# A NOTE ON NEIGHBOR EXPANDED SUM DISTINGUISHING INDEX ${ }^{1}$ 

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#### Abstract

A total $k$-coloring of a graph $G$ is a coloring of vertices and edges of $G$ using colors of the set $[k]=\{1, \ldots, k\}$. These colors can be used to distinguish the vertices of $G$. There are many possibilities of such a distinction. In this paper, we consider the sum of colors on incident edges and adjacent vertices.


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## 1. Introduction and Terminology

Let $G=(V, E)$ be a finite, undirected simple graph with vertex set $V$ and edge set $E$.

Karoński et al. [4] introduced and investigated a coloring of the edges of a graph with positive integers so that adjacent vertices have different sums of incident edge colors. More precisely, let $c: E \rightarrow[k]=\{1,2, \ldots, k\}$ be an edge

[^0]coloring of $G$ (such a coloring is also called an edge $k$-coloring of $G$ ). For $x \in V$, we define
$$
\sigma(x)=\sum_{e \ni x} c(e)
$$
where the expression $e \ni x$ means that $e$ contains $x$. An edge $k$-coloring $c$ of $G$ is called neighbor sum distinguishing if $\sigma(x) \neq \sigma(y)$ whenever $x y \in E$. In other words, the vertex coloring $\sigma$ induced by $c$ in the above must be proper. The minimum integer $k$ for which there is a neighbor sum distinguishing coloring of a graph $G$ will be denoted by $\operatorname{gndi}_{\Sigma}(\mathrm{G})$.

In [4] Karoński et al. posed the following elegant problem, known as the 1-2-3 Conjecture.
Conjecture 1. Let $G$ be a connected graph, $G \neq K_{2}$. Then $\operatorname{gndi}_{\Sigma}(\mathrm{G}) \leq 3$.
Thus far it is known that $\operatorname{gndi}_{\Sigma}(\mathrm{G}) \leq 5$ for any graph $G$ without a connected component isomorphic to $K_{2}$ (see [3]).

In [5] the following problem related to the 1-2-3 Conjecture was introduced. Let $c: E \cup V \rightarrow\{1,2, \ldots, k\}$ be a total $k$-coloring of a graph $G$. For every vertex $x$, we denote by

$$
t(x):=c(x)+\sum_{y \in N(x)} c(x y)=c(x)+\sigma(x)
$$

where $N(x)=\{y \in V \mid x y \in E\}$ denotes an open neighborhood of $x$. Thus, $t(x)$ is the sum of edge colors of incident edges to $x$ and the color of $x$. We say that $c$ is a total neighbor sum distinguishing coloring of $G$ if $t(x) \neq t(y)$ for all adjacent vertices $x, y$ in $G$.

Similarly as above, the minimum value of $k$ for which there exists a total neighbor sum distinguishing coloring of a graph $G$ will be denoted by tgndi ${ }_{\Sigma}(\mathrm{G})$.

In [5] Przybyło and Woźniak posed the following problem, known as the 1-2 Conjecture.

Conjecture 2. Let $G$ be a connected graph. Then $\operatorname{tgndi}_{\Sigma}(\mathrm{G}) \leq 2$.
Thus far it is known that for every graph $G$, $\operatorname{tgndi}_{\Sigma}(\mathrm{G}) \leq 3$ (see [2]).
However, in the case of total coloring of $G$, there are also other possibilities to define the palette of colors i.e., the elements which we take into account. In this paper, for $x \in V$, we define

$$
w(x)=\sum_{e \ni x} c(e)+\sum_{y \in N(x)} c(y)
$$

where $c$ is a total $k$-coloring of $G$. The value $w(x)$ will be called an expanded sum at $x$. A total $k$-coloring $c$ of $G$ is called neighbor expanded sum distinguishing ( $N E S D$ for short) if

$$
w(x) \neq w(y)
$$

whenever $x y \in E$. In other words, the vertex coloring $w$ induced by $c$ in the above must be proper. The corresponding invariant, i.e., the minimum value of $k$ for which such an NESD total $k$-coloring of $G$ exists, is called the neighbor expanded sum distinguishing index of $G$ or simply expanded index of $G$ and denoted by $\operatorname{egndi}_{\Sigma}(G)$.

We state the following conjecture.
Conjecture 3. For every graph $G$, egndi ${ }_{\Sigma}(G) \leq 2$.
Remark 4. Another possibility would be to distinguish vertices by considering full sums defined for a vertex $x$ by

$$
\phi(x)=c(x)+\sum_{e \ni x} c(e)+\sum_{y \in N(x)} c(y),
$$

where $c$ is a total coloring of $G$. The corresponding parameter is denoted by fgndi ${ }_{\Sigma}(G)$. The main reason why we consider expanded sums and not full sums is that the parameter $\operatorname{egndi}_{\Sigma}(G)$ is well defined for each graph $G$, while the parameter fgndi ${ }_{\Sigma}(G)$ does not exist for graphs containing $K_{2}$ as a component. Observe by the way that $\operatorname{fgndi}_{\Sigma}\left(K_{3}\right)=3$ while egndi ${ }_{\Sigma}\left(K_{3}\right)=2$. Thus, in general, we need three colors in order to distinguish adjacent vertices in such a way while, if the above conjecture is true, in the case of expanded sum two colors are sufficient. Therefore, in a sense, the parameter $\operatorname{egndi}_{\Sigma}(G)$ is closer to $\operatorname{tgndi}_{\Sigma}(\mathrm{G})$ than fgndi ${ }_{\Sigma}(G)$.

Remark 5. As we continue to deal with only one parameter, namely egndi ${ }_{\Sigma}(G)$, later on, we use the shorter notation, simply putting $\eta(G):=\operatorname{egndi}_{\Sigma}(G)$.

Let $G$ and $G^{\prime}$ be isomorphic graphs and $c$ and $c^{\prime}$ be two total colorings of $G$ and $G^{\prime}$, respectively. We say that an isomorphism $\phi: V(G) \longmapsto V\left(G^{\prime}\right)$ of $G$ and $G^{\prime}$ is a total isomorphism, respect to $c$ and $c^{\prime}$, if $c^{\prime}(\phi(x))=c(x)$ for any $x \in V(G)$ and $c^{\prime}(\phi(x y))=c(x y)$ for any $x y \in E$.

If $a<b$, where $a, b$ are natural numbers, then by $[a, b]$ we mean the integer interval of ends $a$ and $b$, i.e., the set $\{a, a+1, \ldots, b\}$. Remind that $[n]=[1, n]$. We use Bondy and Murty's book [1] for terminologies and notations not defined here.

## 2. Paths, Cycles and Complete Graphs

The proof of the following proposition is left to the reader.
Proposition 6. If $P_{m}$ is the path of order $m \geq 2$, then $\eta\left(P_{m}\right)=2$, if $m \neq 3$ and $\eta\left(P_{3}\right)=1$.

Proposition 7. For $m \geq 3, \eta\left(C_{m}\right)=2$.
Proof. Let $C_{m}=x_{1}, \ldots, x_{m}, x_{1}$ be the cycle of order $m \geq 4$. Put
$c\left(x_{2 i-1}\right)=1$, if $1 \leq 2 i-1 \leq m ;$
$c\left(x_{2 i}\right)=2$, if $2 \leq 2 i \leq m$;
$c\left(x_{i} x_{i+1}\right)=1$, if $i=1, \ldots, m$ and $m$ even (indices of a cycle $C_{m}$ are taken modulo $m$ );
$c\left(x_{i} x_{i+1}\right)=1$, if $i=1, \ldots, m, i \neq m-1$ and $m$ odd, $c\left(x_{m-1} x_{m}\right)=2$.
It can be easily seen that the above function is a neighbor expanded sum distinguishing total 2 -coloring of $C_{m}$. For $m=3$, the result follows from the next theorem.

Theorem 8. For every $n \geq 2, \eta\left(K_{n}\right)=2$.
Proof. Denote by $x_{1}, \ldots, x_{n}$ the vertices of the complete graph $K_{n}$ and let $c_{n}$ be a total $k$-coloring of $K_{n}$. Let $w\left(c_{n}\right)=\left\{a \in \mathbb{N} \mid\right.$ there is an $i \in[n]$ such that $w\left(x_{i}\right)=$ $a\}$ and let $f\left(c_{n}\right)=\sum_{i=1}^{n} c_{n}\left(x_{i}\right)$.

We claim that for every $n \geq 2$ there is an NESD total 2-coloring $c_{n}$ of $K_{n}$ such that

$$
w\left(c_{n}\right)= \begin{cases}{\left[\frac{5 n-5}{2}, \frac{7 n-7}{2}\right],} & \text { if } n \text { is odd }  \tag{1}\\ {\left[\frac{5 n-6}{2}, \frac{7 n-8}{2}\right],} & \text { if } n \text { is even }\end{cases}
$$

and

$$
f\left(c_{n}\right)=\left\{\begin{array}{l}
\frac{3 n-1}{2}, \text { if } n \text { is odd }  \tag{2}\\
\frac{3 n}{2}, \text { if } n \text { is even }
\end{array}\right.
$$

Note that for each $n \geq 2$ the set $w\left(c_{n}\right)$ contains exactly $n$ different values, so if there exists a coloring $c_{n}$ satisfying (1), then it is NESD.

Now, let $c_{2}$ be the total coloring of $K_{2}$ defined as follows: $c_{2}\left(x_{1}\right)=1$, $c_{2}\left(x_{2}\right)=2$ and $c_{2}\left(x_{1} x_{2}\right)=1$. Obviously, $c_{2}$ is an NESD total 2-coloring, where $w\left(c_{2}\right)=[2,3]=\left[\frac{5 n-6}{2}, \frac{7 n-8}{2}\right]$ and $f\left(c_{2}\right)=3=\frac{3 n}{2}$. The coloring $c_{n}$ will be defined recursively as follows.

Suppose that there exists a total coloring $c_{n-1}$ verifying the conditions (1) and (2) and color the subgraph of $K_{n}$ induced by the set $\left\{x_{1}, x_{2}, \ldots, x_{n-1}\right\}$ using the coloring $c_{n-1}$. If $n$ is odd use 1 to color the vertex $x_{n}$, and 2 to color all $n-1$ edges incident to $x_{n}$. For $n$ even use 2 to color $x_{n}$ and 1 for all edges incident to $x_{n}$. We will denote by $c_{n}$ the total coloring of $K_{n}$ obtained in this way.

Observe that in the coloring $c_{n}$ the weights $w\left(x_{i}\right)$ for $i=1, \ldots, n-1$ increase by 3 (with respect to the weights for $c_{n-1}$ ), and $w\left(x_{n}\right)$ is equal to $f\left(c_{n-1}\right)+2(n-1)$ for $n$ odd and $f\left(c_{n-1}\right)+(n-1)$ for $n$ even. So, if $n$ is odd, then

$$
w\left(c_{n}\right)=\left[\frac{5(n-1)-6}{2}+3, \frac{7(n-1)-8}{2}+3\right] \cup\left\{f\left(c_{n-1}\right)+2(n-1)\right\}
$$

$$
=\left[\frac{5 n-5}{2}, \frac{7 n-9}{2}\right] \cup\left\{\frac{3(n-1)}{2}+2 n-2\right\}=\left[\frac{5 n-5}{2}, \frac{7 n-7}{2}\right]
$$

and

$$
f\left(c_{n}\right)=\frac{3(n-1)}{2}+1=\frac{3 n-1}{2}
$$

If $n$ is even, then we have

$$
\begin{aligned}
w\left(c_{n}\right) & =\left[\frac{5(n-1)-5}{2}+3, \frac{7(n-1)-7}{2}+3\right] \cup\left\{f\left(c_{n-1}\right)+(n-1)\right\} \\
& =\left[\frac{5 n-4}{2}, \frac{7 n-8}{2}\right] \cup\left\{\frac{3(n-1)-1}{2}+n-1\right\}=\left[\frac{5 n-6}{2}, \frac{7 n-8}{2}\right]
\end{aligned}
$$

and

$$
f\left(c_{n}\right)=\frac{3(n-1)-1}{2}+2=\frac{3 n}{2}
$$

It follows that $c_{n}$ is an NESD total 2-coloring of $K_{n}$.
Corollary 9. There exist exactly two (up to total isomorphism) NESD total 2colorings of $K_{n}$.

Proof. We will use the notation of the proof of Theorem 8. Consider another NESD total 2 -coloring $c_{n}^{\prime}$ of $K_{n}$ defined recursively as follows. For $n=2$ let $c_{2}^{\prime}\left(x_{1}\right)=1, c_{2}^{\prime}\left(x_{2}\right)=2$ and $c_{2}^{\prime}\left(x_{1} x_{2}\right)=2$. Suppose the coloring $c_{n-1}^{\prime}$ is given. Now color the subgraph of $K_{n}$ induced by the set $\left\{x_{1}, x_{2}, \ldots, x_{n-1}\right\}$ using the coloring $c_{n-1}^{\prime}$. If $n$ is odd use 2 to color the vertex $x_{n}$, and 1 to color all $n-1$ edges incident to $x_{n}$. A vertex $x$ which is colored by 2 and such that its incident edges are colored by 1 will be called the edge monochromatic vertex of type 2 .

For $n$ even use 1 to color $x_{n}$ and 2 for all edges incident to this vertex. Such a vertex is called the edge monochromatic vertex of type 1.

We denote the total coloring of $K_{n}$ obtained in this way by $c_{n}^{\prime}$. Note that, in each coloring $c_{n}$ or $c_{n}^{\prime}$ of $K_{n}$ there is exactly one edge monochromatic vertex of type 1 or 2 , and if the coloring $c_{n}$ contains one edge monochromatic vertex of type 1 , then $c_{n}^{\prime}$ has one edge monochromatic vertex of type 2 and viceversa. Applying the method described in the proof of Theorem 8 one can easily show that $c_{n}^{\prime}$ is an NESD total 2-coloring of $K_{n}$.

Let $l_{n}$ be another NESD total 2-coloring of $K_{n}, n \geq 2$. Let $d^{n}\left(x_{i}\right)=$ $\sum_{e \ni x_{i}} l_{n}(e), i=1, \ldots, n$.

Put $m=\min _{i} d^{n}\left(x_{i}\right), M=\max _{i} d^{n}\left(x_{i}\right)$. We claim that $m=n-1$ or $M=2 n-2$ and there is exactly one edge monochromatic vertex of type 1 or 2 (so exactly one of these two equalities is true). Suppose for the sake of contradiction that $d^{n}\left(x_{i}\right) \geq n$ and $d^{n}\left(x_{i}\right) \leq 2 n-3$ for all $i$. Without loss of generality, we may
assume that $d^{n}\left(x_{1}\right)=m \geq n$ and $d^{n}\left(x_{2}\right)=M \leq 2 n-3$. Therefore, the expanded sum associated to $l_{n}$, say $w$, satisfies

$$
w\left(x_{1}\right)=d^{n}\left(x_{1}\right)+b \geq n+b
$$

where $b=\sum_{i \neq 1} l_{n}\left(x_{i}\right)$.
Moreover,

$$
w\left(x_{2}\right) \leq d^{n}\left(x_{2}\right)+(b+1) \leq b+2 n-2
$$

so $w\left(x_{2}\right)-w\left(x_{1}\right) \leq n-2$ and we cannot distinguish all the vertices of $K_{n}$ by expanded sums $w\left(x_{i}\right)$, a contradiction.

Thus, we may assume that (for example) $d^{n}\left(x_{1}\right)=n-1$, i.e., all edges incident to $x_{1}$ are painted using the color 1 . It follows that $M \leq 2 n-3$ and $w\left(x_{2}\right)-w\left(x_{1}\right) \leq n-1$. Therefore, $w\left(x_{2}\right)=b+2 n-2=\sum_{i \neq 2} l_{n}\left(x_{i}\right)+d^{n}\left(x_{2}\right)=$ $\sum_{i \neq 1} l_{n}\left(x_{i}\right)+l_{n}\left(x_{1}\right)-l_{n}\left(x_{2}\right)+d^{n}\left(x_{2}\right)=b+l_{n}\left(x_{1}\right)-l_{n}\left(x_{2}\right)+d^{n}\left(x_{2}\right)$. Thus, $d^{n}\left(x_{2}\right)=l_{n}\left(x_{2}\right)-l_{n}\left(x_{1}\right)+2 n-2 \leq 2 n-3$ and this implies $l_{n}\left(x_{1}\right)>l_{n}\left(x_{2}\right)$, so $l_{n}\left(x_{1}\right)=2$. Hence, $x_{1}$ is the only vertex satisfying $d^{n}\left(x_{1}\right)=n-1$ (the second one would have the same color so the same weight as $x_{1}$ ) and this is the edge monochromatic vertex of type 2 . The proof of the case when $d^{n}\left(x_{2}\right)=M=2 n-2$ is analogous, so our claim is true.

Now applying the induction on $n$, we will show that every NESD total 2coloring $l_{n}$ of $K_{n}(n \geq 2)$ is identical with $c_{n}$ or $c_{n}^{\prime}$ (up to total isomorphism). Clearly, this assertion is evident for $n=2$. Assume that it is true for $n^{\prime}<$ $n$ and consider an NESD total 2 -coloring $l_{n}$ of $K_{n}$. Deleting an unique edge monochromatic vertex of type 2 (type 1 , respectively), we get the graph $K_{n-1}$ together with an NESD total 2-coloring $c_{n-1}$ or $c_{n-1}^{\prime}$ having edge monochromatic vertex of type 1 (type 2 , respectively), so our assertion is true.

## 3. Bipartite Graphs

Theorem 10. Let $T$ be a tree of order $n \geq 2$. Then $\eta(T) \leq 2$.
Proof. The proof is by induction on $n$. Observe that the theorem is trivial if $T$ is a star $K_{1, n-1}$, hence, in particular, for every tree of order $n \in\{2,3\}, \eta(T) \leq 2$.

Suppose our assertion is true for all trees of order $n-1 \geq 3$ and let $T$ be a tree of order $n$. We may assume that $T$ is not isomorphic to $K_{1, n-1}$. Let $x$ be an end-vertex of a longest path $P=x y z \cdots$ in $T$ and let $T^{\prime}$ denote the tree $T \backslash\{x\}$. By the choice of $x$ and $T, z$ is the only neighbor of $y$ having the degree greater than or equal to 2 in $T$. Let $d_{T^{\prime}}(t)=d^{\prime}(t)$ for any vertex $t \in V\left(T^{\prime}\right)$. The degree in $T^{\prime}$ of any vertex $t$ is the same as in $T$, except for $t=y$ for which $d^{\prime}(y)=d_{T}(y)-1$.

By induction hypothesis, there is an NESD total 2-coloring $c^{\prime}$ of $T^{\prime}$. We will color the edge $x y$ and the vertex $x$ by $a$ and $b$, respectively, $a, b \in\{1,2\}$, so that the coloring $c$ of $T$ defined as follows

$$
c(\alpha)=\left\{\begin{array}{l}
c^{\prime}(\alpha), \text { if } \alpha \in V\left(T^{\prime}\right) \cup E\left(T^{\prime}\right)  \tag{3}\\
a, \text { if } \alpha=x y \\
b, \text { if } \alpha=x
\end{array}\right.
$$

would be an NESD total 2-coloring of $T$. We prove that this is always possible.
Let $w^{\prime}(v)$ denote the expanded sum at $v \in V\left(T^{\prime}\right)$ with respect to the color$\operatorname{ing} c^{\prime}$.

Suppose now that the degree $d_{T}(y)$ of $y$ in $T$ is at least three and observe that for any total 2-coloring $c$ of $T$ and for any $t \in N_{T}(y) \backslash\{z\}$, we have $w(t)=$ $c(y)+c(y t) \leq 4$ and $w(y) \geq 6$, so the vertices $t$ and $y$ are distinguished. Therefore, we can choose $a$ and $b$ such that $w(z)=w^{\prime}(z) \neq w^{\prime}(y)+a+b=w(y)$ and the new total coloring $c$ of $T$ defined by (3) will distinguish all vertices of $T$.

If $d_{T}(y)=2$, we can also choose $a$ and $b$ such that $w(x)=a+c^{\prime}(y) \neq$ $w^{\prime}(y)+a+b=w(y)$ and $w(z)=w^{\prime}(z) \neq w^{\prime}(y)+a+b=w(y)$, so the total coloring $c$ distinguishes all vertices of $T$.

Proposition 11. Let $G=(X, Y, E)$ be a connected bipartite graph with bipartition classes $X$ and $Y$ such that $|X|$ is even or $G$ has a vertex of odd degree. Then $\eta(G) \leq 2$.

Proof. Suppose that $|X|$ is even. We will follow the idea presented in [4] and show that there exists a coloring of vertices and edges of $G$ with the elements of the group $\mathbb{Z}_{2}$ such that all vertices of $X$ have expanded sums 1 and the expanded sum at any vertex of $Y$ is 0 . Let $X=\left\{x_{1}, \ldots, x_{2 k}\right\}$ and let $P_{j}$ denote a path of end-vertices $x_{2 j-1}$ and $x_{2 j}, j=1, \ldots, k$. Clearly, each $P_{j}$ is of even length. Begin now with color 0 on all vertices and edges of $G$ and modify this coloring along the consecutive paths $P_{j}$ in the following way: start with $P_{1}$ and add 1 (in $\mathbb{Z}_{2}$ ) to the color of every edge of $P_{1}$, then add 1 to the color of every edge of $P_{2}$ and so on. Obviously, in $j$-th step this operation maintains the expanded sums at internal vertices of $P_{j}$, so that of $Y$, and change the expanded sums at end-vertices of $P_{j}$. After $k$ steps we obtain the desired coloring with the elements of $\mathbb{Z}_{2}$.

Replacing the color 0 by 2 and applying the addition in $\mathbb{N}$ we get an NESD total 2-coloring of $G$.

Now, assume that $|X|$ and $|Y|$ are odd (otherwise we could apply the first part of the proof), $X=\left\{x_{1}, \ldots, x_{2 k}, x_{2 k+1}\right\}$, and $d\left(x_{2 k+1}\right)=2 l+1$ is odd. Color the edges and the vertices of $G$ with the elements of $\mathbb{Z}_{2}$ using the same method as in the first part of the proof, taking the set $X^{\prime}=\left\{x_{1}, \ldots, x_{2 k}\right\}$ as the set of end-vertices of paths $P_{j}$. Perhaps some paths $P_{j}$ contain the vertex $x_{2 k+1}$. Now the weight of every vertex of $Y \cup\left\{x_{2 k+1}\right\}$ is 0 and all remaining weights are equal
to 1 . Put 1 on the vertex $x_{2 k+1}$ and add 1 to the color of each edge incident to the vertex $x_{2 k+1}$. Now the weights of vertices which are not adjacent to $x_{2 k+1}$ remain unchanged, we add $1+1$ to the weight of every neighbor of $x_{2 k+1}$ and $(2 l+1) \cdot 1$ to the weight of the vertex $x_{2 k+1}$. Thus the weight of every vertex of $X$ is 1 and all weights of vertices of $Y$ are equal to 0 . Now we change the color 0 for 2, apply the addition in $\mathbb{N}$ and get an NESD total 2-coloring of $G$.

The following proposition is obvious.
Proposition 12. If every two adjacent vertices of $G$ have different degrees, then $\eta(G)=1$.

In some cases, the value of $\eta(G)$ can be determined exactly.
Corollary 13. For any integers $n, p \geq 1, \eta\left(K_{n, p}\right)=2$ for $n=p$ and $\eta\left(K_{n, p}\right)=1$ for $n \neq p$.

Proof. Suppose that $K_{n, n}$ has bipartition $(X, Y)$. If we color the vertices of $X$ by 1 and other vertices and edges of $K_{n, n}$ by 2 , we get an NESD total 2-coloring. For $n \neq p$ our result follows from Proposition 12.

## 4. Some Other Results

In [4] Karoński et al. proved the following result.
Theorem 14. Let $\Gamma$ be a finite abelian group of odd order $k$ and let $G$ be a $k$ colorable graph on $n \geq 3$ vertices. Then there exists a coloring $c$ of the edges of $G$ with the elements of $\Gamma$ such that the resulting vertex coloring $\sigma$ induced by $c$ is a proper coloring of $G$.
Corollary 15. Let $k$ be an odd integer and let $G$ be a connected $k$-colorable graph. Then $\eta(G) \leq k$.

Proof. If we color the edges of $G$ using the method described in [4] with the elements of $\mathbb{Z}_{k}=\mathbb{Z}_{2 l+1}$ and put 0 on the vertices of $G$, then we get an NESD total coloring of $G$ with the elements of $\Gamma=\mathbb{Z}_{k}$. Now we can obtain an NESD total $k$-coloring of $G$ by replacing 0 by $k$ and applying the addition in $\mathbb{N}$.

Thus the following corollary is true.
Corollary 16. If $G$ is a connected $k$-colorable graph, then $\eta(G) \leq k+1$.
As already mentioned, in [3], Kalkowski et al. showed that for every graph $G$ without components isomorphic to $K_{2}$ there exists a coloring of the edges of $G$ with the elements of $\{1, \ldots, 5\}$ such that the resulting vertex weighting is a proper vertex coloring of $G$. This implies at once the following corollary.

Corollary 17. If $G$ is a connected regular graph, then $\eta(G) \leq 5$.
Proof. Color the edges of $G$ with 5 colors in such a way that the obtained vertex coloring is proper. Afterwards put 1 on the vertices of $G$. All weights will increase by a constant.

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