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SIGNED ROMAN EDGE k-DOMINATION IN GRAPHS

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Abstract

Let $k \geq 1$ be an integer, and $G = (V, E)$ be a finite and simple graph. The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and all edges having a common end-vertex with e. A signed Roman edge k-dominating function (SREkDF) on a graph G is a function $f : E \to \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge e of G, $\sum_{x \in N_G[e]} f(x) \geq k$ and (ii) every edge e for which $f(e) = -1$ is adjacent to at least one edge e' for which $f(e') = 2$. The minimum of the values $\sum_{e \in E} f(e)$, taken over all signed Roman edge k-dominating functions f of G is called the signed Roman edge k-domination number of G, and is denoted by $\gamma'_{sRk}(G)$. In this paper we initiate the study of the signed Roman edge k-domination in graphs and present some (sharp) bounds for this parameter.

Keywords: signed Roman edge k-dominating function, signed Roman edge k-domination number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* $N(e) = N_G(e)$ of an edge $e \in E$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N[e] = N_G[e] = N_G(e) \cup \{e\}$. The degree of an edge $e \in E$ is $d_G(e) = d(e) = |N(e)|$. The *minimum* and *maximum edge degree* of a graph G are denoted by $\delta_e = \delta_e(G)$ and $\Delta_e = \Delta_e(G)$, respectively. If v is a vertex, then denote by $E(v)$ the set of edges incident with the vertex v. We write K_n for a *complete graph*, C_n for a *cycle*, P_n for a *path* of order n and $K_{1,n}$ for a *star* of order $n+1$. A *subdivided star*, denoted $K^*_{1,n}$, is a star $K_{1,n}$ whose edges are subdivided once, that is each edge is replaced by a path of length 2 by adding a vertex of degree 2. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ee' \in E(L(G))$ when $e = uv$ and $e' = vw$ in G. It is easy to see that $L(K_{1,n}) = K_n$, $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

A function $f : E \to \{-1, 1\}$ is called a *signed edge k-dominating function* (SEkDF) of G if $\sum_{x\in N[e]} f(x) \geq k$ for each edge $e \in E$. The *weight* of f, denoted $\omega(f)$, is defined to be $\omega(f) = \sum_{e\in E} f(e)$. The *signed edge k-domination number* $\gamma'_{sk}(G)$ is defined as $\gamma'_{sk}(G) = \min\{\omega(f) \mid f \text{ is an SE}k\text{DF of } G\}$. The signed edge k-domination number was first defined in [3].

A *signed Roman k-dominating function* (SRkDF) on a graph G is a function $f : V \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N[v]} f(x) \geq k$ for each vertex $v \in V$, and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an SRkDF f is $\omega(f) = \sum_{v\in V} f(v)$. The *signed Roman k-domination number* of G, denoted γ^k_{sR}, is the minimum weight of an SRkDF in G. The signed Roman k-domination number was introduced by Henning and Volkman in [5] and has been studied in [6]. The special case $k = 1$ was introduced and investigated in [1].

A *signed Roman edge k-dominating function* (SREkDF) on a graph G is a function $f : E \to \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge e of G, $\sum_{x\in N[e]} f(x) \geq k$ and (ii) every edge e for which $f(e) = -1$ is adjacent to at least one edge e' for which $f(e') = 2$. The *weight* of an SREkDF is the sum of its function values over all edges. The *signed Roman edge k-domination number* of G, denoted $\gamma'_{sRk}(G)$, is the minimum weight of an SREkDF in G. For an edge e, we denote $f[e] = f(N[e]) = \sum_{x\in N[e]} f(x)$ for notational convenience. The special case $k = 1$ was introduced by Ahangar *et al.* [2]. If $G_1, G_2, \ldots, G_s$ are the components of G, then

$$\gamma'_{sRk}(G) = \sum_{i=1}^{s} \gamma'_{sRk}(G_i). \tag{1}$$

Since assigning a weight 1 to every edge of G produces an SREkDF, we have

$$\gamma'_{sRk}(G) \le |E(G)|. \tag{2}$$

The signed Roman edge k-domination number exists if $|N_G(e)| \ge \frac{k}{2} - 1$ for every edge $e \in E$. However, for investigations of the signed Roman edge k-domination number it is reasonable to claim that for every edge $e \in E$, $|N_G(e)| \ge k-1$. Thus we assume throughout this paper that $\delta_e(G) \ge k-1$.

In this note we initiate the study of the signed Roman edge k-domination in graphs and present some (sharp) bounds for this parameter. In addition, we determine the signed Roman edge k-domination number of some classes of graphs.

The proof of the following results can be found in [5].

Proposition 1. *If $k = 1$, then $\gamma^1_{sR}(K_3) = 2$ and $\gamma^1_{sR}(K_n) = 1$ for $n \ne 3$. If $n \ge k \ge 2$, then $\gamma^k_{sR}(K_n) = k$.*

The case $k = 1$ in Proposition 1 was proved in [1]. A set $S \subseteq V$ is a 2-*packing set* of G if $N[u] \cap N[v] = \emptyset$ holds for any two distinct vertices $u, v \in S$. The 2-*packing number* of G, denoted $\rho(G)$, is defined as follows:

$$\rho(G) = \max\{|S| \mid S \text{ is a 2-packing set of } G\}.$$

Proposition 2. *If G is a graph of order n with $\delta \ge k-1$, then*

$$\gamma^k_{sR}(G) \ge (\delta + k + 1)\rho(G) - n.$$

Proposition 3. $\gamma^2_{sR}(P_n) = \begin{cases} n & \text{if } 1 \le n \le 7, \\ \lceil \frac{2n+5}{3} \rceil & \text{if } n \ge 8. \end{cases}$

Proposition 4. *For $n \ge 3$, we have $\gamma^2_{sR}(C_n) = \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$.*

The proof of the following result is straightforward and therefore omitted.

Observation 5. *For any nonempty graph G of order $n \ge 2$ and any integer $k \ge 1$,*

$$\gamma'_{sRk}(G) = \gamma^k_{sR}(L(G)).$$

Observation 6. *Let G be a graph and f be a $\gamma'_{sR2}(G)$-function. If $e = uv$ is a pendant edge in G with $d(v) = 2$ and $w \in N(v)\setminus\{u\}$, then* $\min\{f(uv), f(vw)\} \ge 1$.

Observation 5 and Propositions 1, 2, 3 and 4 lead to

Corollary 7. *If* $k=1$*, then* $\gamma'_{sR1}(K_{1,3})=2$ *and* $\gamma'_{sR1}(K_{1,n})=1$ *for* $n\neq 3$*. If* $n\geq k\geq 2$*, then* $\gamma'_{sRk}(K_{1,n})=k$*.*

Corollary 8. *Let* G *be a graph of size* m*. Then*

$$\gamma'_{sRk}(G)\geq(2\delta+k-1)\rho(L(G))-m.$$

Corollary 9. $\gamma'_{sR2}(P_n)=\begin{cases} n-1 & \text{if } 2\leq n\leq 8,\\ \lceil\frac{2n}{3}\rceil+1 & \text{if } n\geq 9.\end{cases}$

Corollary 10. *For* $n\geq 3$*, we have* $\gamma'_{sR2}(C_n)=\lceil\frac{2n}{3}\rceil+\lceil\frac{n}{3}\rceil-\lfloor\frac{n}{3}\rfloor$*.*

Next we show that for every two positive integers k and t, there exists a connected graph G whose signed Roman edge k-domination number is at most $-t$.

Proposition 11. *For every positive integers* k *and* t*, there exists a connected graph* G *such that* $\gamma'_{sRk}(G)\leq -t$*.*

Proof. Let $n\geq\max\{k+5,t/3\}$, and let G be the graph obtained from the complete graph K_n by adding $n+2$ pendant edges at each vertex of K_n. Define $f:E(G)\to\{-1,1,2\}$ by $f(e)=2$ if $e\in E(K_n)$ and $f(e)=-1$ otherwise. Obviously, f is an SREkDF on G of weight $-3n$. This completes the proof. ■

We close this section by determining the signed Roman edge k-domination number of two classes of graphs.

Example 12. For $n\geq 2$, $\gamma'_{sR2}(K_{2,n})=\begin{cases} 4 & \text{if } n=2,\\ 5 & \text{if } n=3,4,\\ 6 & \text{otherwise.}\end{cases}$

Proof. Let $X=\{u_1,u_2\}$ and $Y=\{v_1,v_2,\ldots,v_n\}$ be the partite sets of $K_{2,n}$ and let f be a $\gamma'_{sR2}(K_{2,n})$-function such that $r=\min\{\sum_{i=1}^n f(u_1v_i),\sum_{i=1}^n f(u_2v_i)\}$ is as small as possible. Assume that $r=\sum_{i=1}^n f(u_1v_i)$. The result is immediate for $n=2$ by Corollary 10. Assume that $n\geq 3$. Since $f[u_1v_1]=f(u_2v_1)+\sum_{i=1}^n f(u_1v_i)\geq 2$, we have $\sum_{i=1}^n f(u_1v_i)\geq 0$. Consider three cases.

Case 1. $n\geq 5$. Define $g:E(K_{2,n})\to\{-1,1,2\}$ by $g(u_1v_1)=g(u_2v_2)=2$, $g(u_1v_2)=g(u_2v_1)=1$ and $g(u_1v_i)=(-1)^i$, $g(u_2v_i)=(-1)^{i+1}$ for $3\leq i\leq n$. Obviously, g is an SRE2DF of $K_{2,n}$ of weight 6 and so $\gamma'_{sR2}(K_{2,n})\leq 6$. Now, we show that $\gamma'_{sR2}(K_{2,n})=6$. If $r\geq 3$, then we obtain $\gamma'_{sR2}(K_{2,n})=r+\sum_{i=1}^n f(u_2v_i)\geq 6$ implying that $\gamma'_{sR2}(K_{2,n})=6$. Assume that $r\leq 2$. If $r=0$, then we deduce from $f[u_1v_i]=f(u_2v_i)+\sum_{i=1}^n f(u_1v_i)\geq 2$ that $f(u_2v_i)\geq 2$ for each i and hence $\gamma'_{sR2}(K_{2,n})=r+\sum_{i=1}^n f(u_2v_i)=2n>6$, a contradiction.

Thus $r = 1$ or $r = 2$. Then it follows from $f[u_1v_i] = f(u_2v_i) + \sum_{i=1}^{n} f(u_1v_i) \geq 2$ that $f(u_2v_i) \geq 1$ for each i. Hence, $\gamma'_{sR2}(K_{2,n}) = \sum_{i=1}^{n} f(u_1v_i) + \sum_{i=1}^{n} f(u_2v_i) \geq 1 + n \geq 6$ that implies $\gamma'_{sR2}(K_{2,n}) = 6$.

Case 2. $n = 3$. Define $g : E(K_{2,n}) \to \{-1, 1, 2\}$ by $g(u_1v_2) = 2, g(u_1v_3) = -1, g(u_1v_1) = 1$ and $g(u_2v_i) = 1$ for $1 \leq i \leq 3$. Obviously, g is an SRE2DF of $K_{2,3}$ of weight 5 and hence $\gamma'_{sR2}(K_{2,3}) \leq 5$. Now we show that $\gamma'_{sR2}(K_{2,3}) = 5$. Since $\gamma'_{sR2}(K_{2,3}) = r + \sum_{i=1}^{3} f(u_2v_i) \leq 5$, we have $r \leq 2$. If $r = 2$, then it follows from $f[u_1v_i] = f(u_2v_i) + r \geq 2$ that $f(u_2v_i) \geq 1$ for each $i = 1, 2, 3$. Hence, $\gamma'_{sR2}(K_{2,3}) = r + \sum_{i=1}^{3} f(u_2v_i) \geq 5$ that implies $\gamma'_{sR2}(K_{2,3}) = 5$. If $r = 0$, then as above we must have $f(u_2v_i) = 2$ for each i. But then $\gamma'_{sR2}(K_{2,3}) = r + \sum_{i=1}^{3} f(u_2v_i) = 6$, a contradiction. Let $r = 1$. We may assume without loss of generality, that $f(u_1v_1) = -1$ and $f(u_1v_2) = f(u_1v_3) = 1$. It follows from $f[u_1v_i] = f(u_2v_i) + \sum_{i=1}^{3} f(u_1v_i) = f(u_2v_i) + 1 \geq 2$ that $f(u_2v_i) \geq 1$ for each i. Since u_1v_1 must be adjacent to an edge with label 2, we have $\sum_{i=1}^{3} f(u_2v_i) \geq 4$ implying that $\gamma'_{sR2}(K_{2,3}) = 5$.

Case 3. $n = 4$. Define $g : E(K_{2,4}) \to \{-1, 1, 2\}$ by $g(u_1v_1) = 2$, $g(u_1v_2) = g(u_1v_3) = -1$ and $g(u_1v_4) = g(u_2v_i) = 1$ for $1 \leq i \leq 4$. Obviously, g is an SRE2DF of $K_{2,4}$ of weight 5 and hence $\gamma'_{sR2}(K_{2,4}) \leq 5$. Using an argument similar to that described in Case 2, we obtain $\gamma'_{sR2}(K_{2,4}) = 5$ and the proof is complete. ■

A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf. For $r, s \geq 1$, a *double star* $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves.

Example 13. For positive integers $r \geq s \geq k - 1 \geq 1$,

$$\gamma'_{sRk}(S(r, s)) = \begin{cases} 3 & \text{if } s = 1, \\ 2k - 2 & \text{if } s \geq 2. \end{cases}$$

Proof. Let u and v be the central vertices of $S(r, s)$ and let $N(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_r\}$ and $N(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_s\}$. Suppose that f is a $\gamma'_{sRk}(S(r, s))$-function. Consider two cases.

Case 1. $s = 1$. By assumption, we have $k = 2$. We deduce from $f[vv_1] = f(vv_1) + f(uv) \geq 2$ that $f(vv_1) \geq 1$. Hence,

$$\gamma'_{sRk}(S(r, s)) = f(vv_1) + f[uu_1] = 1 + f[uu_1] \geq 3.$$

If $r = 1$, then define $f : E(S(r, s)) \to \{-1, 1, 2\}$ by $f(x) = 1$ for each $x \in E(S(r, s))$. If r is even, then define $f : E(S(r, s)) \to \{-1, 1, 2\}$ by $f(vv_1) = 1$, $f(uv) = 2$ and $f(uu_i) = (-1)^i$ for $1 \leq i \leq r$, and if $r \geq 3$ is odd, then define $f : E(S(r, s)) \to \{-1, 1, 2\}$ by $f(vv_1) = 1, f(uv) = f(uu_1) = 2, f(uu_2) = f(uu_3) =$

-1 and $f(uu_i) = (-1)^i$ for $i \geq 4$. Clearly, f is an SREkDF of $S(r,s)$ of weight 3 and so $\gamma'_{sRk}(S(r,s)) = 3$.

Case 2. $s \geq 2$. We have $\gamma'_{sRk}(S(r,s)) = f[uu_1]+f[vv_1]-f(uv) \geq 2k-f(uv) \geq 2k-2$. To prove $\gamma'_{sRk}(S(r,s)) \leq 2k-2$, we distinguish the following subcases.

Subcase 2.1. $r-k+2$ and $s-k+2$ are even. Define $f : E(S(r,s)) \to \{-1, 1, 2\}$ by $f(uv) = 2$, $f(uu_i) = f(vv_i) = 1$ for $1 \leq i \leq k-2$, $f(uu_i) = (-1)^i$ for each $k-1 \leq i \leq r$ and $f(vv_j) = (-1)^j$ for each $k-1 \leq j \leq s$. Obviously, f is an SREkDF of $S(r,s)$ of weight $2k-2$ and so $\gamma'_{sRk}(S(r,s)) = 2k-2$.

Subcase 2.2. $r-k+2$ and $s-k+2$ are odd. Define $f : E(S(r,s)) \to \{-1,1,2\}$ by $f(uv) = f(uu_1) = f(vv_1) = 2, f(uu_2) = f(vv_2) = -1$, $f(uu_i) = f(vv_i) = 1$ for $3 \leq i \leq k-1$, $f(uu_i) = (-1)^i$ for each $i \geq k$ and $f(vv_j) = (-1)^j$ for each $j \geq k$. Clearly, f is an SREkDF of $S(r,s)$ of weight $2k-2$ and so $\gamma'_{sRk}(S(r,s)) = 2k-2$.

Subcase 2.3. $r-k+2$ and $s-k+2$ have opposite parity. Assume, without loss of generality, that $r-k+2$ is even and $s-k+2$ is odd. Define $f : E(S(r,s)) \to \{-1,1,2\}$ by $f(uv) = f(vv_1) = 2, f(vv_2) = -1$, $f(vv_i) = 1$ for $3 \leq i \leq k-1$, $f(vv_i) = (-1)^i$ for each $k \leq i \leq s$ and $f(uu_i) = 1$ for $1 \leq i \leq k-2$, $f(uu_i) = (-1)^i$ for each $i \geq k-1$. Clearly, f is an SREkDF of $S(r,s)$ of weight $2k-2$ and so $\gamma'_{sRk}(S(r,s)) = 2k-2$. This completes the proof. ■

2. Trees

In this section we first present a lower bound on the signed Roman edge k-domination number of trees and then we characterize all extremal trees.

Theorem 14. *Let $k \geq 2$ be an integer and T be a tree of order $n \geq k$. Then $\gamma'_{sRk}(T) \geq k$. Moreover, this bound is sharp for stars.*

Proof. We proceed by induction on n. The base step handles trees with few vertices or diameter 2 and 3. If $\mathrm{diam}(T) \leq 3$, then by Corollary 7 and Example 13, we have $\gamma'_{sRk}(T) \geq k$. Assume that T is an arbitrary tree of order n and that the statements holds for all trees of order less than n. We may assume, that $\mathrm{diam}(T) \geq 4$. Let f be a $\gamma'_{sRk}(T)$-function.

If T has a non-pendant edge $e = u_1u_2$ with $f(u_1u_2) = -1$, then let $T-u_1u_2 = T_1 \cup T_2$ where T_i is the component of $T-u_1u_2$ containing u_i for $i = 1,2$. It is easy to verify that the function f, restricted to T_i is an SREkDF of T_i for $i = 1,2$. It follows from the induction hypothesis that

$$\gamma'_{sRk}(T) = f(E(T_1)) + f(E(T_2)) - 1 \geq \gamma'_{sRk}(T_1) + \gamma'_{sRk}(T_2) - 1 \geq 2k-1 > k.$$

Henceforth, we may assume that every edge with label -1 is a pendant edge.

Let $P = u_1u_2\cdots u_k$ be a diametral path in T such that $d_T(u_2)$ is as large as possible. Root T at u_k. Since $f[u_1u_2] \geq k$, we have $d_T(u_2) \geq \lceil \frac{k}{2} \rceil$. By assumption $f(u_2u_3) \geq 1$. Let T_1 and T_2 be the components of $T - u_2u_3$ containing u_2 and u_3, respectively. Assume that T_1' is the tree obtained from T_1 by adding a new pendant edge u_2w and define $f_1 : E(T_1') \to \{-1, 1, 2\}$ by $f_1(u_2w) = f(u_2u_3)$ and $f_1(x) = f(x)$ otherwise. Clearly, f_1 is an SREkDF of T_1' and by the induction hypothesis we have $\omega(f_1) \geq k$. Consider two cases.

Case 1. $k = 2$. Let T_2' be the tree obtained from T_2 by adding a new pendant edge u_3w_1 and define $f_2 : E(T_2') \to \{-1, 1, 2\}$ by $f_2(u_3w_1) = f(u_2u_3)$ and $f_2(x) = f(x)$ otherwise. Clearly, f_2 is an SRE2DF of T_1' and by the induction hypothesis we have $\omega(f_2) \geq 2$. Since $\omega(f) = \omega(f_1) + \omega(f_2) - f(u_2u_3)$, we have

$$\gamma'_{sR2}(T) = \omega(f_1) + \omega(f_2) - f(u_2u_3) \geq 4 - f(u_2u_3) \geq 2.$$

Case 2. $k \geq 3$. Let T_2' be the tree obtained from T_2 by adding $\lceil \frac{k-2}{2} \rceil$ new pendant edges $u_3w_1, \ldots, u_3w_{\lceil \frac{k-2}{2} \rceil}$. Clearly, $|V(T_2')| < n$. First let k be odd. Define $f_2 : E(T_2') \to \{-1, 1, 2\}$ by $f_2(u_3w_i) = 2$ for each i and $f_2(x) = f(x)$ otherwise. It is easy to verify that f_2 is an SREkDF of T_2' and by the induction hypothesis we have $\omega(f_2) \geq k$. Now we have

$$\gamma'_{sRk}(T) = \omega(f) = \omega(f_1) + \omega(f_2) - (k - 2) \geq k + (\omega(f_2) - k) + 2 > k.$$

Now let k be even. Define $f_2 : E(T_2') \to \{-1, 1, 2\}$ by $f_2(u_3u_4) = f_2(u_3w_i) = 2$ for each i and $f_2(x) = f(x)$ otherwise. It is not hard to see that f_2 is an SREkDF of T_2' and by the induction hypothesis we have $\omega(f_2) \geq k$. Then

$$\begin{aligned}\gamma'_{sR2}(T) &= \omega(f) = \omega(f_1) + \omega(f_2) - (k - 2) - (2 - f(u_3u_4)) \\ &\geq k + (\omega(f_2) - k) + f(u_3u_4) > k.\end{aligned}$$

■

Using Corollary 7, Example 13 and a closer look at the proof of Theorem 14, we obtain the next result.

Corollary 15. *If $k \geq 3$ and T is a tree of order $n \geq k$, then $\gamma'_{sR2}(T) = k$ if and only if T is a star.*

In what follows, we provide a constructive characterization of all trees T for which $\gamma'_{sR2}(T) = 2$. To do this, we describe a procedure to build a family $\mathcal{F}$ that attains the bound in Theorem 14 when $k = 2$. First we define the following operations. Let $\mathcal{F}$ be the family of trees that:

1. contains P_2, and
2. is closed under the operations $\mathfrak{T}_1, \mathfrak{T}_2$ and $\mathfrak{T}_3$, which extend the tree T by attaching a tree to the vertex $y \in V(T)$, called the *attacher*.

Operation $\mathfrak{T}_1$. If $T \in \mathcal{F}$, uv is a pendant edge with $d(u) = 1$, and there is a $\gamma'_{sR2}(T)$-function with $f(uv) = 2$ and either no -1-edge at v or a 2-edge at v other than uv, then $\mathfrak{T}_1$ adds a pendant edge vv'.

Operation $\mathfrak{T}_2$. If $T \in \mathcal{F}$, uv is a pendant edge with $d(u) = 1$, and there is a $\gamma'_{sR2}(T)$-function with $f(uv) = 1$, then $\mathfrak{T}_2$ adds a pendant edge vw_1.

Operation $\mathfrak{T}_3$. If $T \in \mathcal{F}$, $uv \in E(T)$, and there is a $\gamma'_{sR2}(T)$-function with $f(uv) = 2$, then $\mathfrak{T}_3$ adds two pendant edges vw_1, vw_2.

Lemma 16. *If $T \in \mathcal{F}$, then $\gamma'_{sR2}(T) = 2$.*

Proof. Let $T \in \mathcal{F}$ be obtained from a path P_2 by successive operations $\mathcal{T}^1, \mathcal{T}^2, \ldots, \mathcal{T}^m$, where $\mathcal{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$ if $m \geq 1$ and $T = P_2$ if $m = 0$. The proof is by induction on m. If $m = 0$, then clearly the statement is true. Let $m \geq 1$ and assume that the statement holds for all trees which are obtained from P_2 by applying at most $m - 1$ operations. Let T_{m-1} be the tree obtained from P_2 by the first $m - 1$ operations $\mathcal{T}^1, \mathcal{T}^2, \ldots, \mathcal{T}^{m-1}$. We consider the following cases.

Case 1. $\mathcal{T}^m = \mathfrak{T}_1$. Assume that $uv \in T_{m-1}$ is a pendant edge with $d(u) = 1$, f a $\gamma'_{sR2}(T)$-function with $f(uv) = 2$ such that either no -1-edge at v or a 2-edge at v other than uv, and $\mathcal{T}^m$ adds a pendant edge vv'. Define $g : E(T) \to \{-1, 1, 2\}$ by $g(uv) = g(vv') = 1$ and $g(x) = f(x)$ otherwise. Obviously, g is an SRE2DF of $T = T_m$ of weight 2 and so $\gamma'_{sR2}(T) = 2$ by Theorem 14.

Case 2. $\mathcal{T}^m = \mathfrak{T}_2$. Let $uv \in T_{m-1}$ be a pendant edge with $d(u) = 1$, f a $\gamma'_{sR2}(T)$-function with $f(uv) = 1$, and $\mathcal{T}^m$ adds a pendant edge vw_1. Then the function $g : E(T) \to \{-1, 1, 2\}$ defined by $g(uv) = 2, g(vw_1) = -1$ and $g(x) = f(x)$ otherwise, is an SRE2DF of $T = T_m$ of weight 2 that implies $\gamma'_{sR2}(T) = 2$ by Theorem 14.

Case 3. $\mathcal{T}^m = \mathfrak{T}_3$. Let $uv \in T_{m-1}$, f be a $\gamma'_{sR2}(T)$-function with $f(uv) = 2$, and $\mathcal{T}^m$ adds two pendant edges vw_1, vw_2. Define $g : E(T) \to \{-1, 1, 2\}$ by $g(vw_1) = 1$, $g(vw_2) = -1$ and $g(x) = f(x)$ otherwise. Obviously, g is an SRE2DF of $T = T_m$ of weight 2 implying that $\gamma'_{sR2}(T) = 2$. This completes the proof. ∎

Theorem 17. *Let T be a tree of order $n \geq 2$. Then $\gamma'_{sR2}(T) = 2$ if and only if $T \in \mathcal{F}$.*

Proof. By Lemma 16, we only need to prove that every tree T with $\gamma'_{sR2}(T) = 2$ is in $\mathcal{F}$. We prove this by induction on n. If $n = 2$, then the only tree T of order 2 and $\gamma'_{sR2}(T) = 2$ is $P_2 \in \mathcal{F}$. If $\operatorname{diam}(T) = 2$, then T is a star and obviously T can be obtained from P_2 by applying Operations $\mathfrak{T}_1$ and $\mathfrak{T}_2$. Let $n \geq 4$ and assume that the statement holds for every tree of order less than n with $\gamma'_{sR2}(T) = 2$. Let T be a tree of order n and $\gamma'_{sR2}(T) = 2$. We may assume that $\operatorname{diam}(T) \geq 3$.

Suppose f is a $\gamma'_{sR2}(T)$-function. Then $f(v) = \sum_{e\in E(v)} f(e) \geq 2$ for every support vertex v.

Claim 1. *T has no non-pendant edge $e = u_1u_2$ with $f(u_1u_2) = -1$.*

Proof. Assume, to the contrary, that T has a non-pendant edge $e = u_1u_2$ such that $f(u_1u_2) = -1$. Assume $T-e = T_{u_1} \cup T_{u_2}$, where T_{u_i} is the component of $T-e$ containing u_i, for $i = 1, 2$. Obviously, $\gamma'_{sR2}(T) = f(E(T_{u_1})) - 1 + f(E(T_{u_2}))$ and the function f, restricted to T_{u_i} is an SRE2DF and hence $\gamma'_{sR2}(T_{u_i}) \leq f(E(T))$ for $i = 1, 2$. By Theorem 14, we get

$$\gamma'_{sR2}(T) \geq \gamma'_{sR2}(T_{u_1}) + \gamma'_{sR2}(T_{u_2}) - 1 \geq 3,$$

a contradiction. □

Claim 2. *T has no non-pendant edge with label* 1.

Proof. Assume, to the contrary, that T has a non-pendant edge $e = u_1u_2$ such that $f(u_1u_2) = 1$. Let T_{u_1} and T_{u_2} be the components of $T - e$ containing u_1 and u_2, respectively, and let T'_{u_i} be the tree obtained from T_{u_i} by adding a new pendant edge $u_iu'_i$ for $i = 1, 2$. Define $f_i : E(T'_i) \to \{-1, 1, 2\}$ by $f_i(u_iu'_i) = 1$ and $f_i(e) = f(e)$ if $e \in E(T_i)$, for $i = 1, 2$. Clearly, f_i is an SRE2DF of T'_i for each i, and $\omega(f) = \omega(f_1) + \omega(f_2) - 1$. Similar to Case 2, we can get the contradiction $\gamma'_{sR2}(T) = \omega(f_1) + \omega(f_2) - 1 \geq 3$. □

Thus, all -1-edges and 1-edges are pendant edges and hence all non-pendant edges are 2-edges.

Let $v_1v_2 \cdots v_D$ be a diametral path in T and root T at v_D. Obviously, $d(v_1) = d(v_D) = 1$.

Claim 3. *$d(v_2) \geq 3$.*

Proof. Assume, to the contrary, that $d(v_2) = 2$. By Observation 6, we have $f(v_1v_2) \geq 1$. If there is a pendant -1-edge at v_3, then let $T' = T - v_1$. It is easy to see that the function $h = f|_{E(T')}$ is an SRE2DF on $T' = T - v_1$ of weight less than $\omega(f)$, and it follows from Theorem 14 that $\gamma'_{sR2}(T) = \omega(f) > \omega(f|_{E(T')}) \geq \gamma'_{sR2}(T') \geq 2$. Assume that there is no pendant -1-edge at v_3. Let $T' = T - v_1$. Since $f(v_1v_2) \geq 1$, we have $\omega(f) \geq \omega(f|_{E(T')}) + 1$ and the function f restricted to T' is an SRE2DF of T'. This implies $\gamma'_{sR2}(T) > 2$ which is a contradiction. □

Now we consider three cases.

Case 1. T has two pendant edges v_2u_1 and v_2u_2 with $f(v_2u_1) = 1$ and $f(v_2u_2) = -1$. Assume $T' = T - \{u_1, u_2\}$. Clearly, the function f restricted to T' is an SRE2DF on T'. So $\gamma'_{sR2}(T') = 2$ and by the induction hypothesis $T' \in \mathcal{F}$. Obviously T can be obtained from T' by operation $\mathfrak{T}_3$. Thus $T \in \mathcal{F}$.

Case 2. T has two pendant edges v_2u_1 and v_2u_2 with $f(v_2u_1) = 2$ and $f(v_2u_2) = -1$. Since T is not a star, we deduce that there is an edge v_2v_3 such that $f(v_2v_3) = 2$ and $v_3 \neq u_1$. Assume that $T' = T - \{u_1\}$ and define $g : E(T') \to \{-1, 1, 2\}$ by $f(v_2u_2) = 1$ and $g(e) = f(e)$ for $e \in E(T') \setminus \{v_2u_2\}$. Obviously, g is an SRE2DF on T' of weight 2 and by the induction hypothesis we have $T' \in \mathcal{F}$. Clearly, T can be obtained from T' by operation $\mathfrak{T}_2$. This implies $T \in \mathcal{F}$.

Case 3. T has two pendant edges v_2u_1 and v_2u_2 with $f(v_2u_1) = f(v_2u_2) = 1$. Assume $T' = T - \{u_1\}$ and define $g : E(T') \to \{-1, 1, 2\}$ by $g(v_2u_2) = 2$ and $g(e) = f(e)$ for $e \in E(T') \setminus \{v_2u_2\}$. Obviously, g is an SRE2DF on T' of weight 2 and by the induction hypothesis we have $T' \in \mathcal{F}$. Then T can be obtained from T' by operation $\mathfrak{T}_1$. Thus $T \in \mathcal{F}$ and the proof is complete. ■

3. Bounds on the Signed Roman Edge k-Domination

In this section we establish some sharp bounds on the signed Roman edge k-domination number and we characterize all connected graphs whose signed Roman edge k-domination number is equal to their size.

Proposition 18. *If G is a graph of size m, then*

$$\gamma'_{sRk}(G) \geq k + \Delta + \delta - m - 1.$$

This bound is sharp for stars $K_{1,r}$ with $r \neq 3$ when $k = 1$.

Proof. Let f be a $\gamma'_{sRk}(G)$-function, v a vertex of maximum degree Δ and $u \in N(v)$. By definition $f[uv] \geq k$ and the least possible weight for f will now be achieved if $f(e') = -1$ for each $e' \in E(G) \setminus N[uv]$. Thus $\gamma'_{sRk}(G) \geq k - [m - (d(u) + d(v) - 1)] \geq k - m + \Delta + \delta - 1$. ■

Theorem 19. *Let G be a graph of size m. Then*

$$\gamma'_{sRk}(G) \geq \frac{m(2(\delta - \Delta) + k)}{2\Delta - 1}.$$

Proof. Assume that g is a $\gamma'_{sRk}(G)$-function. Define $f : E(G) \to \{0, 2, 3\}$ by $f(e) = g(e) + 1$ for each $e \in E$. We have

$$\begin{aligned}\sum_{e\in E(G)} f(N[e]) &= \sum_{e=uv\in E(G)} (g(N[e]) + d(u) + d(v) - 1)\\ &\geq \sum_{e=uv\in E(G)} (g(N[e]) - 1) + 2m\delta = m(2\delta + k - 1).\end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned}\sum_{e\in E(G)} f(N[e]) &= \sum_{e=uv\in E(G)} (d(u)+d(v)-1)f(e) \\ &\leq \sum_{e\in E(G)} (2\Delta-1)f(e) = (2\Delta-1)f(E(G)).\end{aligned} \tag{4}$$

By (3) and (4), we have $f(E(G)) \geq \frac{m(2\delta+k-1)}{2\Delta-1}$. Since $g(E(G)) = f(E(G)) - m$, we have

$$\gamma'_{sRk}(G) = g(E(G)) \geq \frac{m(2\delta+k-1)}{2\Delta-1} - m,$$

as desired. ■

Corollary 20. *For any r-regular graph G, $(r \geq 1)$, $\gamma'_{sRk}(G) \geq \frac{km}{2r-1}$.*

The special case $k = 1$ of Theorem 19 and Corollary 20 can be found in [2]. Corollary 10 shows that Corollary 20 is sharp for $k = 2$ and $m \equiv 0 \pmod 3$.

Theorem 21. *Let G be a connected graph of size $m \geq 2$. Then*

$$\gamma'_{sRk}(G) \leq \frac{\gamma'_{sk}(G)+m}{2}.$$

Proof. Let f be a $\gamma'_{sk}(G)$-function, and let $P = \{e \mid f(e) = 1\}$ and $M = \{e \mid f(e) = -1\} = \{e_1, e_2, \ldots, e_{|M|}\}$. Suppose $e'_i \in P$ is an edge adjacent to e_i for each i. Define $g : E(G) \to \{-1, 1, 2\}$ by $g(e'_i) = 2$ for $1 \leq i \leq |M|$ and $g(e) = f(e)$ otherwise. It is easy to see that g is an SREkDF on G of weight at most $\gamma'_{sk}(G) + |M|$. It follows from $\gamma'_{sk}(G) = |P| - |M|$ and $m = |P| + |M|$ that $|P| = \frac{\gamma'_{sk}(G)+m}{2}$ and hence

$$\gamma'_{sRk}(G) \leq \omega(g) \leq \gamma'_{sk}(G) + |M| = |P| = \frac{\gamma'_{sk}(G)+m}{2},$$

as desired. ■

Theorem 22. *Let G be a connected graph of order $n \geq 3$ and size m. Then*

$$\gamma'_{sR2}(G) \geq 2(n-m).$$

Furthermore, this bound is sharp.

Proof. Let p be the number of cycles of G. The proof is by induction on p. The statement is true for $p = 0$ by Theorem 14. Assume the statement is true for all simple connected graphs G for which the number of cycles is less than p, where $p \geq 1$. Let G be a simple connected graph with p cycles. Assume that f is a

$\gamma'_{sR2}(G)$-function and let $e = uv$ be a non-cut edge. If $f(e) = -1$, then obviously $f|_{G-e}$ is an SRE2DF for $G - e$ and by the induction hypothesis, we have

$$2(n-m) < 2(n-(m-1)) - 1 \le f(E(G-e)) - 1 = f(E(G)) = \gamma'_{sR2}(G).$$

Thus, we may assume that all non-cut edges are assigned 1 or 2 by f. We consider two cases.

Case 1. $f(uv) = 1$. Consider two subcases.

Subcase 1.1. $f(E(u)) \le 1$ (the case $f(E(v)) \le 1$ is similar). Then u has at least one neighbor u' such that $f(uu') = -1$. Assume that G' is the graph obtained from $G - \{uv, uu'\}$ by adding a new pendant edge vv'. Define $g : E(G') \to \{-1, 1, 2\}$ by $g(vv') = 1, g(a) = f(a)$ for $a \in E(G) \setminus \{uv, uu'\}$. Clearly, g is an SRE2DF for G' and it follows from the induction hypothesis and (1) that

$$\omega(f) = -1 + \omega(g) \ge -1 + 2(n(G') - m(G')) = -1 + 2(n - (m-1)) > 2(n-m).$$

Subcase 1.2. $f(E(u)) \ge 2$ and $f(E(v)) \ge 2$. Let G' be the graph obtained from $G - \{e\}$ by adding two new pendant edges vv' and uu' and define $g : E(G') \to \{-1, 1, 2\}$ by $g(vv') = g(uu') = 1$ and $g(a) = f(a)$ otherwise. Clearly, g is an SRE2DF for G'. It follows from the induction hypothesis that

$$\omega(f) = -1 + \omega(g) \ge -1 + 2(n(G') - m(G')) = -1 + 2(n+2-(m+1)) > 2(n-m).$$

By Case 1, we may assume that all non-cut edges are assigned 2 by f.

Case 2. $f(uv) = 2$. Consider two subcases.

Subcase 2.1. $f(E(u)) \le 2$ (the case $f(E(v)) \le 2$ is similar). Then clearly $f(E(v)) \ge 2$. Since all non-cut edges are assigned 2 by f (by assumption) and since uv belongs to a cycle in G, it follows from $f(E(u)) \le 2$ that there are two -1-edges at u, say e', e''. Assume that G' is the graph obtained from $G - \{e, e', e''\}$ by adding a new pendant edge vv' at v. Define $g : E(G') \to \{-1, 1, 2\}$ by $g(vv') = 2$ and $g(a) = f(a)$ otherwise. It is easy to see that g is an SRE2DF of G' and we deduce from the induction hypothesis and (1) that

$$\omega(f) = -2 + \omega(g) \ge -2 + 2(n(G') - m(G')) = -2 + 2(n-1-(m-2)) = 2(n-m).$$

Subcase 2.2. $f(E(u)) \ge 3$ and $f(E(v)) \ge 3$. Let G' be the graph obtained from $G - \{e\}$ by adding two new pendant edges vv' and uu'. Define $g : E(G') \to \{-1, 1, 2\}$ by $g(vv') = g(uu') = 2$ and $g(a) = f(a)$ otherwise. Clearly, g is an SRE2DF for G' and by the induction hypothesis, we obtain

$$\omega(f) = -2 + \omega(g) \ge -2 + 2(n(G') - m(G')) = -2 + 2(n+2-(m+1)) = 2(n-m).$$

■

Theorem 23. *Let $k \geq 1$ be an integer, and let G be a graph of size m and minimum degree δ. If $2\delta - k \geq 3$, then $\gamma'_{sRk}(G) \leq m - 1$.*

Proof. Let $v \in V(G)$ be an arbitrary vertex, and let $u_1, u_2, \ldots, u_p$ be the neighbors of v. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(vu_1) = -1$, $f(vu_2) = 2$ and $f(x) = 1$ otherwise. If $e = wz$ is an arbitrary edge, then $f[wz] \geq d(w)+d(z)-3 \geq 2\delta-3 \geq k$. Therefore f is an SREkDF on G of weight $m-1$ and so $\gamma'_{sRk}(G) \leq m - 1$. ■

Theorem 24. *Let $k \geq 1$ be an integer, and let G be a graph of size m and minimum degree δ. If $2\delta - k \geq 5$, then*

$$\gamma'_{sRk}(G) \leq m - 2\left\lfloor \frac{2\delta - k}{2} \right\rfloor + 1.$$

Proof. Let $t = \left\lfloor \frac{2\delta-k}{2} \right\rfloor$, and let $v \in V(G)$ be an arbitrary vertex. Now let $A = \{u_1, u_2, \ldots, u_t\}$ be a set of t neighbors of v. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(vu_i) = -1$ for $1 \leq i \leq t$, $f(vu_{t+1}) = 2$ and $f(x) = 1$ otherwise. Then $f[vu_i] = -t+1+(d(v)-t)+d(u_i)-1 \geq 2\delta-2t \geq k$ for $1 \leq i \leq d(v)$. If $e = wz$ is an edge different from vu_i, then $f[wz] \geq d(w) + d(z) - 5 \geq 2\delta - 5 \geq k$. Therefore f is an SREkDF on G of weight $m - 2t + 1$ and so $\gamma'_{sRk}(G) \leq m - 2t + 1$. ■

Theorem 25. *Let $k \geq 1$ be an integer, and let G be a graph of size m, minimum degree δ and maximum matching M. If $2\delta - k \geq 5$, then $\gamma'_{sRk}(G) \leq m - |M|$.*

Proof. Let $M = \{e_1, e_2, \ldots, e_{|M|}\}$ be a maximum matching, and let $x_1, x_2, \ldots, x_t$ be a minimum edge set such that each e_i is adjacent to an edge x_j for $1 \leq i \leq |M|$ and $1 \leq j \leq t$. Then $t \leq |M|$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(e_i) = -1$ for $1 \leq i \leq |M|$, $f(x_j) = 2$ for $1 \leq j \leq t$ and $f(x) = 1$ otherwise. If $e = uv$ is an arbitrary edge of G, then $f[e] \geq d(u) + d(v) - 5 \geq 2\delta - 5 \geq k$. Therefore f is an SREkDF on G of weight $m-2|M|+t \leq m-|M|$ and so $\gamma'_{sRk}(G) \leq m-|M|$. ■

In what follows, we characterize all connected graphs attaining the bound in (2).

Theorem 26. *Let G be a connected graph of size $m \geq 2$. Then $\gamma'_{sR2}(G) = m$ if and only if $G = C_4$, $G = C_5$, $G = P_n$ $(3 \leq n \leq 8)$ or G is a subdivided star $K^*_{1,r}$ $(r \geq 1)$.*

Proof. If $G = C_4$, $G = C_5$, $G = P_n$ $(3 \leq n \leq 7)$ or G is a subdivided star $K^*_{1,r}$ $(r \geq 1)$, then the result is immediate by Corollary 9 and Observation 6. Let $\gamma'_{sR2}(G) = m$. If $\Delta \leq 2$, then it follows from Corollaries 9 and 10 that $G = P_n$ $(3 \leq n \leq 8)$ or $G = C_4$ or $G = C_5$. Assume that $\Delta \geq 3$.

Claim 1. *G has no support vertex of degree at least* 3.

Proof. Let G have a support vertex u with $d(u) \geq 3$ and let $v, w \in N(u)$ where $d(v) = 1$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(uv) = -1$, $f(uw) = 2$ and $f(x) = 1$ for $x \in E(G) \setminus \{uv, uw\}$. Obviously, f is an SRE2DF of weight less than m, a contradiction. □

Claim 2. *G is acyclic.*

Proof. Let $C_g = (v_1 v_2 \cdots v_g)$ be a cycle of G of length $g = \text{girth}(G)$. Since $\Delta \geq 3$, we observe that $G \neq C_g$. By Claim 1, v_i is not a support vertex for each $1 \leq i \leq g$. Since $G \neq C_g$, we may assume that $d(v_1) \geq 3$ and $u \in N(v_1) \setminus \{v_2, v_g\}$. Then the function $f : E(G) \to \{-1, 1, 2\}$ defined by $f(v_1v_2) = -1, f(v_2v_3) = 2$ and $f(x) = 1$ otherwise, is an SRE2DF of weight less than m, a contradiction. □

Claim 3. *For each non pendant edge $e = uv$, $\min\{d(u), d(v)\} = 2$.*

Proof. Let $e = uv$ be a non pendant edge of G such that $\min\{d(u), d(v)\} \geq 3$. By Claim 1, both u and v are not support vertices. Let $v_1 \in N(v) \setminus \{u\}$ and define $f : E(G) \to \{-1, 1, 2\}$ by $f(vv_1) = 2$, $f(uv) = -1$ and $f(x) = 1$ otherwise. Clearly, f is an SRE2DF of weight $m - 1$, a contradiction. □

Let v be a vertex of maximum degree Δ and let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. By Claims 1 and 3, we deduce that $d(v_i) = 2$ for each i. If v_i is a support vertex for each i, then $G = K^*_{1,\Delta}$ and we are done. Assume that v_1 is not a support vertex. Let $u \in N(v_1) \setminus \{v\}$. Define $f : E(G) \to \{-1, 1, 2\}$ by $f(vv_1) = -1$, $f(uv_1) = 2$ and $f(x) = 1$ otherwise. Clearly, f is an SRE2DF of weight $m - 1$, a contradiction. This completes the proof. ■

We conclude this paper with an open problem.

Problem 27. Characterize all connected graphs G of order n and size m attaining the bound of Theorem 22.

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