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# A NOTE ON NEIGHBOR EXPANDED SUM DISTINGUISHING INDEX[1]

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**Abstract**

A total $k$-coloring of a graph $G$ is a coloring of vertices and edges of $G$ using colors of the set $[k] = \{1, \ldots, k\}$. These colors can be used to distinguish the vertices of $G$. There are many possibilities of such a distinction. In this paper, we consider the sum of colors on incident edges and adjacent vertices.

**Keywords:** general edge coloring, total coloring, neighbor-distinguishing index, neighbor sum distinguishing coloring.

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## 1. Introduction and Terminology

Let $G = (V, E)$ be a finite, undirected simple graph with vertex set $V$ and edge set $E$.

Karoński *et al.* [4] introduced and investigated a coloring of the edges of a graph with positive integers so that adjacent vertices have different sums of incident edge colors. More precisely, let $c : E \to [k] = \{1, 2, \ldots, k\}$ be an edge

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coloring of $G$ (such a coloring is also called an edge $k$-coloring of $G$). For $x \in V$, we define

$$\sigma(x) = \sum_{e \ni x} c(e),$$

where the expression $e \ni x$ means that $e$ contains $x$. An edge $k$-coloring $c$ of $G$ is called *neighbor sum distinguishing* if $\sigma(x) \neq \sigma(y)$ whenever $xy \in E$. In other words, the vertex coloring $\sigma$ induced by $c$ in the above must be proper. The minimum integer $k$ for which there is a neighbor sum distinguishing coloring of a graph $G$ will be denoted by $\mathrm{gndi}_\Sigma(\mathrm{G})$.

In [4] Karoński *et al.* posed the following elegant problem, known as the 1-2-3 Conjecture.

**Conjecture 1.** *Let $G$ be a connected graph, $G \neq K_2$. Then* $\mathrm{gndi}_\Sigma(\mathrm{G}) \leq 3$.

Thus far it is known that $\mathrm{gndi}_\Sigma(\mathrm{G}) \leq 5$ for any graph $G$ without a connected component isomorphic to $K_2$ (see [3]).

In [5] the following problem related to the 1-2-3 Conjecture was introduced. Let $c : E \cup V \to \{1, 2, \dots, k\}$ be a total $k$-coloring of a graph $G$. For every vertex $x$, we denote by

$$t(x) := c(x) + \sum_{y \in N(x)} c(xy) = c(x) + \sigma(x),$$

where $N(x) = \{y \in V | \; xy \in E\}$ denotes an open neighborhood of $x$. Thus, $t(x)$ is the sum of edge colors of incident edges to $x$ and the color of $x$. We say that $c$ is a *total neighbor sum distinguishing* coloring of $G$ if $t(x) \neq t(y)$ for all adjacent vertices $x, y$ in $G$.

Similarly as above, the minimum value of $k$ for which there exists a total neighbor sum distinguishing coloring of a graph $G$ will be denoted by $\mathrm{tgndi}_\Sigma(\mathrm{G})$.

In [5] Przybyło and Woźniak posed the following problem, known as the 1-2 Conjecture.

**Conjecture 2.** *Let $G$ be a connected graph. Then* $\mathrm{tgndi}_\Sigma(\mathrm{G}) \leq 2$.

Thus far it is known that for every graph $G$, $\mathrm{tgndi}_\Sigma(\mathrm{G}) \leq 3$ (see [2]).

However, in the case of total coloring of $G$, there are also other possibilities to define the *palette* of colors i.e., the elements which we take into account. In this paper, for $x \in V$, we define

$$w(x) = \sum_{e \ni x} c(e) + \sum_{y \in N(x)} c(y),$$

where $c$ is a total $k$-coloring of $G$. The value $w(x)$ will be called an *expanded sum at $x$*. A total $k$-coloring $c$ of $G$ is called *neighbor expanded sum distinguishing* (*NESD* for short) if

$$w(x) \neq w(y)$$

whenever $xy \in E$. In other words, the vertex coloring $w$ induced by $c$ in the above must be proper. The corresponding invariant, i.e., the minimum value of $k$ for which such an NESD total $k$-coloring of $G$ exists, is called the *neighbor expanded sum distinguishing index of* $G$ or simply *expanded index of* $G$ and denoted by $\mathrm{egndi}_{\Sigma}(G)$.

We state the following conjecture.

**Conjecture 3.** *For every graph* $G$, $\mathrm{egndi}_{\Sigma}(G) \leq 2$.

**Remark 4.** Another possibility would be to distinguish vertices by considering *full sums* defined for a vertex $x$ by

$$\phi(x) = c(x) + \sum_{e \ni x} c(e) + \sum_{y \in N(x)} c(y),$$

where $c$ is a total coloring of $G$. The corresponding parameter is denoted by $\mathrm{fgndi}_{\Sigma}(G)$. The main reason why we consider expanded sums and not full sums is that the parameter $\mathrm{egndi}_{\Sigma}(G)$ is well defined for each graph $G$, while the parameter $\mathrm{fgndi}_{\Sigma}(G)$ does not exist for graphs containing $K_2$ as a component. Observe by the way that $\mathrm{fgndi}_{\Sigma}(K_3) = 3$ while $\mathrm{egndi}_{\Sigma}(K_3) = 2$. Thus, in general, we need three colors in order to distinguish adjacent vertices in such a way while, if the above conjecture is true, in the case of expanded sum two colors are sufficient. Therefore, in a sense, the parameter $\mathrm{egndi}_{\Sigma}(G)$ is closer to $\mathrm{tgndi}_{\Sigma}(\mathrm{G})$ than $\mathrm{fgndi}_{\Sigma}(G)$.

**Remark 5.** As we continue to deal with only one parameter, namely $\mathrm{egndi}_{\Sigma}(G)$, later on, we use the shorter notation, simply putting $\eta(G) := \mathrm{egndi}_{\Sigma}(G)$.

Let $G$ and $G'$ be isomorphic graphs and $c$ and $c'$ be two total colorings of $G$ and $G'$, respectively. We say that an isomorphism $\phi : V(G) \longmapsto V(G')$ of $G$ and $G'$ is a *total isomorphism*, respect to $c$ and $c'$, if $c'(\phi(x)) = c(x)$ for any $x \in V(G)$ and $c'(\phi(xy)) = c(xy)$ for any $xy \in E$.

If $a < b$, where $a$, $b$ are natural numbers, then by $[a, b]$ we mean the integer interval of ends $a$ and $b$, i.e., the set $\{a, a+1, \ldots, b\}$. Remind that $[n] = [1, n]$. We use Bondy and Murty's book [1] for terminologies and notations not defined here.

## 2. Paths, Cycles and Complete Graphs

The proof of the following proposition is left to the reader.

**Proposition 6.** *If* $P_m$ *is the path of order* $m \geq 2$, *then* $\eta(P_m) = 2$, *if* $m \neq 3$ *and* $\eta(P_3) = 1$.

**Proposition 7.** *For $m \geq 3$, $\eta(C_m) = 2$.*

***Proof.*** Let $C_m = x_1, \ldots, x_m, x_1$ be the cycle of order $m \geq 4$. Put

$c(x_{2i-1}) = 1$, if $1 \leq 2i - 1 \leq m$;
$c(x_{2i}) = 2$, if $2 \leq 2i \leq m$;
$c(x_i x_{i+1}) = 1$, if $i = 1, \ldots, m$ and $m$ even (indices of a cycle $C_m$ are taken modulo $m$);
$c(x_i x_{i+1}) = 1$, if $i = 1, \ldots, m$, $i \neq m-1$ and $m$ odd, $c(x_{m-1}x_m) = 2$.

It can be easily seen that the above function is a neighbor expanded sum distinguishing total 2-coloring of $C_m$. For $m = 3$, the result follows from the next theorem. ■

**Theorem 8.** *For every $n \geq 2$, $\eta(K_n) = 2$.*

***Proof.*** Denote by $x_1, \ldots, x_n$ the vertices of the complete graph $K_n$ and let $c_n$ be a total $k$-coloring of $K_n$. Let $w(c_n) = \{a \in \mathbb{N} \mid \text{there is an}\, i \in [n] \text{ such that } w(x_i) = a\}$ and let $f(c_n) = \sum_{i=1}^{n} c_n(x_i)$.

We claim that for every $n \geq 2$ there is an NESD total 2-coloring $c_n$ of $K_n$ such that

$$w(c_n) = \begin{cases} \left[\frac{5n-5}{2}, \frac{7n-7}{2}\right], & \text{if } n \text{ is odd,} \\ \left[\frac{5n-6}{2}, \frac{7n-8}{2}\right], & \text{if } n \text{ is even,} \end{cases} \tag{1}$$

and

$$f(c_n) = \begin{cases} \frac{3n-1}{2}, & \text{if } n \text{ is odd,} \\ \frac{3n}{2}, & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Note that for each $n \geq 2$ the set $w(c_n)$ contains exactly $n$ different values, so if there exists a coloring $c_n$ satisfying (1), then it is NESD.

Now, let $c_2$ be the total coloring of $K_2$ defined as follows: $c_2(x_1) = 1$, $c_2(x_2) = 2$ and $c_2(x_1x_2) = 1$. Obviously, $c_2$ is an NESD total 2-coloring, where $w(c_2) = [2, 3] = [\frac{5n-6}{2}, \frac{7n-8}{2}]$ and $f(c_2) = 3 = \frac{3n}{2}$. The coloring $c_n$ will be defined recursively as follows.

Suppose that there exists a total coloring $c_{n-1}$ verifying the conditions (1) and (2) and color the subgraph of $K_n$ induced by the set $\{x_1, x_2, \ldots, x_{n-1}\}$ using the coloring $c_{n-1}$. If $n$ is odd use 1 to color the vertex $x_n$, and 2 to color all $n-1$ edges incident to $x_n$. For $n$ even use 2 to color $x_n$ and 1 for all edges incident to $x_n$. We will denote by $c_n$ the total coloring of $K_n$ obtained in this way.

Observe that in the coloring $c_n$ the weights $w(x_i)$ for $i = 1, \ldots, n-1$ increase by 3 (with respect to the weights for $c_{n-1}$), and $w(x_n)$ is equal to $f(c_{n-1}) + 2(n-1)$ for $n$ odd and $f(c_{n-1}) + (n-1)$ for $n$ even. So, if $n$ is odd, then

$$w(c_n) = \left[\frac{5(n-1)-6}{2} + 3, \frac{7(n-1)-8}{2} + 3\right] \cup \{f(c_{n-1}) + 2(n-1)\}$$

$$= \left[\frac{5n-5}{2}, \frac{7n-9}{2}\right] \cup \left\{\frac{3(n-1)}{2} + 2n - 2\right\} = \left[\frac{5n-5}{2}, \frac{7n-7}{2}\right],$$

and

$$f(c_n) = \frac{3(n-1)}{2} + 1 = \frac{3n-1}{2}.$$

If $n$ is even, then we have

$$\begin{aligned} w(c_n) &= \left[\frac{5(n-1)-5}{2} + 3, \frac{7(n-1)-7}{2} + 3\right] \cup \{f(c_{n-1}) + (n-1)\} \\ &= \left[\frac{5n-4}{2}, \frac{7n-8}{2}\right] \cup \left\{\frac{3(n-1)-1}{2} + n - 1\right\} = \left[\frac{5n-6}{2}, \frac{7n-8}{2}\right] \end{aligned}$$

and

$$f(c_n) = \frac{3(n-1)-1}{2} + 2 = \frac{3n}{2}.$$

It follows that $c_n$ is an NESD total 2-coloring of $K_n$. ■

**Corollary 9.** *There exist exactly two* (*up to total isomorphism*) NESD *total* 2-*colorings of* $K_n$.

***Proof.*** We will use the notation of the proof of Theorem 8. Consider another NESD total 2-coloring $c'_n$ of $K_n$ defined recursively as follows. For $n = 2$ let $c'_2(x_1) = 1$, $c'_2(x_2) = 2$ and $c'_2(x_1x_2) = 2$. Suppose the coloring $c'_{n-1}$ is given. Now color the subgraph of $K_n$ induced by the set $\{x_1, x_2, \ldots, x_{n-1}\}$ using the coloring $c'_{n-1}$. If $n$ is odd use 2 to color the vertex $x_n$, and 1 to color all $n-1$ edges incident to $x_n$. A vertex $x$ which is colored by 2 and such that its incident edges are colored by 1 will be called the *edge monochromatic vertex of type* 2.

For $n$ even use 1 to color $x_n$ and 2 for all edges incident to this vertex. Such a vertex is called the *edge monochromatic vertex of type* 1.

We denote the total coloring of $K_n$ obtained in this way by $c'_n$. Note that, in each coloring $c_n$ or $c'_n$ of $K_n$ there is exactly one edge monochromatic vertex of type 1 or 2, and if the coloring $c_n$ contains one edge monochromatic vertex of type 1, then $c'_n$ has one edge monochromatic vertex of type 2 and viceversa. Applying the method described in the proof of Theorem 8 one can easily show that $c'_n$ is an NESD total 2-coloring of $K_n$.

Let $l_n$ be another NESD total 2-coloring of $K_n$, $n \geq 2$. Let $d^n(x_i) = \sum_{e \ni x_i} l_n(e)$, $i = 1, \ldots, n$.

Put $m = \min_i d^n(x_i)$, $M = \max_i d^n(x_i)$. We claim that $m = n-1$ or $M = 2n-2$ and there is exactly one edge monochromatic vertex of type 1 or 2 (so exactly one of these two equalities is true). Suppose for the sake of contradiction that $d^n(x_i) \geq n$ and $d^n(x_i) \leq 2n-3$ for all $i$. Without loss of generality, we may assume that $d^n(x_1) = m \geq n$ and $d^n(x_2) = M \leq 2n-3$. Therefore, the expanded sum associated to $l_n$, say $w$, satisfies

$$w(x_1) = d^n(x_1) + b \geq n + b,$$

where $b = \sum_{i \neq 1} l_n(x_i)$.

Moreover,

$$w(x_2) \leq d^n(x_2) + (b+1) \leq b + 2n - 2,$$

so $w(x_2) - w(x_1) \leq n - 2$ and we cannot distinguish all the vertices of $K_n$ by expanded sums $w(x_i)$, a contradiction.

Thus, we may assume that (for example) $d^n(x_1) = n - 1$, i.e., all edges incident to $x_1$ are painted using the color 1. It follows that $M \leq 2n - 3$ and $w(x_2) - w(x_1) \leq n - 1$. Therefore, $w(x_2) = b + 2n - 2 = \sum_{i \neq 2} l_n(x_i) + d^n(x_2) = \sum_{i \neq 1} l_n(x_i) + l_n(x_1) - l_n(x_2) + d^n(x_2) = b + l_n(x_1) - l_n(x_2) + d^n(x_2)$. Thus, $d^n(x_2) = l_n(x_2) - l_n(x_1) + 2n - 2 \leq 2n - 3$ and this implies $l_n(x_1) > l_n(x_2)$, so $l_n(x_1) = 2$. Hence, $x_1$ is the only vertex satisfying $d^n(x_1) = n - 1$ (the second one would have the same color so the same weight as $x_1$) and this is the edge monochromatic vertex of type 2. The proof of the case when $d^n(x_2) = M = 2n-2$ is analogous, so our claim is true.

Now applying the induction on $n$, we will show that every NESD total 2-coloring $l_n$ of $K_n$ ($n \geq 2$) is identical with $c_n$ or $c'_n$ (up to total isomorphism). Clearly, this assertion is evident for $n = 2$. Assume that it is true for $n' < n$ and consider an NESD total 2-coloring $l_n$ of $K_n$. Deleting an unique edge monochromatic vertex of type 2 (type 1, respectively), we get the graph $K_{n-1}$ together with an NESD total 2-coloring $c_{n-1}$ or $c'_{n-1}$ having edge monochromatic vertex of type 1 (type 2, respectively), so our assertion is true. ■

## 3. Bipartite Graphs

**Theorem 10.** *Let $T$ be a tree of order $n \geq 2$. Then $\eta(T) \leq 2$.*

***Proof.*** The proof is by induction on $n$. Observe that the theorem is trivial if $T$ is a star $K_{1,n-1}$, hence, in particular, for every tree of order $n \in \{2, 3\}$, $\eta(T) \leq 2$.

Suppose our assertion is true for all trees of order $n - 1 \geq 3$ and let $T$ be a tree of order $n$. We may assume that $T$ is not isomorphic to $K_{1,n-1}$. Let $x$ be an end-vertex of a longest path $P = xyz \cdots$ in $T$ and let $T'$ denote the tree $T \setminus \{x\}$. By the choice of $x$ and $T$, $z$ is the only neighbor of $y$ having the degree greater than or equal to 2 in $T$. Let $d_{T'}(t) = d'(t)$ for any vertex $t \in V(T')$. The degree in $T'$ of any vertex $t$ is the same as in $T$, except for $t = y$ for which $d'(y) = d_T(y) - 1$.

By induction hypothesis, there is an NESD total 2-coloring $c'$ of $T'$. We will color the edge $xy$ and the vertex $x$ by $a$ and $b$, respectively, $a$, $b \in \{1, 2\}$, so that the coloring $c$ of $T$ defined as follows

$$c(\alpha) = \begin{cases} c'(\alpha), & \text{if } \alpha \in V(T') \cup E(T'), \\ a, & \text{if } \alpha = xy, \\ b, & \text{if } \alpha = x, \end{cases} \tag{3}$$

would be an NESD total 2-coloring of $T$. We prove that this is always possible.

Let $w'(v)$ denote the expanded sum at $v \in V(T')$ with respect to the coloring $c'$.

Suppose now that the degree $d_T(y)$ of $y$ in $T$ is at least three and observe that for any total 2-coloring $c$ of $T$ and for any $t \in N_T(y) \setminus \{z\}$, we have $w(t) = c(y)+c(yt) \leq 4$ and $w(y) \geq 6$, so the vertices $t$ and $y$ are distinguished. Therefore, we can choose $a$ and $b$ such that $w(z) = w'(z) \neq w'(y) + a + b = w(y)$ and the new total coloring $c$ of $T$ defined by (3) will distinguish all vertices of $T$.

If $d_T(y) = 2$, we can also choose $a$ and $b$ such that $w(x) = a + c'(y) \neq w'(y) + a + b = w(y)$ and $w(z) = w'(z) \neq w'(y) + a + b = w(y)$, so the total coloring $c$ distinguishes all vertices of $T$. ■

**Proposition 11.** *Let $G = (X, Y, E)$ be a connected bipartite graph with bipartition classes $X$ and $Y$ such that $|X|$ is even or $G$ has a vertex of odd degree. Then $\eta(G) \leq 2$.*

***Proof.*** Suppose that $|X|$ is even. We will follow the idea presented in [4] and show that there exists a coloring of vertices and edges of $G$ with the elements of the group $\mathbb{Z}_2$ such that all vertices of $X$ have expanded sums 1 and the expanded sum at any vertex of $Y$ is 0. Let $X = \{x_1, \ldots, x_{2k}\}$ and let $P_j$ denote a path of end-vertices $x_{2j-1}$ and $x_{2j}$, $j = 1, \ldots, k$. Clearly, each $P_j$ is of even length. Begin now with color 0 on all vertices and edges of $G$ and modify this coloring along the consecutive paths $P_j$ in the following way: start with $P_1$ and add 1 (in $\mathbb{Z}_2$) to the color of every edge of $P_1$, then add 1 to the color of every edge of $P_2$ and so on. Obviously, in $j$-th step this operation maintains the expanded sums at internal vertices of $P_j$, so that of $Y$, and change the expanded sums at end-vertices of $P_j$. After $k$ steps we obtain the desired coloring with the elements of $\mathbb{Z}_2$.

Replacing the color 0 by 2 and applying the addition in $\mathbb{N}$ we get an NESD total 2-coloring of $G$.

Now, assume that $|X|$ and $|Y|$ are odd (otherwise we could apply the first part of the proof), $X = \{x_1, \ldots, x_{2k}, x_{2k+1}\}$, and $d(x_{2k+1}) = 2l + 1$ is odd. Color the edges and the vertices of $G$ with the elements of $\mathbb{Z}_2$ using the same method as in the first part of the proof, taking the set $X' = \{x_1, \ldots, x_{2k}\}$ as the set of end-vertices of paths $P_j$. Perhaps some paths $P_j$ contain the vertex $x_{2k+1}$. Now the weight of every vertex of $Y \cup \{x_{2k+1}\}$ is 0 and all remaining weights are equal to 1. Put 1 on the vertex $x_{2k+1}$ and add 1 to the color of each edge incident to the vertex $x_{2k+1}$. Now the weights of vertices which are not adjacent to $x_{2k+1}$ remain unchanged, we add 1+1 to the weight of every neighbor of $x_{2k+1}$ and $(2l+1)\cdot 1$ to the weight of the vertex $x_{2k+1}$. Thus the weight of every vertex of $X$ is 1 and all weights of vertices of $Y$ are equal to 0. Now we change the color 0 for 2, apply the addition in $\mathbb{N}$ and get an NESD total 2-coloring of $G$. ■

The following proposition is obvious.

**Proposition 12.** *If every two adjacent vertices of $G$ have different degrees, then $\eta(G) = 1$.*

In some cases, the value of $\eta(G)$ can be determined exactly.

**Corollary 13.** *For any integers $n, p \geq 1$, $\eta(K_{n,p}) = 2$ for $n = p$ and $\eta(K_{n,p}) = 1$ for $n \neq p$.*

***Proof.*** Suppose that $K_{n,n}$ has bipartition $(X, Y)$. If we color the vertices of $X$ by 1 and other vertices and edges of $K_{n,n}$ by 2, we get an NESD total 2-coloring. For $n \neq p$ our result follows from Proposition 12. ■

## 4. Some Other Results

In [4] Karoński *et al.* proved the following result.

**Theorem 14.** *Let $\Gamma$ be a finite abelian group of odd order $k$ and let $G$ be a $k$-colorable graph on $n \geq 3$ vertices. Then there exists a coloring $c$ of the edges of $G$ with the elements of $\Gamma$ such that the resulting vertex coloring $\sigma$ induced by $c$ is a proper coloring of $G$.*

**Corollary 15.** *Let $k$ be an odd integer and let $G$ be a connected $k$-colorable graph. Then $\eta(G) \leq k$.*

***Proof.*** If we color the edges of $G$ using the method described in [4] with the elements of $\mathbb{Z}_k = \mathbb{Z}_{2l+1}$ and put 0 on the vertices of $G$, then we get an NESD total coloring of $G$ with the elements of $\Gamma = \mathbb{Z}_k$. Now we can obtain an NESD total $k$-coloring of $G$ by replacing 0 by $k$ and applying the addition in $\mathbb{N}$. ■

Thus the following corollary is true.

**Corollary 16.** *If $G$ is a connected $k$-colorable graph, then $\eta(G) \leq k + 1$.*

As already mentioned, in [3], Kalkowski *et al.* showed that for every graph $G$ without components isomorphic to $K_2$ there exists a coloring of the edges of $G$ with the elements of $\{1, \ldots, 5\}$ such that the resulting vertex weighting is a proper vertex coloring of $G$. This implies at once the following corollary.

**Corollary 17.** *If $G$ is a connected regular graph, then $\eta(G) \leq 5$.*

***Proof.*** Color the edges of $G$ with 5 colors in such a way that the obtained vertex coloring is proper. Afterwards put 1 on the vertices of $G$. All weights will increase by a constant. ■

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