – 605014, India*

**e-mail:** selvafrancis@gmail.com
francisraj_s@yahoo.com

**Abstract**

A $b$-coloring of a graph $G$ with $k$ colors is a proper coloring of $G$ using $k$ colors in which each color class contains a color dominating vertex, that is, a vertex which has a neighbor in each of the other color classes. The largest positive integer $k$ for which $G$ has a $b$-coloring using $k$ colors is the $b$-chromatic number $b(G)$ of $G$. In this paper, we obtain bounds for the $b$-chromatic number of induced subgraphs in terms of the $b$-chromatic number of the original graph. This turns out to be a generalization of the result due to R. Balakrishnan *et al.* [*Bounds for the b-chromatic number of $G-v$*, Discrete Appl. Math. 161 (2013) 1173–1179]. Also we show that for any connected graph $G$ and any $e \in E(G)$, $b(G-e) \leq b(G) + \left\lceil \frac{n}{2} \right\rceil - 2$. Further, we determine all graphs which attain the upper bound. Finally, we conclude by finding bound for the $b$-chromatic number of any subgraph.

**Keywords:** $b$-coloring, $b$-chromatic number.

**2010 Mathematics Subject Classification:** 05C15.

## 1. Introduction

All graphs considered in this paper are simple, finite and undirected. A $b$-coloring of a graph is a proper coloring of the vertices of $G$ such that each color class contains a color dominating vertex (c.d.v.), that is, a vertex adjacent to at least one vertex of every other color class. The largest positive integer $k$ for which $G$ has a $b$-coloring using $k$ colors is the $b$-chromatic number $b(G)$ of $G$. A $b$-chromatic

[1]Research supported by Council of Scientific and Industrial Research, New Delhi, India.

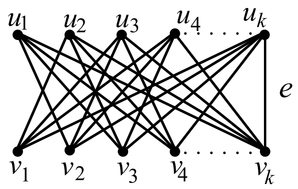


Figure 1. $b(G) = 2$ and $b(G-e) = k$.

coloring of $G$ denotes a $b$-coloring using $b(G)$ colors. From the definition of $\chi(G)$, we observe that each color class of a $\chi$-coloring contains a c.d.v. Thus $\omega(G) \leq \chi(G) \leq b(G)$, where $\omega(G)$ is the size of a maximum clique of $G$.

The concept of $b$-coloring was introduced by Irving and Manlove [9] in analogy to the achromatic number of a graph $G$ (which gives the maximum number of color classes in a complete coloring of $G$). They have shown that determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. There has been an increasing interest in the study of $b$-coloring since the publication of [9]. Some of the references are [2, 4–6, 8, 10–14].

Let $e$ be any edge of a graph $G$. We know that for the chromatic number of the edge-deleted subgraph $G-e$ of $G$, $\chi(G-e) = \chi(G)$ or $\chi(G-e) = \chi(G)-1$. Similarly, for the achromatic number $\psi(G)$, $\psi(G-e) = \psi(G)$ or $\psi(G-e) = \psi(G)-1$. Surprisingly, a similar statement does not hold for the $b$-chromatic number $b(G)$ of $G$. Indeed, the gap between $b(G-e)$ and $b(G)$ can be arbitrarily large. For example, consider the graph in Figure 1.

The bounds for the $b$-chromatic number of vertex-deleted subgraphs has been already determined in [1]. In Section 2, we find bounds for the $b$-chromatic number of induced subgraphs in terms of the $b$-chromatic number of the original graph. This actually generalizes the result in [1]. Also in Section 3, for any connected graph $G$ and $e \in E(G)$, we find upper bound for $b(G-e)$ in terms of $b(G)$. In addition, in Section 4, we completely characterize graphs for which the upper bound is attained. Finally in Section 5, we conclude by finding bound for the $b$-chromatic number of subgraphs in terms of the $b$-chromatic number of the original graph.

Note that in the figures, dotted lines indicate consecutive vertices and broken lines indicate possible edges. Throughout this paper, a color dominating vertex is in short written as c.d.v. and color dominating vertices is in short written as c.d.vs.

## 2. Bounds for the $b$-Chromatic Number of Induced Subgraphs

In this Section, let us find bounds for the $b$-chromatic number of induced subgraphs of $G$ in terms of $b(G)$. Note that if $H$ is an induced subgraph of $G$, then

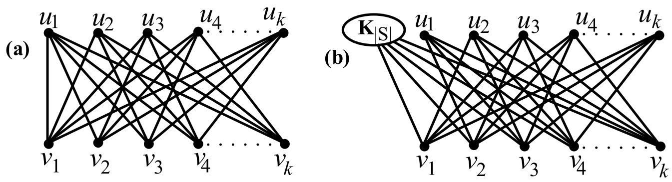


Figure 2. Graphs which attain the bounds.

there exist a subset $S$ of $V(G)$, such that $H$ is isomorphic to the subgraph induced by $V(G)-S$, which we denote by $G-S$. In [3], M. Blidia *et al.* have got an upper bound for the $b$-chromatic number in terms of the clique number.

**Theorem 1** [3]**.** *Every graph $G$ of order $n$ that is not a complete graph satisfies*

$$b(G) \leq \left\lfloor \frac{n+\omega(G)-1}{2} \right\rfloor .$$

As a consequence of Theorem 1, we get bounds for the $b$-chromatic number of induced subgraphs.

**Corollary 2.** *For any graph $G$ other than the complete graph and for any induced subgraph $G-S$ which is not a clique of $G$,*

$$2b(G)-(n+|S|)+1 \leq b(G-S) \leq \left\lfloor \frac{n-|S|+b(G)-1}{2} \right\rfloor .$$

***Proof.*** By using Theorem 1 and the fact that $\omega(G) \leq \omega(G-S)+|S| \leq b(G-S)+|S|$, we get the lower bound. The upper bound can be observed from the fact that $\omega(G-S) \leq \omega(G) \leq b(G)$. ■

The bounds given in Corollary 2 are sharp. For instance, consider $G$ to be the graph given in Figure 2(a). For $S=\{u_1,u_2,\ldots,u_{\lceil \frac{|S|}{2}\rceil}, v_1,v_2,\ldots,v_{\lfloor \frac{|S|}{2}\rfloor}\}$, we see that the upper bound is attained. Consider the graph $K_n-e$ where $e=uv$ and $S$ is any subset of $V(K_n-e)\backslash\{u,v\}$; we see that the lower bound is attained.

Next let us find one more lower bound for the $b$-chromatic number of induced subgraphs of $G$ in terms of $b(G)$.

**Theorem 3.** *For any connected graph $G$ with $n \geq 5$ vertices and for any $S \subset V(G)$ such that $1 \leq |S| \leq n-4$,*

$$b(G-S) \geq b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2.$$

***Proof.*** Let us first consider the case when $b(G-S)=1$. For $b(G)$ to be greater than or equal to $|S|+2$, there should be at least $|S|+2$ vertices of degree at least $|S|+1$. But the vertices of $G-S$ have degree at most $|S|$ in $G$ and hence the number of vertices with degree at least $|S|+1$ can be at most $|S|$, (namely the vertices of $S$). Thus $b(G) \leq |S|+1$. Also $n \geq |S|+4$. Therefore $b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2 \leq |S|+1 - \left\lfloor \frac{2|S|+4}{2} \right\rfloor + 2 = |S|+3-|S|-2 = 1 = b(G-S)$. Hence the bound is true for $b(G-S)=1$. Let us next consider the case when $b(G-S) \geq 2$. Suppose $b(G-S) < b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2$, then

$$b(G-S) = b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2 - k, \ k \geq 1,$$

$$b(G) = b(G-S) + \left\lfloor \frac{n+|S|}{2} \right\rfloor - 2 + k. \tag{2.1}$$

Let $c$ be a $b$-chromatic coloring of $G$ and $P$ denote the set of singleton classes of $c$ and $Q$ denote the remaining classes of $c$, so that $|V(P)| = |P|$ and $|V(Q)| \geq 2|Q|$. Further $n \geq |S|+4$, $b(G)-b(G-S) = \left\lfloor \frac{n+|S|}{2} \right\rfloor - 2 + k \geq \frac{2|S|+4}{2} - 2 + k = |S|+k \geq |S|+1$. As $c$ is a $b$-coloring, the vertices of $P$ induces a clique in $G$ and hence $|Q| \geq 1$ (if $|Q|=0$, then $G$ is complete and hence $b(G)-b(G-S) = |S|$, a contradiction).

*Case* (i) Both $n$ and $|S|$ are of the same parity. Suppose $|Q| > \frac{n-|S|}{2} - b(G-S)+1$, say $|Q| = \frac{n-|S|}{2} - b(G-S)+1+l$, $l \geq 1$. Then by equation (2.1), $|P| = 2b(G-S)+|S|-3+k-l$ and hence $|V(G)| = |V(P)|+|V(Q)| \geq n+1$, a contradiction. Therefore

$$|Q| \leq \frac{n-|S|}{2} - b(G-S)+1, \text{ and} \tag{2.2}$$

$$|P| \geq 2b(G-S)-2+|S|. \tag{2.3}$$

Rewrite equation (2.3) as $|P| \geq b(G-S)+(b(G-S)-2+|S|)$. Since $b(G-S) \geq 2$, $|P| \geq b(G-S)+|S|$. If all the vertices of $S$ belong to $P$, then the coloring $c$ for the remaining graph $G-S$ forms a $b$-coloring using $b(G)-|S|$ colors. Thus $b(G-S) \geq b(G)-|S|$ which implies $b(G)-b(G-S) \leq |S|$, a contradiction to $b(G)-b(G-S) \geq |S|+1$. If at least one of the vertex of $S$ belongs to $Q$, then $|P \backslash S| \geq b(G-S)+1$ and $P$ forms a clique in $G$. Therefore $\omega(G-S) \geq b(G-S)+1$, a contradiction.

*Case* (ii) Both $n$ and $|S|$ are of different parity. By arguments similar to Case (i), we can prove that

$$|Q| \leq \frac{n-|S|-1}{2} - b(G-S)+2, \text{ and} \tag{2.4}$$

$$|P| \geq 2b(G-S) - 3 + |S|. \tag{2.5}$$

If $b(G-S) \geq 3$ or $|P| > 2b(G-S) - 3 + |S|$, then $|P| \geq b(G-S) + |S|$. Again we get the same contradiction as mentioned in Case (i). Therefore $b(G-S) = 2$ and $|P| = 1 + |S|$. Now by using equation (2.1), we get $|Q| \geq \frac{n-|S|-1}{2}$. Also by equation (2.4) we get $|Q| \leq \frac{n-|S|-1}{2}$. Thus $|Q| = \frac{n-|S|-1}{2}$. Since $n \geq |S| + 4$ and the parity of $n$ and $|S|$ are different, $n - |S| \geq 5$ which in turns implies that $|Q| \geq 2$. If all the vertices of $S$ belong to $P$, then $b(G) - b(G-S) \leq |S|$, a contradiction to $b(G) - b(G-S) \geq |S| + 1$. If more than one vertex of $S$ belongs to $Q$, then in $G-S$ we have $|P \backslash S| \geq 3$, and $P \backslash S$ induces a clique of size $\geq 3$, a contradiction to $b(G-S) = 2$. Thus the only remaining possibility is $|S| - 1$ vertices of $S$ belong to $P$ and one vertex belongs to $Q$. Since $|Q| \geq 2$, in this case also we get a $K_3$ in $G-S$, a contradiction to $b(G-S) = 2$. ■

Here also we see that, the bound given in Theorem 3 is sharp. For instance, consider $G$ to be the graph given in Figure 2(b). In Figure 2(b), the circle denotes the clique with vertices $w_1, w_2, \ldots, w_{|S|}$ and every vertex in this clique is adjacent to every $v_i$, $i \in \{1, 2, \ldots, k\}$. For $S = \{u_1, w_2, \ldots, w_{|S|}\}$, we see that the lower bound is attained. Note that, we have two lower bounds for $b(G-S)$, one given in Corollary 2 and the other given in Theorem 3. Let us compare them and find out which is better and under what condition it happens. Consider $b(G) < \frac{n+|S|}{2} + 1$. Here $b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2 - (2b(G) - (n+|S|) + 1) \geq -b(G) - \left(\frac{n+|S|}{2}\right) + (n+|S|) + 1 > -\left(\frac{n+|S|}{2}\right) - \left(\frac{n+|S|}{2}\right) + (n+|S|) - 1 + 1 = 0$. Next consider $b(G) \geq \frac{n+|S|}{2} + 1$. Here it is easy to show that $2b(G) - (n+|S|) + 1 - b(G) + \left\lfloor \frac{n+|S|}{2} \right\rfloor - 2 \geq 0$. Therefore $b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2$ is a better lower bound for $b(G-S)$ when $b(G) < \frac{n+|S|}{2} + 1$ and $2b(G) - (n+|S|) + 1$ is a better lower bound when $b(G) \geq \frac{n+|S|}{2} + 1$.

As a consequence of Corollary 2 and Theorem 3, we get the bounds for $b(G-v)$ in terms of $b(G)$ which was determined in [1].

**Corollary 4** [1]**.** *For any connected graph $G$ with $n \geq 5$ vertices and for any $v \in V(G)$,*

$$b(G) - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) \leq b(G-v) \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

***Proof.*** The lower bound follows immediately from Theorem 3 by taking $S = \{v\}$. Let us next consider the upper bound. From Corollary 2, by taking $S = \{v\}$ and for $G-v$ which is not a clique, we get that $b(G-v) \leq \left\lfloor \frac{n-2+b(G)}{2} \right\rfloor \leq \frac{n-2+b(G)}{2} \leq \frac{b(G)}{2} + \frac{n}{2} - 1 = b(G) - \frac{b(G)}{2} + \frac{n}{2} - 1 \leq b(G) + \frac{n}{2} - \frac{b(G)+2}{2} \leq b(G) + \frac{n}{2} - 2$ (since $b(G) \geq 2$). When $G-v$ forms a clique, the upper bound can be immediately verified. ■

## 3. Bound for $b(G-e)$ in Terms of $b(G)$

Let $G$ be a bipartite graph with bipartition $X$ and $Y$. Connected graphs $G$ for which $b(G) = 2$ have been completely characterized by Kratochvíl *et al.* in [13]. A vertex $x \in X$ ($y \in Y$) is called a full vertex (or a charismatic vertex) of $X$ ($Y$) if it is adjacent to all the vertices of $Y$ ($X$).

**Lemma 5** [13]**.** *Let $G$ be a non-trivial connected graph. Then $b(G) = 2$ if and only if $G$ is bipartite and has a full vertex in each part of the bipartition.*

Next we shall see the bounds for the $b$-chromatic number of an edge-deleted subgraphs. It has already been proved by Faik [7] that $b(G-e) \geq b(G) - 1$ for any $e \in E(G)$. Thus let us consider the upper bound.

**Theorem 6.** *For any non-trivial connected graph $G$ with $n$ vertices and for any $e \in E(G)$,*

$$b(G-e) \leq b(G) + \left\lceil \frac{n}{2} \right\rceil - 2.$$

***Proof.*** Let us start with $n = 2$. Then $G = K_2$ and hence $b(G) = 2$ and $b(G-e) = 1$ which satisfies the inequality. Now let us consider $n \geq 3$ and $e \in E(G)$, where $e = uv$. Suppose $b(G-e) > b(G) + \lceil \frac{n}{2} \rceil - 2$. Then

$$b(G-e) = b(G) + \left\lceil \frac{n}{2} \right\rceil - 2 + k, \quad k \geq 1. \tag{3.1}$$

Let $c'$ be a $b$-chromatic coloring of $G-e$. Let $S'$ denote the set of singleton classes and $T'$ denote the set of remaining classes of $c'$. Since $b(G-e) - b(G) \geq 1$, $u$ and $v$ must be in the same class of $G-e$ and hence $|T'| \geq 1$. Here $|S'| \leq b(G) - 1$. If not, $\omega(G) > b(G)$, a contradiction. Also we know that $b(G-e) = |S'| + |T'|$. Thus from equation (3.1), we get $|T'| \geq b(G) + \lceil \frac{n}{2} \rceil - 2 + k - b(G) + 1 = \lceil \frac{n}{2} \rceil - 1 + k \geq \lceil \frac{n}{2} \rceil$

*Case* (i) $n$ is even. Here $|T'| \geq \frac{n}{2}$, and thus $|V(T')| \geq n$ and $|S'| = 0$. Also $|T'| \leq \frac{n}{2}$, therefore $|T'| = \frac{n}{2}$. As $b(G-e) = |S'| + |T'|$, by using equation (3.1) we get $b(G) = 2 - k \leq 1$, a contradiction.

*Case* (ii) $n$ is odd. Here $|T'| \geq \frac{n+1}{2}$ and thus $|V(G)| \geq |V(T')| \geq n + 1$, a contradiction. ■

## 4. Extremal Graphs

For $n = 2, 3$ and $4$, the extremal graphs which satisfy $b(G) = b(G-e) - \lceil \frac{n}{2} \rceil + 2$, for some $e = uv \in E(G)$ are given in Figure 3. In this Section, we use the same notations as given in the proof of Theorem 6. Let us characterize the connected

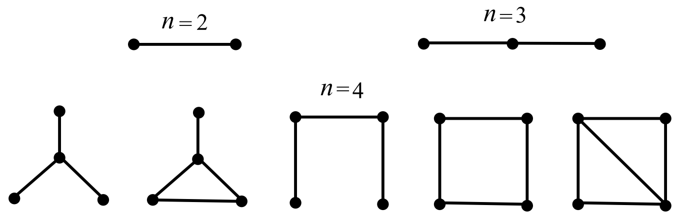


Figure 3. Extremal graphs when $n = 2, 3, 4$.

graphs $G$ with $n \geq 5$ for which $b(G-e) = b(G) + \lceil \frac{n}{2} \rceil - 2$, for some $e = uv \in E(G)$. In other words,

$$b(G) = b(G-e) - \left\lceil \frac{n}{2} \right\rceil + 2, \text{ for some } e = uv \in E(G). \tag{4.1}$$

Our arguments require $n \geq 5$. By arguments similar to the ones used in the proof of Theorem 6, we can make the following observations in this case.

**Observation 7.** (i) *$b(G-e) \geq b(G) + 1$.*

(ii) *$u$ and $v$ belong to the same class of $c'$.*

(iii) *$|T'| \geq 1$, $|S'| \leq b(G) - 1$ and therefore $|T'| \geq \lceil \frac{n}{2} \rceil - 1$, where $S'$ denotes the set of singleton classes and $T'$ denotes the set of remaining classes of a $b$-chromatic coloring $c'$ of $G - e$.*

Let us divide this characterization into two cases depending upon $n$ being odd or even.

*Case* (i) $n$ is odd. Here $|T'| \geq \frac{n+1}{2} - 1$, and thus $|V(T')| \geq n - 1$.

*Subcase* (a) $|V(T')| = n$. Here $|S'| = 0$ and $|T'| = \frac{n+1}{2} - 1$. By using equation (4.1) we get $b(G) = 1$, a contradiction.

*Subcase* (b) $|V(T')| = n - 1$. Now $|S'| = 1$, $|T'| = \frac{n+1}{2} - 1$, and by using equation (4.1), we get $b(G) = 2$. Also each color class of $T'$ has exactly two vertices. Let $S' = \{x\}$ and $T' = \{\{u_i, v_i\}\colon 1 \leq i \leq b(G-e) - 1\}$. For each $i \in \{1, 2, \ldots, b(G-e) - 1\}$, let $u_i$ be a c.d.v. of the color class $\{u_i, v_i\}$ of $T'$. Clearly, each $u_i$ must be adjacent to $x$, for $i \in \{1, 2, \ldots, b(G-e) - 1\}$ . Also by (ii) of Observation 7, $u$ and $v$ must be in the same class, and hence $e = uv = u_i v_i$ for some $i \in \{1, 2, \ldots, b(G-e) - 1\}$. Without loss of generality, let $u = u_{b(G-e)-1}$ and $v = v_{b(G-e)-1}$. There is no edge between two $u_i$, $i \in \{1, 2, \ldots, b(G-e) - 1\}$, as that would yield a $K_3$ in $G - e$, a contradiction to $b(G) = 2$. Hence for each $i \in \{1, 2, \ldots, b(G-e) - 1\}$, $u_i$ is adjacent to every $v_j$, $j \in \{1, 2, \ldots, b(G-e) - 1\} \backslash \{i\}$. Thus $G$ is isomorphic to the graph given in Figure 4 (where $u$ and $v$ are full vertices).

*Case* (ii) $n$ is even. Here $|T'| \geq \frac{n}{2} - 1$, and thus $|V(T')| \geq n - 2$.

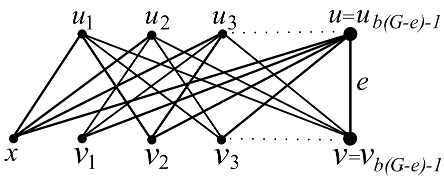


Figure 4. $n$ is odd and $|V(T')| = n-1$.

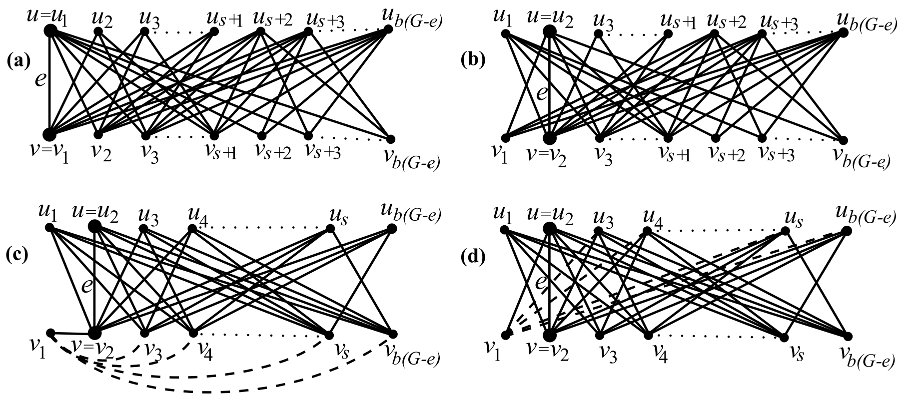


Figure 5. $n$ is even, $|S'| = 0$ and $u_1$ has one neighbor in each class and $s \geq 2$.

*Subcase* (a) $|V(T')| = n$. In this case, $|S'| = 0$ and $|T'| = \frac{n}{2} - 1$ or $|T'| = \frac{n}{2}$. If $|T'| = \frac{n}{2} - 1$, then by equation (4.1) we get $b(G) = 1$, a contradiction. Therefore $|T'| = \frac{n}{2}$ and hence $b(G) = 2$. In $T'$ each color class contains exactly two vertices, say $\{u_i, v_i\}$, $i \in \{1, 2, \ldots, b(G-e)\}$. Without loss of generality, let $u_1$ be a c.d.v. of the class $\{u_1, v_1\}$ of $G - e$. Suppose $u_1$ is adjacent to both the vertices in at least two classes of $T'$, then the c.d.v. of one of these classes will be adjacent to at least one of the vertex of the other class, which induce a $K_3$ in $G$, a contradiction to $b(G) = 2$. Thus $u_1$ cannot be adjacent to both the vertices in more than one class of $T'$. Let us first consider the case when $u_1$ has exactly one neighbor in each of the color class of $T'$ and let them be $v_i$ for $i \in \{2, 3, \ldots, b(G-e)\}$.

Let us assume that $s$ denote the number of $v_i$ which are c.d.vs. of $c'$, $i \in \{2, 3, \ldots, b(G-e)\}$, and without loss of generality let them be $v_2, v_3, \ldots, v_{s+1}$. Let $\mathcal{I} = \{1, s+2, s+3, \ldots, b(G-e)\}$ and $\mathcal{J} = \{2, 3, \ldots, s+1\}$. Clearly as $b(G) = 2$, there cannot be an edge between any two $v_i$, $i \in \{2, 3, \ldots, b(G-e)\}$. Let us first consider the case when $s \geq 2$. Here for $i \in \{2, 3, \ldots, s+1\}$, $v_i$ is adjacent to $u_j$, for all $j \in \{1, 2, \ldots, b(G-e)\} \setminus \{i\}$. Since $s \geq 2$, there are at least two c.d.vs. in $v_i$, $i \in \{2, 3, \ldots, s+1\}$. Thus an edge between any two $u_l$, $l \in \{2, 3, \ldots, b(G-e)\}$ would yield a $K_3$ or $C_5$, a contradiction to $b(G) = 2$. Thus there cannot be an edge between any two $u_l$, $l \in \{2, 3, \ldots, b(G-e)\}$. If $e = uv = u_i v_i$ for some $i \in \mathcal{I}$, say $i = b(G-e)$, then $u_i = u$ must be adjacent to $v_j$, for all $j \in \{1, 2, \ldots, b(G-e)-1\}$.

This is because $b(G) = 2$ and the only vertices that can be made full vertices are $u$ and $v$. Thus $v_{b(G-e)}$ becomes a c.d.v. in $G - e$ and hence the number of $v_i$ which are c.d.vs. of $c'$ is $s + 1$, a contradiction to the assumption of $s$. Thus $e \neq u_i v_i$, $i \in \{s+2, \ldots, b(G-e)\}$ and the only remaining possibility in this case is $e = uv = u_1 v_1$ and hence $v_1$ must be adjacent to $u_j$, for all $j \in \{2, 3, \ldots, b(G-e)\}$ for the same reason. Thus $G$ would be isomorphic to the graph given in Figure 5(a) together with some edges between $u_2, u_3, \ldots, u_{s+1}$ and $v_{s+2}, v_{s+3}, \ldots, v_{b(G-e)}$ (where $u$ and $v$ are full vertices). Next let us consider the possibility when $e = uv = u_i v_i$ for any $i \in \mathcal{J}$, say $i = 2$. Here if $s < b(G-e)-1$, then $u_2 = u$ must be adjacent to $v_j$ and $v_2 = v$ must be adjacent to $u_j$, for all $j \in \{1, 2, \ldots, b(G-e)\}$. Thus $G$ would be isomorphic to the graph given in Figure 5(b) together with some edges between $u_3, u_4, \ldots, u_{s+1}$ and $v_1, v_{s+2}, v_{s+3}, \ldots, v_{b(G-e)}$ (where $u$ and $v$ are full vertices). Next if $s = b(G) - 1$, then for $i \in \{2, 3, \ldots, b(G-e)\}$, $v_i$ must be adjacent to $u_j$ for all $j \in \{1, 2, \ldots, b(G-e)\} \setminus \{i\}$. While considering $v_1$, it is either adjacent to $u_2$ or $v_2$ (otherwise we cannot get full vertices in both the partition of $G$, a contradiction to $b(G) = 2$, see Lemma 5). If $v_1$ is adjacent to $v_2$, then $G$ will be isomorphic to the graph given in Figure 5(c) (where $u$ and $v$ are full vertices). If $v_1$ is adjacent to $u_2$, then $G$ will be isomorphic to the graph given in Figure 5(d) (where $u$ and $v$ are full vertices).

Let $\mathcal{L} = \{3, 4, \ldots, b(G-e)\}$. Let us next consider the case when $s = 1$. If $e = uv = u_i v_i$ for some $i \in \mathcal{L}$, say $i = b(G-e)$, then $u_2$ must be adjacent to either $u$ or $v$ (otherwise we cannot get full vertices in both the partition of $G$, a contradiction to $b(G) = 2$). If $u_2$ is adjacent to $v = v_{b(G-e)}$, then $v = v_{b(G-e)}$ becomes a c.d.v. of $G - e$ and hence $s \geq 2$, a contradiction. Thus $u_2$ must be adjacent to $u$ and hence in this case, $G$ will be isomorphic to the graph given in Figure 6(a). If $e = uv = u_2 v_2$, then $G$ will be isomorphic to the graph given in Figure 6(b) (where $u$ and $v$ are full vertices) and if $e = uv = u_1 v_1$, $G$ will be isomorphic to the graph given in Figure 6(c) (where $u$ and $v$ are full vertices). Suppose $s = 0$ . Then none of the $v_i$, $i \in \{2, 3, \ldots, b(G-e)\}$ is a c.d.v. Thus each $u_i$, $i \in \{2, 3, \ldots, b(G-e)\}$ is a c.d.v. of $c'$ and hence has to be adjacent to $v_1$. Since $b(G) = 2$, $\{u_i : i = 2, 3, \ldots, b(G-e)\}$ form an independent set. Now for $u_i$ to be a c.d.v. it should be adjacent to $v_j$ for all $j \neq i$ and $i, j \in \{1, 2, \ldots, b(G-e)\}$. This in turn makes each $v_i$ a c.d.v., a contradiction to $s = 0$.

Now let us consider the case when $u_1$ has two neighbors in one class, say $\{u_2, v_2\}$. For $i \in \mathcal{L}$, no $v_i$ can be adjacent to either $u_2$ or $v_2$ (otherwise $\{v_i, u_1, u_2\}$ or $\{v_i, u_1, v_2\}$ will induce a $K_3$ in $G$, a contradiction to $b(G) = 2$). Hence for each $i \in \mathcal{L}$, $u_i$ is the c.d.v. of the color class $\{u_i, v_i\}$ in $T'$. Suppose $e = uv = u_i v_i$ for some $i \in \mathcal{L}$, say $i = b(G-e)$ . Then $u_2$ has to be adjacent to $u$ (otherwise we cannot get full vertices in both the partition of $G$, a contradiction to $b(G) = 2$), and hence $G$ will be isomorphic to the graph given in Figure 6(d). If $e = uv = u_1 v_1$, then $G$ will be isomorphic to the graph given in Figure 6(e) (where $u$ and $v$

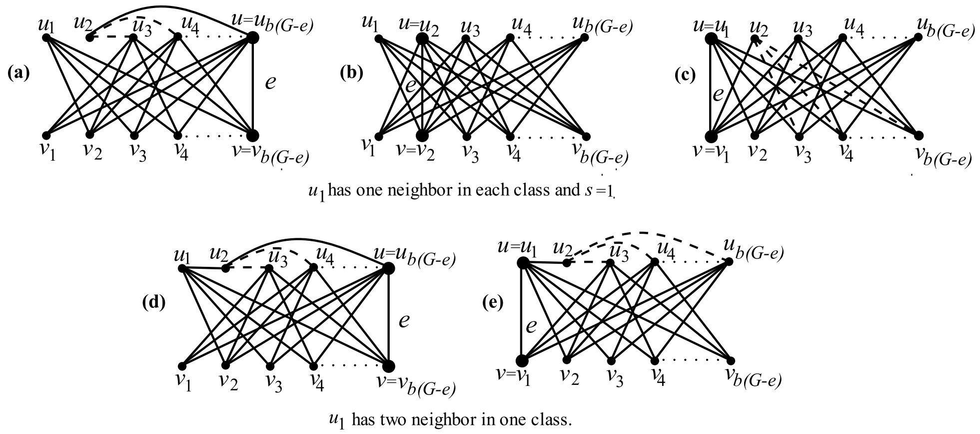


Figure 6. $n$ is even, $b(G) = 2$ and $|V(T')| = n$.

are full vertices). Note that $e = u_2v_2$ yield a $K_3$ in $G$, a contradiction to $b(G) = 2$ and hence not possible.

*Subcase* (b) $|V(T')| = n - 1$. Here $|S'| = 1$ and $|T'| = (\frac{n}{2} - 1) \geq 2$. Hence from equation (4.1) we get $b(G) = 2$. Also each class in $T'$ contains exactly two vertices except one which contains three vertices. Let $S' = \{x\}$, $T' = \{\{u_i, v_i\} : 2 \leq i \leq b(G-e)-1\} \cup \{u_1, v_1, w\}$. Since $c'$ is a $b$-coloring, each color class contains a c.d.v. Without loss of generality, for every $i \in \{1, 2, \ldots, b(G-e)-1\}$, let $u_i$ be a c.d.v. of the color class $\{u_i, v_i\}$ in $T'$. Clearly, each $u_i$ must be adjacent to $x$. Since $b(G) = 2$, no two $u_i$ are adjacent for $i \in \{1, 2, \ldots, b(G-e)-1\}$ and hence each $u_i$ must be adjacent to $v_j$, for all $j \in \{2, 3, \ldots, b(G-e)-1\} \backslash \{i\}$. Also for $i \in \{2, 3, \ldots, b(G-e)-1\}$, $u_i$ is adjacent to at least one of the vertex in $\{w, v_1\}$. In addition, no two $v_j$ are adjacent for $j \in \{2, 3, \ldots, b(G-e)-1\}$. Also $x$ cannot have two neighbors in any class of $T'$ except $\{u_1, v_1, w\}$ and $x$ cannot be adjacent to both $w$ and $v_1$ (as $\{x, w, u_2\}$ or $\{x, v_1, u_2\}$ would yield a $K_3$ in $G$, a contradiction). While considering $w$ and $v_1$ we have two possibilities: (i) $x$ is adjacent to either $w$ or $v_1$, say $w$, and (ii) $x$ is non-adjacent to both $w$ and $v_1$.

Let us first consider the case when $x$ is adjacent to $w$. Since $b(G) = 2$, $w$ cannot be adjacent to any of the $u_i$ (otherwise yields $K_3$ in $G$) and $w$ may be adjacent to some $v_i$. To make $u_2, u_3, \ldots, u_{b(G-e)-1}$ as c.d.vs., they must be adjacent to $v_1$. If $e = uv = u_iv_i$ for some $i \in \{2, 3, \ldots, b(G-e)-1\}$, say $u = u_{b(G-e)-1}$ and $v = v_{b(G-e)-1}$, then $G$ will be isomorphic to the graph given in Figure 7(a) (where $x$ and $u$ are full vertices). Next if $e$ belongs to the class $\{u_1, v_1, w\}$, then $e = u_1w$ is not possible (this induces a $K_3$ in $G$). Thus the only possibilities here are $e = uv = u_1v_1$ or $e = wv_1$. For $e = u_1v_1$, we can easily observe that $G$ has to be isomorphic to the graph given in Figure 7(b) (where $x$ and $u$ are full vertices) and when $e = wv_1$, $G$ will be isomorphic to the graph

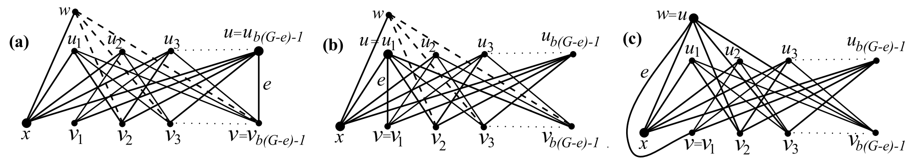

Figure 7. $n$ is even, $|V(T')| = n-1$ and $x$ is adjacent to $w$.

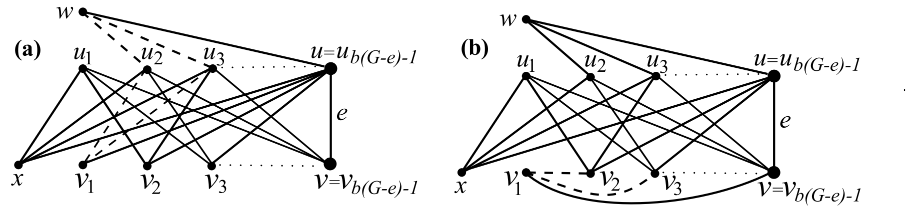

Figure 8. $n$ is even, $|V(T')| = n-1$ and $x$ is non-adjacent to $w$ and $v_1$.

given in Figure 7(c) (where $x$ and $w$ are full vertices).

Let us next consider the case when $x$ is non-adjacent to both $w$ and $v_1$. Here, for each $j \in \{2, 3, \ldots, b(G-e)-1\}$, $u_j$ is adjacent to $v_1$ or $w$ (or to both). Also neither $w$ nor $v_1$ can be adjacent to both $u_i$ and $v_j$, $i, j \in \{2, 3, \ldots, b(G-e)-1\}$, as this would yield a $K_3$ or a $C_5$, a contradiction to $b(G) = 2$. If $e = uv = u_iv_i$ for some $i \in \{2, \ldots, b(G-e)-1\}$, say $u = u_{b(G-e)-1}$ and $v = v_{b(G-e)-1}$, then by using the fact that $b(G) = 2$ we can come to the conclusion that (i) both $v_1$ and $w$ must be adjacent to $u$, (ii) $w$ ($v_1$) is adjacent to $u$, and $v_1$ ($w$) is adjacent to $v$. The possibility that both $w$ and $v_1$ are adjacent to $v$ will yield a $K_3$ and hence discarded. Thus $G$ will be isomorphic to one of the graphs given in Figure 8(a) and Figure 8(b) (where $u$ and $v$ are full vertices).

Since $b(G) = 2$, $G$ is a bipartite graph with bipartition, say $(X, Y)$. Next let us consider the case when $e$ belongs to the class $\{u_1, v_1, w\}$. There are two possibilities for $e$: (i) $e = u_1v_1$ (the same for $e = wu_1$) (ii) $e = wv_1$. Let us start with $e = u_1v_1$. If $x \in X$, then $\{u_1, u_2, \ldots, u_{b(G-e)-1}\} \subseteq Y$ and $\{v_1, v_2, \ldots, v_{b(G-e)-1}\} \subseteq X$. If $w \in X$, then there is no $u_i \in Y$ which is adjacent to all the vertices in $X$ for $i \in \{1, 2, \ldots, b(G-e)-1\}$, hence there is no full vertex in $Y$, and if $w \in Y$, then there is no full vertex in $X$, a contradiction to $b(G) = 2$.

Next let $e = wv_1$ and $x \in X$. Then $\{u_1, u_2, \ldots, u_{b(G-e)-1}\} \subseteq Y$ and $\{v_2, v_3, \ldots, v_{b(G-e)-1}\} \subseteq X$. If both $w$ and $v_1$ have neighbors in $\{u_1, u_2, \ldots, u_{b(G-e)-1}\}$, then it will yield a $K_3$ or $C_5$ in $G$, therefore one of $w$ or $v_1$ must be adjacent to $u_i$ for all $i \in \{2, 3, \ldots, b(G-e)-1\}$, say $v_1$, then $w \in Y$ and hence there is no full vertex in $X$, a contradiction to $b(G) = 2$. Therefore $e$ does not belong to $\{u_1, v_1, w\}$.

*Subcase* (c) $|V(T')| = n-2$. Here $|S'| = 2, |T'| = \frac{n}{2} - 1$ and therefore by using equation (4.1), we get $b(G) = 3$. Also each color class of $T'$ has exactly

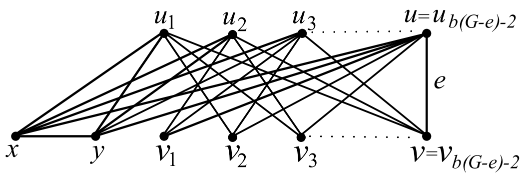


Figure 9. $n$ is even and $|V(T')| = n-2$.

two vertices. Let $S' = \{x, y\}$ and $T' = \{\{u_i, v_i\}\colon 1 \le i \le b(G-e)-2\}$. For $i \in \{1, 2, \ldots, b(G-e)-2\}$, let $u_i$ be a c.d.v. of the color class $\{u_i, v_i\}$ of $T'$. Clearly, each $u_i$ must be adjacent to both $x$ and $y$, for $i \in \{1, 2, \ldots, b(G-e)-2\}$. Here $e = uv = u_iv_i$ for some $i \in \{1, 2, \ldots, b(G-e)-2\}$. Without loss of generality, let $u = u_{b(G-e)-2}$ and $v = v_{b(G-e)-2}$. Also there can be no edges between any two $u_i$, $i \in \{1, 2, \ldots, b(G-e)-2\}$, as that would yield a $K_4$ in $G-e$, a contradiction to $b(G) = 3$. Hence for each $i \in \{1, 2, \ldots, b(G-e)-2\}$, $u_i$ is adjacent to every $v_j$, $j \in \{1, 2, \ldots, b(G-e)-2\} \backslash \{i\}$. Therefore $G$ contains the graph given in Figure 9 as a spanning subgraph.

We observe that there can be a few more edges between $x, y, v_1, \ldots, v_{b(G-e)-2}$. Also for $i \in \{1, 2, \ldots, b(G-e)-2\}$, no $v_i$ can be adjacent to both $x$ and $y$. Also the subgraph induced by $\{x, y, v_1, \ldots, v_{b(G-e)-2}\}$ is a bipartite graph (else, $b(G) \ge 4$, a contradiction). For $i \in \{1, 2, \ldots, b(G-e)-2\}$, let $A_i = \{N_G(v_i) \setminus N_G(x)\}$, $B_i = \{N_G(v_i) \setminus N_G(y)\}$. In any $b$-coloring of $G$, the vertices $x, y$ and $u$ must have different colors. Without loss of generality, let the colors of $x, y$ and $u$ be $1, 2$ and 3, respectively. Also we know that $u$ is adjacent to $v_i$, for all $i \in \{1, 2, \ldots, b(G-e)-2\}$ and hence none of the $u_j$, $j \in \{1, 2, \ldots, b(G-e)-3\}$, can be a c.d.v. of any new color class. We shall now formulate the condition on how the additional edges should be so that $b(G)$ does not exceed 3.

*Possibility* 1. $v$ has no neighbor in $\{x, y\}$. Here, suppose there exists a vertex $v_i \neq v$, $i \in \{1, 2, \ldots, b(G-e)-3\}$ such that $v_i$ satisfies one of the following conditions.

$(C1)$ $A_i \setminus N_G(v) \neq \emptyset$ and $B_i \neq \emptyset$ with $w \in A_i \setminus N_G(v)$ and $w' \in B_i$ such that $w \neq w'$ (this we write as distinct representatives).
$(C2)$ $A_i \neq \emptyset$ and $B_i \setminus N_G(v) \neq \emptyset$ with $w \in A_i$ and $w' \in B_i \setminus N_G(v)$ such that $w \neq w'$.

We shall first show that, if any $v_i \neq v$ satisfies $(C2)$, then either $v_i$ or a neighbor of $v_i$ satisfies $(C1)$ and vice versa. Let $v_i \neq v$ satisfy $(C2)$ and let $w \in A_i$ and $w' \in B_i \setminus N_G(v)$ where $w \neq w'$ and $w, w' \in \{x, y, v_1, v_2, \ldots, v_{b(G-e)-2}\}$. Now if $v_i$ satisfies $(C1)$, then we are done. If not, $v_i$ satisfies at least one of the following: (i) $A_i \setminus N_G(v) = B_i$ and $|B_i| = 1$ (ii) $A_i \setminus N_G(v) = \emptyset$ or $B_i = \emptyset$. Suppose $v_i$ satisfy (ii) $A_i \setminus N_G(v) = \emptyset$ or $B_i = \emptyset$. Since $v_i$ satisfies $(C2)$, $B_i \neq \emptyset$ and hence $A_i \setminus N_G(v) = \emptyset$. That is, every neighbor of $v_i$ is either adjacent to $x$

or to $v$. Since $w'$ is non adjacent to $v$, and $w$ is non adjacent to $x$, $w'$ has to be adjacent to $x$ and $w$ has to be adjacent to $v$. Thus $w$ cannot be $x$ or $y$ and hence $w = v_k$, for some $k \in \{1, 2, \ldots, b(G-e)-2\} \backslash \{i\}$. While considering $v_i$, it cannot be adjacent to both $x$ and $v$. This implies $v_i \in A_k \setminus N_G(v)$ and $v \in B_k$ and hence $w$, a neighbor of $v_i$, satisfies $(C1)$. If $v_i$ satisfies (i) $A_i - N_G(v) = B_i$ and $|B_i| = 1$, then also in a similar way we can show that $w$ satisfies $(C1)$. Next if any $v_i \neq v$ satisfies $(C1)$, then the fact that either $v_i$ or a neighbor of $v_i$ satisfies $(C2)$ can also be proved by similar arguments.

Now let us consider the case when there exists a vertex $v_i \neq v$ satisfying $(C1)$ with $w \in A_i \setminus N_G(v)$ and $w' \in B_i$, where $w, w' \in \{x, y, v_1, v_2, \ldots, v_{b(G-e)-2}\}$ are distinct. Let us show that in this case there exists a $b$-coloring using at least 4 colors. Let us start by giving color 1 to $v$ and $w$, 2 to $w'$, and 4 to $u_i$ and $v_i$. If $w$ or $w'$ (or both) belongs to $\{v_1, v_2, \ldots, v_{b(G-e)-2}\}$, then give the corresponding $u_j$ color 3. For $l \in \{1, 2, \ldots, b(G)-3\}$, if $v_l$ is uncolored and is adjacent to all used colors, then give a new color (the same) to both $u_l$ and $v_l$. If not, give 3 to $u_l$ and color $v_l$ with the color to which it is not adjacent. This procedure yields a $b$-coloring using at least 4 colors for $G$, a contradiction to $b(G) = 3$.

Thus for every $i \in \{1, 2, \ldots, b(G-e)-3\}$, $v_i \neq v$ satisfies both the following conditions.

(i) $(C1)$ is not satisfied,

(ii) $(C2)$ is not satisfied.

That is,

(i) $(D1)$ $A_i \setminus N_G(v) = B_i$ and $|B_i| = 1$ or $(D2)$ $A_i \setminus N_G(v) = \emptyset$ or $B_i = \emptyset$,

(ii) $(E1)$ $A_i = B_i \setminus N_G(v)$ and $|A_i| = 1$ or $(E2)$ $A_i = \emptyset$ or $B_i \setminus N_G(v) = \emptyset$.

Therefore each $v_i$ satisfies at least one of the following: (1) $(D1)$ and $(E1)$, (2) $(D1)$ and $(E2)$, (3) $(D2)$ and $(E1)$, (4) $(D2)$ and $(E2)$. One can easily observe that if $(D1)$ is satisfied, then $(E2)$ will not be satisfied. Similarly if $(E1)$ is satisfied, then $(D2)$ cannot be satisfied. When $v_i$ satisfies $(D1)$ and $(E1)$, $v_i$ has only one neighbor in $\{x, y, v_1, v_2, \ldots, v_{b(G-e)-2}\}$ and hence cannot form a c.d.v. of a new color class. Now let us consider the final possibility when $(D2)$ and $(E2)$ are satisfied. Here if $v_i$ is a vertex such that both $A_i \neq \emptyset$ and $B_i \neq \emptyset$ (with distinct representatives), say $w \in A_i$ and $w' \in B_i$ where $w, w' \in \{x, y, v_1, v_2, \ldots, v_{b(G-e)-2}\}$. Then by using $(D2)$ and $(E2)$, $w$ and $w'$ are adjacent to $v$. Clearly $w \neq x$ or $w \neq y$ and hence $w = v_k$, $k \in \{1, 2, \ldots, b(G-e)-3\}$. This $v_k$ satisfies $(C1)$, a contradiction. Thus for every $v_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$, $A_i = \emptyset$ or $B_i = \emptyset$ or ($A_i = B_i$ and $|B_i| = 1$) or (($D1$) and ($E1$)) are satisfied. But in none of these cases $v_i$ can be a c.d.v. of a new color class. Thus for every $v_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$, one of the following is possible: (i) $A_i = \emptyset$ or $B_i = \emptyset$ (ii) $v_i$ has only one neighbor in $\{x, y, v_1, v_2, \ldots, v_{b(G-e)-2}\}$.

If $v = v_{b(G-e)-2}$ is such that $A_{b(G-e)-2} = \emptyset$ or $B_{b(G-e)-2} = \emptyset$ or ($A_{b(G-e)-2} = B_{b(G-e)-2}$ and $|A_{b(G-e)-2}| = 1$ ), then $v_{b(G-e)-2}$ cannot form a c.d.v. of a new color class. Now suppose $v = v_{b(G-e)-2}$ is such that $A_{b(G-e)-2} \neq \emptyset$ and $B_{b(G-e)-2} \neq \emptyset$ (with distinct representatives), say $v_j \in A_{b(G-e)-2}$ and $v_k \in B_{b(G-e)-2}$, $j \neq k$ and $j, k \in \{1, 2, \ldots, b(G-e)-3\}$. Let us find those graphs $G$ with $b(G) \geq 4$ in this case and eliminate those possibilities. Here it is impossible to get a c.d.v. for a new color class in $v_i$, with $v$ receiving color 1 or 2 as none of the vertices $v_i$ satisfies $(C1)$ or $(C2)$ where $i \in \{1, 2, \ldots, b(G-e)-3\}$. Moreover, if there is a c.d.v. $v_i$ for a new color class, say 4, then the two neighbors with color 1 and 2 should also be adjacent to $v$ (since $v_i$ does not satisfy both $(C1)$ and $(C2)$) and hence by giving color 4 to $v$, it becomes a c.d.v. of the color class 4. Thus without loss of generality, let us start by coloring $v$ with 4, $v_j \in A_{b(G-e)-2}$ and $v_k \in B_{b(G-e)-2}$ with colors 1 and 2 respectively, and $u_j$, $u_k$ with 3. If one of $v_j$ or $v_k$ is a c.d.v., say $v_j$, then $v_j$ should be adjacent to $y$ or to some $v_{j'}$ which is not a neighbor of $y$ and $v_k$. But in this case $v_j$ satisfies $(C1)$, a contradiction. Thus $v_j$ and $v_k$ cannot be c.d.vs. of color classes 1 and 2, respectively. For extending this to a $b$-coloring using at least 4 colors, we need c.d.vs. for color classes 1 and 2. Since $v$ is given color 4, 4 cannot be given to any $u_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$. Thus for both color classes 1 and 2, we need c.d.vs. with neighbors colored 4 in $v_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$. If $v_p$ is a non-neighbor of $v$ and $x$ which is a c.d.v. of the color class 1, then $v_p$ must have neighbors $v_q \notin N_G(v)$ with color 4 and $y$ or $v_{p'}$ which is not a neighbor of $y$ and $v_k$ with color 2. For $u_p, u_q, u_{p'}$ give color 3. Here $B_p \neq \emptyset$. Suppose $A_p \setminus N_G(v) \neq \emptyset$ (with distinct representatives), then condition $(C1)$ is satisfied by $v_p$, a contradiction. Therefore $A_p \setminus N_G(v) = \emptyset$ or $A_p \setminus N_G(v) = B_p$ and $|B_p| = 1$. But $A_p \setminus N_G(v) = B_p$ and $|B_p| = 1$ means $v_q$ is adjacent to both $x$ and $y$, a contradiction. Thus $A_p \setminus N_G(v) = \emptyset$ and $v_q$ must be adjacent to $x$ and hence $x$ becomes a c.d.v. of color class 1. By a similar argument, we can show that if there exist a c.d.v. for color class 2, then $y$ will become a c.d.v. of color class 2. For $l \in \{1, 2, \ldots, b(G)-3\}$, if $v_l$ is uncolored and is adjacent to all used colors, then give a new color (the same) to both $u_l$ and $v_l$. If not, give 3 to $u_l$ and color $v_l$ with the color to which it is not adjacent. These are the graphs in this case which have $b(G) \geq 4$. That is, for $b(G)$ to be greater than or equal to 4, we need a neighbor for $x$ which is not adjacent to $v$ and a neighbor for $y$ which is not adjacent to $v$. But we know that $b(G) = 3$. Therefore $N_G(x) \setminus N_G(v) = \emptyset$ or $N_G(y) \setminus N_G(v) = \emptyset$ in this case.

*Possibility* 2. $v$ has neighbors in $\{x, y\}$. It is easy to observe that both $x$ and $y$ cannot be adjacent to $v$, as that would yield a $K_4$ in $G$, a contradiction. Hence $v$ can be adjacent only to one vertex in $\{x, y\}$. Without loss of generality, let it be $y$. Suppose there exists a vertex $v_i \neq v$ satisfying $(C1)$, then we can obtain a $b$-coloring of $G$ using at least 4 colors by a similar argument as in Possibility 1, which is a contradiction to the fact that $b(G) = 3$. Hence there cannot be a vertex $v_i \neq v$ such that it satisfies $(C1)$.

If $v = v_{b(G-e)-2}$ is such that $A_{b(G-e)-2} = \emptyset$, then $v_{b(G-e)-2}$ cannot form a c.d.v. of a new color class. Now suppose $v = v_{b(G-e)-2}$ is such that $A_{b(G-e)-2} \neq \emptyset$, say $v_j \in A_{b(G-e)-2}$, $j \in \{1, 2, \ldots, b(G-e)-3\}$. Here note that $y \in B_{b(G-e)-2}$ and hence this set is non-empty. Let us find those graphs $G$ with $b(G) \geq 4$ in this case and eliminate those possibilities. Here as seen in Possibility 1, it is impossible to get a c.d.v for a new color class in $v_i$, with $v$ receiving color 1 as none of the vertices $v_i$ satisfies $(C1)$ where $i \in \{1, 2, \ldots, b(G-e)-3\}$. Moreover, if there is a c.d.v. $v_i$ for a new color class, say 4, then the neighbor with color 1 should also be adjacent to $v$ (since $v_i$ does not satisfy $(C1)$) and hence by giving color 4 to $v$, it becomes a c.d.v. of the color class 4. Thus without loss of generality, let us start by coloring $v$ with 4, $v_j$ with colors 1, and $u_j$ with 3. Note that $v$, $u$ and $y$ are c.d.vs. of color classes 4, 3 and 2, respectively. If $v_j$ is a c.d.v., then $v_j$ should be adjacent to some $v_{j'}$ which is not a neighbor of $y$. But in this case $v_j$ satisfies $(C1)$, a contradiction. Thus $v_j$ cannot be a c.d.v. of color class 1. For extending this to a $b$-coloring using at least 4 colors, we need a c.d.v. for the color class 1. Since $v$ is given color 4, 4 cannot be given to any $u_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$. Thus for color class 1, we need a c.d.v. with neighbors colored 4 in $v_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$. If $v_p$ is a non-neighbor of $v$ and $x$ which is a c.d.v. of the color class 1, then $v_p$ must have a neighbor $v_q \notin N_G(v)$ with color 4 and a neighbor $y$ or $v_{p'}$ which is not a neighbor of $y$ with color 2. For $u_p, u_q, u_{p'}$ give color 3. Here $B_p \neq \emptyset$. Suppose $A_p \setminus N_G(v) \neq \emptyset$ (with distinct representatives), then condition $(C1)$ is satisfied by $v_p$, a contradiction. Therefore $A_p \setminus N_G(v) = \emptyset$ or $A_p \setminus N_G(v) = B_p$ and $|B_p| = 1$. But $A_p \setminus N_G(v) = B_p$ and $|B_p| = 1$ means $v_q$ is adjacent to both $x$ and $y$, a contradiction. Thus $A_p \setminus N_G(v) = \emptyset$ and hence $v_q$ must be adjacent to $x$ and hence $x$ becomes a c.d.v. of color class 1. For $l \in \{1, 2, \ldots, b(G)-3\}$, if $v_l$ is uncolored and is adjacent to all used colors, then give a new color (the same) to both $u_l$ and $v_l$. If not, give 3 to $u_l$ and color $v_l$ with the color to which it is not adjacent. These are graphs in this case which have $b(G) \geq 4$. That is, for $b(G)$ to be greater than or equal to 4, we need a neighbor for $x$ which is not adjacent to $v$. But we know that $b(G) = 3$. Therefore $N_G(x) \setminus N_G(v) = \emptyset$ in this case.

While considering $v_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$, we have already observed that $v_i$ does not satisfy $(C1)$. That is each $v_i$ satisfies at least one of the following.

$(D1)$ $A_i \setminus N_G(v) = B_i$ and $|B_i| = 1$ or

$(D2)$ $A_i \setminus N_G(v) = \emptyset$ or $B_i = \emptyset$.

If $v_i$ satisfies $(D1)$, then $v_i$ cannot form a c.d.v. of a new color class. Next let us assume that $v_i$ satisfies $(D2)$. Here if (i) $A_i = \emptyset$ or $B_i = \emptyset$ or (ii) $A_i = B_i$ and $|B_i| = 1$, then also $v_i$ cannot form a c.d.v. of a new color class. If not, $A_i \neq \emptyset$ and $B_i \neq \emptyset$ (with distinct representatives), say $w \in A_i$ and $w' \in B_i$ where $w, w' \in \{x, y, v_1, v_2, \ldots, v_{b(G-e)-2}\}$. Since $v_i$ satisfies $(D2)$, $w$ is adjacent to $v$, which in turn implies that $A_{b(G-e)-2} \neq \emptyset$. Hence $N_G(x) \setminus N_G(v) = \emptyset$ (from the above conclusion). Therefore for every $v_i$, $i \in \{1, 2, \ldots, b(G-e)-3\}$, one of the following is possible: (i) $A_i = \emptyset$ or $B_i = \emptyset$ (ii) $(D1)$ is satisfied (iii) $A_i \neq \emptyset$ and

$B_i \neq \emptyset$ (with distinct representatives) where $N_G(x) \setminus N_G(v) = \emptyset$ (iv) $A_i = B_i$ and $|B_i| = 1$.

## 5. Bounds for the $b$-Chromatic Number of any Subgraphs

In Corollary 2 even if the subgraph is not induced still the result works with a minor change.

**Corollary 8.** *For any graph $G$ other than the complete graph and for any subgraph $H$ which is not a clique of $G$ with $k = |E(G)| - |E(H)|$,*

$$2b(G) - (n+k) + 1 \leq b(H) \leq \left\lfloor \frac{n + b(G) - 1}{2} \right\rfloor.$$

***Proof.*** By using Theorem 1 and the fact that $\omega(G) \leq \omega(H) + k \leq b(H) + k$, we get the lower bound. The upper bound can be observed from the fact that $\omega(H) \leq \omega(G) \leq b(G)$. ■

As a consequence of Corollary 8, we can get bounds for the $b$-chromatic number of edge-deleted subgraphs.

**Corollary 9.** *For any graph $G$ other than the complete graph and for any $e \in E(G)$,*

$$b(G - e) \leq \left\lfloor \frac{n + b(G) - 1}{2} \right\rfloor.$$

Thus we have two upper bounds for $b(G-e)$: one given in Theorem 6 and the other given in Corollary 9.

For any $n$ and for $b(G) = 2$, both the upper bounds are the same. Also when $b(G) = 3$ and $n$ is even, both the upper bounds are the same. Thus for these cases the graphs attaining the upper bound given in Corollary 9, are the same as the graphs got in Section 4. For all the other values, the bound given in Corollary 9 is better than that given in Theorem 6.

Let us try to characterize the extremal graphs when $b(G) = 3$ and $n$ is odd. When $n = 3$ or $n = 5$, without much difficulty we can find the graphs attaining the bound. We now consider the graphs $G \neq K_n$ with $n \geq 7$, which attain the upper bound $b(G-e) = \left\lfloor \frac{n+b(G)-1}{2} \right\rfloor = \left\lfloor \frac{n+2}{2} \right\rfloor$, for some $e = uv \in E(G)$. Here if $c'$ is a $b$-chromatic coloring of $G - e$ and $S'$ denote the singleton classes and $T'$ denote the remaining classes of $c'$, then by similar observations as in Section 4, we get that $|S'| \leq b(G) - 1 = 2$ and $|T'| \geq b(G-e) - |S'| = \left\lfloor \frac{n+2}{2} \right\rfloor - |S'| \geq \frac{n+1}{2} - 2$. If $|S'| = 2$, then $|T'| = \frac{n+1}{2} - 2$ and $|V(T')| = n - 2$. Thus every class of $T'$ contains exactly two vertices except one which has three vertices. In this case the extremal graphs can be obtained in a similar way as done in Subcase (c) of Section 4 but with little more involvement. Also note that $|S'| = 0$ is not possible. The final case to be considered is $|S'| = 1$ and $|T'| = \frac{n-1}{2}$. Let $S' = \{x\}$ and $T' = \{\{u_i, v_i\} : 1 \leq i \leq b(G - e) - 1\}$. Without loss of generality let $u_i$, $i \in \{1, 2, \ldots, b(G - e) - 1\}$ be the c.d.vs. of the color classes in $T'$. Clearly each $u_i$ must be adjacent to $x$. Since $b(G) = 3$ and there is only one singleton color class, each $u_i$ will be adjacent to $u_j$ or $v_j$ (or to both) where $j \neq i$ and $i, j \in \{1, 2, \ldots, b(G - e) - 1\}$. Also there is no characterization available for graphs with $b(G) = 3$. Thus it turns out to be a difficult problem to obtain the extremal graphs by the techniques used in Section 4. Also for any graph $G$ with $b(G) \geq 4$ and in the case when $|S'| < b(G) - 1$ the difficulties arise in a similar way. Thus we conclude by posing this as an open problem.

**Open Problem**

Characterize graphs $G$ for which $b(G - e) = \left\lfloor \frac{n+b(G)-1}{2} \right\rfloor$ when $b(G) \geq 4$.

## References


[1] R. Balakrishnan and S. Francis Raj, *Bounds for the b-chromatic number of $G - v$*, Discrete Appl. Math. **161** (2013) 1173–1179.
doi:10.1016/j.dam.2011.08.022

[2] D. Barth, J. Cohen and T. Faik, *On the b-continuity property of graphs*, Discrete Appl. Math. **155** (2007) 1761–1768.
doi:10.1016/j.dam.2007.04.011

[3] M. Blidia, N.I. Eschouf and F. Maffray, *b-coloring of some bipartite graphs*, Australas. J. Combin. **53** (2012) 67–76.

[4] S. Corteel, M. Valencia-Pabon and J.-C. Vera, *On approximating the b-chromatic number*, Discrete Appl. Math. **146** (2005) 106–110.
doi:10.1016/j.dam.2004.09.006

[5] B. Effantin and H. Kheddouci, *The b-chromatic number of some power graphs*, Discrete Math. Theor. Comput. Sci. **6** (2003) 45–54.

[6] T. Faik, *About the b-continuity of graph*, Electron. Notes Discrete Math. **17** (2004) 151–156.
doi:10.1016/j.endm.2004.03.030

[7] T. Faik, La $b$-continuite des $b$-colorations: complexité, propriétés structurelles et algorithmes, Ph.D. Thesis (University of Paris XI Orsay, 2005).

[8] C.T. Hoàng and M. Kouider, *On the b-dominating coloring of graphs*, Discrete Appl. Math. **152** (2005) 176–186.
doi:10.1016/j.dam.2005.04.001

[9] R.W. Irving and D.F. Manlove, *The b-chromatic number of a graph*, Discrete Appl. Math. **91** (1999) 127–141.
doi:10.1016/S0166-218X(98)00146-2

[10] M. Jakovac and S. Klavžar, *The b-chromatic number of cubic graphs*, Graphs Combin. **26** (2010) 107–118.
doi:10.1007/s00373-010-0898-9

[11] M. Kouider and M. Mahéo, *Some bounds for the b-chromatic number of a graph*, Discrete Math. **256** (2002) 267–277.
doi:10.1016/S0012-365X(01)00469-1

[12] M. Kouider and M. Zaker, *Bounds for the b-chromatic number of some families of graphs*, Discrete Math. **306** (2006) 617–623.
doi:10.1016/j.disc.2006.01.012

[13] J. Kratochvíl, Zs. Tuza and M. Voigt,*On the b-chromatic number of graphs*, Lecture Notes in Comput. Sci. **2573** (2002) 310–320.
doi:10.1007/3-540-36379-3_27

[14] F. Maffray and A. Silva, *b-colouring outerplanar graphs with large girth*, Discrete Math. **312** (2012) 1796–1803.
doi:10.1016/j.disc.2012.01.035

Received 28 April 2015
Revised 11 November 2015
Accepted 8 January 2016