# ON THE EDGE-HYPER-HAMILTONIAN LACEABILITY OF BALANCED HYPERCUBES 

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#### Abstract

The balanced hypercube $B H_{n}$, defined by Wu and Huang, is a variant of the hypercube network $Q_{n}$, and has been proved to have better properties than $Q_{n}$ with the same number of links and processors. For a bipartite graph $G=\left(V_{0} \cup V_{1}, E\right)$, we say $G$ is edge-hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_{i}, i \in\{0,1\}$, any edge $e \in E(G-v)$, there is a Hamiltonian path containing $e$ in $G-v$ between any two vertices of $V_{1-i}$. In this paper, we prove that $B H_{n}$ is edge-hyperHamiltonian laceable.


Keywords: balanced hypercubes, hyper-Hamiltonian laceability, edge-hyper-Hamiltonian laceability.
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## 1. Introduction

An interconnection network is usually represented by an undirected graph, where the vertices represent the processors and the edges represent the communication links between processors. Let $G=(V, E)$ be a simple undirected graph with

[^0]vertex set $V$ and edge set $E$. A graph $G=(V, E)$ is bipartite if $V=V_{0} \cup V_{1}$ and $V_{0} \cap V_{1}=\emptyset$ such that the two ends of any edge come from different set. For $e=$ $(u, v)$ (or alternatively $e=u v$ ), $u$ (resp. $v$ ) is said to be incident with $e$, and $e$ is said to be incident with $u$ and $v$. A path $P\left[v_{0}, v_{m}\right]=\left\langle v_{0}, v_{1}, \ldots, v_{m}\right\rangle$ is a sequence of distinct vertices from $v_{0}$ to $v_{m}$ such that two consecutive vertices are adjacent. A Hamiltonian path (resp. Hamiltonian cycle) of $G$ is a path (resp. cycle) that traverses each vertex of $G$ exactly once. For $x, y \in V$, a Hamiltonian path between $x$ and $y$ in $G$ is called an $(x, y)$-Hamiltonian path. A graph $G$ is Hamiltonian if it has a Hamiltonian cycle. In a Hamiltonian bipartite graph $G$, there exists no Hamiltonian path between two vertices in the same partite set. Simmons [9] introduced the notation of Hamiltonian laceability of a bipartite graph. A bipartite graph $G=\left(V_{0} \cup V_{1}, E\right)$ is Hamiltonian laceable if there is a Hamiltonian path between any two vertices $x$ and $y$ with $x \in V_{0}$ and $y \in V_{1}$. Hsieh et al. [3] extended this concept to strongly Hamiltonian laceable. A Hamiltonian laceable graph $G=\left(V_{0} \cup V_{1}, E\right)$ is strongly Hamiltonian laceable if there is a path of length $\left|V_{0} \cup V_{1}\right|-2$ between any two distinct vertices of the same partite set. Lewinter and Widulski [6] further proposed the concept of hyper-Hamiltonian laceability. A graph $G$ is hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_{i}, i \in\{0,1\}$, there exists a Hamiltonian path in $G-v$ between any pair of vertices in $V_{1-i}$. A graph $G$ is edge-hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_{i}, i \in\{0,1\}$, any edge $e \in E(G-v)$, there is a Hamiltonian path containing $e$ in $G-v$ between any pair of vertices in $V_{1-i} . G$ is a vertex transitive graph (resp. edge transitive graph), if for any two vertices $x$ and $y$ (resp. edges $e_{1}$ and $e_{2}$ ) of $G$, there is an automorphism $T$ of $G$ such that $T(x)=y$ (resp., $T\left(e_{1}\right)=e_{2}$ ). Some other definitions and notations not given in this paper are referred to [1, 10, 12].

Interconnection networks play an important role in parallel and distributed systems. The hypercube network has proved to be one of the most popular interconnection networks. The balanced hypercube $B H_{n}$, proposed by Huang and Wu [5], is a variant of the hypercube. Like hypercubes, the balanced hypercubes are bipartite, vertex-transitive and edge transitive [4, 11, 15]. The balanced hypercubes are superior to the hypercube in having smaller diameter, supporting an efficient reconfiguration without changing the adjacent relationship among tasks [11]. Plenty of properties of balanced hypercubes have been studied extensively $[2,4,5,8,14]$. Xu et al. [13] showed that the balanced hypercube is edge-bipancyclic and Hamiltonian laceable. Lv and Zhang [7] obtained that $B H_{n}$ is hyper-Hamiltonian laceable. This means that $B H_{n}$ is Hamiltonian laceable, and for any vertex $v \in V_{i}, i \in\{0,1\}$, there exists a Hamiltonian path in $G-v$ between any pair of vertices in $V_{1-i}$. So it is natural to propose the following problem.

For any vertex $v \in V_{i}, i \in\{0,1\}$, any edge $e \in E(G-v)$, does there exist a Hamiltonian path containing e in $G-v$ between any pair of vertices in $V_{1-i}$ ?

This is the main motivation of this paper, and our answer is yes.
This paper is organized as follows. Section 2 introduces some definitions of balanced hypercubes and their basic properties. The proof of our main result is presented in Section 3. In Section 4, we draw a conclusion of this paper.

## 2. Preliminaries

In the following ' + ' is an operation with modular 4.
Definition [5]. An $n$-dimensional balanced hypercube, denoted by $B H_{n}$, is defined as follows. For $n \geq 1, B H_{n}$ has $4^{n}$ vertices with addresses $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$, where $a_{i} \in\{0,1,2,3\}$ for each $0 \leq i \leq n-1$. For $1 \leq i \leq n-1$, an arbitrary vertex ( $a_{0}, \ldots, a_{i-1}, a_{i}, a_{i+1}, \ldots, a_{n-1}$ ) in $B H_{n}$ has the following $2 n$ neighbors:

$$
\begin{aligned}
& \left(a_{0} \pm 1, a_{1}, \ldots, a_{i-1}, a_{i}, a_{i+1}, \ldots, a_{n-1}\right), \text { and } \\
& \left(a_{0} \pm 1, a_{1}, \ldots, a_{i-1}, a_{i}+(-1)^{a_{0}}, a_{i+1}, \ldots, a_{n-1}\right) .
\end{aligned}
$$

$B H_{1}$ is a cycle of length 4 (see Figure 1), since $B H_{n}$ is vertex transitive and edge transitive, there are some automorphisms of $B H_{n}$, Figure 2 shows four graphs that are isomorphic to $\mathrm{BH}_{2}$.


Figure 1. $B H_{1}$.


Figure 2. Four automorphisms of $\mathrm{BH}_{2}$.

The balanced hypercube $B H_{n}$ can be recursively defined.
Definition. $B H_{n}$ is constructed recursively as follows:
(1) $B H_{1}$ is a cycle with 4 vertices labeled as $0,1,2,3$, respectively.
(2) For $n \geq 2, B H_{n}$ consists of four copies of $B H_{n-1}$, denoted by $B H_{n-1}^{i}$, for each integer $i$ with $0 \leq i \leq 3$. The vertex in $B H_{n-1}^{i}$ corresponding to a vertex $\left(a_{0}, a_{1}, \ldots, a_{n-2}\right)$ in $B H_{n-1}$ is denoted by $\left(a_{0}, a_{1}, \ldots, a_{n-2}, i\right)$, where $a_{j} \in\{0,1$, $2,3\}$ for every $0 \leq j \leq n-2$. Each vertex $\left(a_{0}, a_{1}, \ldots, a_{n-2}, i\right)$ of $B H_{n-1}^{i}$ has the following two extra neighbors:
$\left(a_{0} \pm 1, a_{1}, \ldots, a_{n-2}, i+1\right)$ in $B H_{n-1}^{i+1}$ if $a_{0}$ is even.
$\left(a_{0} \pm 1, a_{1}, \ldots, a_{n-2}, i-1\right)$ in $B H_{n-1}^{i-1}$ if $a_{0}$ is odd.
The first coordinate $a_{0}$ of vertex $\left(a_{0}, a_{1}, \ldots, a_{j}, \ldots, a_{n-1}\right)$ is named inner index, and the other coordinates $a_{j}(1 \leq j \leq n-1)$ are named $j$-dimension index. In [11], it is seen that the balanced hypercube is bipartite. The vertex sets

$$
V_{0}=\left\{x=\left(a_{0}, a_{1}, \ldots, a_{n-1}\right) \mid x \in V\left(B H_{n}\right), \text { and } a_{0} \text { is odd }\right\}
$$

and

$$
V_{1}=\left\{x=\left(a_{0}, a_{1}, \ldots, a_{n-1}\right) \mid x \in V\left(B H_{n}\right), \text { and } a_{0} \text { is even }\right\}
$$

give the desired partition.
In the following, we use black vertices to denote the vertices in $V_{0}$ and white vertices to denote the vertices in $V_{1}$. Now we classify the edges of $B H_{n}$. If two adjacent vertices $u, v$ differ only in the inner index, the edge $(u, v)$ is said to be a 0 -dimension edge and $v$ is a 0 -dimension neighbor of $u$. If two adjacent vertices $u, v$ not only differ in the inner index, but also differ in some $i$-dimension index $(1 \leq i \leq n-1)$, then the edge $(u, v)$ is said to be an $i$-dimension edge and $v$ is an $i$-dimension neighbor of $u$. Let $E_{i}$ denote the set of all edges of $i$-dimension edges for $i \in\{0,1,2, \ldots, n-1\}$. Then $E\left(B H_{n}\right)=\bigcup_{i=0}^{n-1} E_{i}$. For $i \in\{0,1,2,3\}$ and $1 \leq$ $j \leq n-1$, we use $B H_{n-1}^{j, i}$ to denote the ( $n-1$ )-dimension sub-balanced hypercubes of the $B H_{n}$ induced by all vertices labeled by $\left(a_{0}, a_{1}, \ldots, a_{j-1}, i, a_{j+1}, \ldots, a_{n-1}\right)$. Obviously, $B H_{n}-E_{j}=\bigcup_{i=0}^{3} B H_{n-1}^{j, i}$ and $B H_{n-1}^{j, i} \cong B H_{n-1}$. If $j=n-1, B H_{n-1}^{j, i}$ and $E_{j}$ are denoted by $B H_{n-1}^{i}$ and $E_{c}$, respectively, where $i \in\{0,1,2,3\}$. One has $B H_{n}-E_{c}=\bigcup_{i=0}^{3} B H_{n-1}^{i}$.

In the rest of this paper we often use $w_{i j}$ and $b_{i j}$ to denote white and black vertices in $B H_{n}$, respectively, where $i \in N$ and $j \in\{0,1,2,3\}$, and $j$ means that the corresponding vertex lies in $B H_{n-1}^{j}$. By definition, any white vertex in $B H_{n-1}^{j}$ has two black $(n-1)$-dimension neighbors in $B H_{n-1}^{j+1}, j \in\{0,1,2,3\}$.

Some basic properties of $B H_{n}$ are given below and will be used in the sequel.
Lemma 1 [11]. The balanced hypercube $B H_{n}$ is bipartite and vertex transitive.
Lemma 2 [15]. The balanced hypercube $B H_{n}$ is edge transitive.
Lemma 3 [13]. Let $(u, v)$ be an edge of $B H_{n}$. Then $(u, v)$ is contained in a cycle $C$ of length 8 in $B H_{n}$ such that $\left|E(C) \cap E\left(B H_{n-1}^{i}\right)\right|=1$, where $i \in\{0,1,2,3\}$.

Lemma 4 [11]. The vertices $\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$ and $\left(a_{0}+2, a_{1}, \ldots, a_{n-1}\right)$ of $B H_{n}$ have the same neighborhood.

Lemma 5 [13]. The balanced hypercube $B H_{n}$ is Hamiltonian laceable for $n \geq 1$.
Lemma 6 [7]. The balanced hypercube $B H_{n}$ is hyper-Hamiltonian laceable for $n \geq 1$.

## 3. Edge-Hyper-Hamiltonian Lacebility of Balanced Hypercube

Lemma 7. The balanced hypercube $\mathrm{BH}_{2}$ is edge-hyper-Hamiltonian laceable.
Proof. Let $u$ be any vertex of $B H_{2}$. From Lemma $1, B H_{2}$ is vertex transitive, therefore we may suppose $u=(0,0)$ without loss of generality. For any $e \in$ $B H_{2}-u$ and any two black vertices $x$ and $y$, we can prove there exists an $(x, y)$ Hamiltonian path containing $e$ in $\mathrm{BH}_{2}-u$. Here we only prove the result for $e=(1,0)(2,0)$ without loss of generality. By the relative positions of $e, x$ and $y$ in $\mathrm{BH}_{2}$, we distinguish into the following two cases.

Case 1. $x$ and $y$ are in the same $B H_{1}^{i}, i \in\{0,1,2,3\}$. Since there are only two black vertices in $B H_{1}^{i}, i \in\{0,1,2,3\}$, so if $x$ and $y$ are in the same $B H_{1}^{i}$, then one is $x$, the other is $y$. Let $x, y \in B H_{1}^{0}, x=(1,0), y=(3,0)$. Then $\langle(1,0)$, $(2,0),(1,1),(0,1),(3,1),(2,1),(1,2),(2,2),(3,2),(0,2),(3,3),(2,3),(1,3),(0,3)$, $(3,0)\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $\mathrm{BH}_{2}-u$.

Case 2. $x$ and $y$ are not in the same $B H_{1}^{i}, i \in\{0,1,2,3\}$. Let $x \in B H_{1}^{0}$, $y \in B H_{1}^{1}$. Suppose $x=(1,0), y=(1,1)$. Then $\langle(1,0),(2,0),(3,1),(0,1),(1,2)$, $(2,2),(3,3),(2,3),(3,0),(0,3),(1,3),(0,2),(3,2),(2,1),(1,1)\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $\mathrm{BH}_{2}-u$.

For other cases, we can similarly find an $(x, y)$-Hamiltonian path containing $e$ in $\mathrm{BH}_{2}-u$, their proofs are omitted here.

Theorem 8. The balanced hypercube $B H_{n}$ is edge-hyper-Hamiltonian laceable for $n \geq 2$.

Proof. We prove the result by induction on $n \geq 2$. The case $n=2$ follows from Lemma 7. Next we consider $n \geq 3$. Suppose that the theorem is true for $B H_{k}, 3 \leq k \leq n-1$. We suppose that $u \in V_{0}$ is a white vertex. For any edge $e \in B H_{n}-u$, then $e \in E\left(B H_{n-1}^{i}\right), i \in\{0,1,2,3\}$, or $e$ is an $(n-1)$-dimension edge. If $e$ is an $(n-1)$-dimension edge, since $B H_{n}$ is vertex and edge transitive, there is an automorphism of $B H_{n}$ so that $e \in E\left(B H_{n-1}^{i}\right), i \in\{0,1,2,3\}$, therefore in the following we only consider $e \in B H_{n-1}^{i}, i \in\{0,1,2,3\}$. For any pair of vertices in $V_{1}$, say $x$ and $y$, they are black vertices. By the relative positions of $e, x$ and $y$ in $B H_{n}$, we distinguish into the following cases.

Case 1. $e \in E\left(B H_{n-1}^{0}\right)$.
Subcase 1.1. $x$ and $y$ are in the same $B H_{n-1}^{i}, i \in\{0,1,2,3\}$.


Figure 3. Subcase 1.1.1.


Figure 4. Subcase 1.1.2.

Subcase 1.1.1. $x, y \in B H_{n-1}^{0}$ (see Figure 3). By the induction hypothesis, the graph $B H_{n-1}^{0}-u$ contains a Hamiltonian path $P_{0}$ from $x$ to $y$ containing $e$. Since $n \geq 3$, there exists an edge different from $e$, say $w_{00} b_{00} \in E\left(P_{0}\right)$, such that the removal of the edge $w_{00} b_{00}$ decomposes $P_{0}$ into two sections $P_{00}\left[x, w_{00}\right]$ and $P_{10}\left[b_{00}, y\right]$. Let $b_{01} \in B H_{n-1}^{1}$ (resp. $\left.w_{03} \in B H_{n-1}^{3}\right)$ be $(n-1)$-dimension neighbors of $w_{00}$ (resp. $b_{00}$ ). By Lemma $3, w_{00} b_{00}$ is contained in a cycle $C$ of length 8 in $B H_{n}$ such that $\left|E(C) \cap E\left(B H_{n-1}^{i}\right)\right|=1, i \in\{0,1,2,3\}$. We denote the cycle $C=\left(w_{00}, b_{01}, w_{01}, b_{02}, w_{02}, b_{03}, w_{03}, b_{00}, w_{00}\right)$. By Lemma $5, B H_{n}$ is Hamiltonian laceable for $n \geq 1$, so $B H_{n-1}^{i}$ contains a $\left(b_{0 i}, w_{0 i}\right)$-Hamiltonian path $P_{i}, i \in$ $\{1,2,3\}$. Then $\left\langle x, P_{00}, w_{00}, b_{01}, P_{1}, w_{01}, b_{02}, P_{2}, w_{02}, b_{03}, P_{3}, w_{03}, b_{00}, P_{10}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Subcase 1.1.2. $x, y \in B H_{n-1}^{1}$ (see Figure 4). Choose an arbitrary white vertex, say $w_{01}$, in $B H_{n-1}^{1}$. By Lemma 5, there exists an $\left(x, w_{01}\right)$-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$. Then there exists an edge, say $w_{11} y \in E\left(P_{1}\right)$, whose removal divides $P_{1}$ into two sections $P_{11}\left[x, w_{11}\right]$ and $P_{01}\left[y, w_{01}\right]$. Let $b_{02}$ and $b_{12}$ be $(n-1)$-dimension neighbors of $w_{01}$ and $w_{11}$ in $B H_{n-1}^{2}$ respectively. Since every white vertex of $B H_{n-1}^{1}$ has two $(n-1)$-dimension neighbors in $B H_{n-1}^{2}$, so we can always choose two vertices $b_{02}$ and $b_{12}$ such that $b_{02} \neq b_{12}$. Similarly, choose an arbitrary white vertex, say $w_{02}$, in $B H_{n-1}^{2}$. By Lemma 5 , there exists a $\left(b_{12}, w_{02}\right)$-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}$ via the edge $b_{02} w_{12}$. We delete $b_{02} w_{12}$, then $P_{2}$ is divided to two sections $P_{02}\left[w_{02}, b_{02}\right]$ and $P_{12}\left[w_{12}, b_{12}\right]$. Let $b_{03}$ and $b_{13}$ be $(n-1)$-dimension neighbors of $w_{02}$ and $w_{12}$ in $B H_{n-1}^{3}$ respectively, and $b_{03} \neq b_{13}$. Choose any white vertex $w_{03}$ in $B H_{n-1}^{3}$; by Lemma 5, there is a $\left(b_{13}, w_{03}\right)$-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}$. Therefore there exists an edge, say $b_{03} w_{13}$, in $P_{3}$ whose deletion divides $P_{3}$ into two parts $P_{03}\left[w_{03}, b_{03}\right]$ and $P_{13}\left[w_{13}, b_{13}\right]$. Let $b_{00}$ and $b_{10}$ be $(n-1)$-dimension neighbors of $w_{03}$ and $w_{13}$ in $B H_{n-1}^{1}$ respectively, and $b_{00} \neq b_{10}$. By the induction hypothesis, there exists a
$\left(b_{00}, b_{10}\right)$-Hamiltonian path $P_{0}$ containing $e$ in $B H_{n-1}^{0}-u$. Then $\left\langle x, P_{11}, w_{11}\right.$, $\left.b_{12}, P_{12}, w_{12}, b_{13}, P_{13}, w_{13}, b_{10}, P_{0}, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, P_{02}, b_{02}, w_{01}, P_{01}, y\right\rangle$ is an ( $x, y$ )-Hamiltonian path containing $e$ in $B H_{n}-u$.


Figure 5. Subcase 1.1.3.


Figure 6. Subcase 1.1.4.

Subcase 1.1.3. $x, y \in V\left(B H_{n-1}^{2}\right)$ (see Figure 5). Choose an arbitrary white vertex, say $w_{03}$, in $B H_{n-1}^{3}$. Let $b_{00}$ and $b_{20}$ be two $(n-1)$-dimension neighbors (different from $u$ ) of $w_{03}$ in $B H_{n-1}^{0}$. By the induction hypothesis, for any $e \in$ $B H_{n-1}^{0}$, the graph $B H_{n-1}^{0}-u$ contains a ( $b_{00}, b_{20}$ )-Hamiltonian path $P_{0}$ containing $e$. Now we choose two arbitrary white vertices in $B H_{n-1}^{2}$ with different ( $n-1$ )dimension neighbors in $B H_{n-1}^{3}$, say $w_{02}$ and $w_{12}$. By Lemma 6, there is a $\left(w_{02}, w_{12}\right)$-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}-x$. There exists an edge, say $y w_{22} \in$ $E\left(P_{2}\right)$, whose deletion divides $P_{2}$ into two sections $P_{12}\left[w_{12}, y\right]$ and $P_{02}\left[w_{22}, w_{02}\right]$. Let $b_{03}, b_{13}$ and $b_{23}$ (they are different from each other) be ( $n-1$ )-dimension neighbors of $w_{02}, w_{12}$ and $w_{22}$ in $B H_{n-1}^{3}$, respectively. By Lemma 6 , there exists a $\left(b_{03}, b_{23}\right)$-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}-w_{03}$. Then there exists an edge $w_{13} b_{13} \in E\left(P_{3}\right)$, whose deletion divides $P_{3}$ into two sections $P_{03}\left[b_{03}, w_{13}\right]$ and $P_{13}\left[b_{13}, b_{23}\right]$. Let $b_{10}$ be an $(n-1)$-dimension neighbor of $w_{13}$ in $B H_{n-1}^{0}$ such that $b_{10}$ is not incident with $e$. Then there exists an edge $b_{10} w_{00} \in E\left(P_{0}\right)$, whose deletion divides $P_{0}$ into two sections $P_{00}\left[b_{00}, w_{00}\right]$ and $P_{10}\left[b_{10}, b_{20}\right]$. Let $b_{01}$ and $w_{01}$ be ( $n-1$ )-dimension neighbors of $w_{00}$ and $x$ in $B H_{n-1}^{1}$, respectively. By Lemma 5 , there exists a ( $w_{01}, b_{01}$ )-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$. Then $\left\langle x, w_{01}, P_{1}, b_{01}\right.$, $\left.w_{00}, P_{00}, b_{00}, w_{03}, b_{20}, P_{10}, b_{10}, w_{13}, P_{03}, b_{03}, w_{02}, P_{02}, w_{22}, b_{23}, P_{13}, b_{13}, w_{12}, P_{12}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Subcase 1.1.4. $x, y \in V\left(B H_{n-1}^{3}\right)$ (see Figure 6). Let $w_{03}$ and $w_{13}$ be any two white vertices in $B H_{n-1}^{3}$, with different ( $n-1$ )-dimension neighbors in $B H_{n-1}^{0}$. By the induction hypothesis, for any edge, say $y w_{23} \in E\left(B H_{n-1}^{3}-x\right)$, there exists a ( $w_{03}, w_{13}$ )-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}-x$. We choose such an edge $y w_{23}$ so that $b_{20}$ is not incident with $e$ and $b_{20} \neq b_{00}$, where $b_{00}$ and $b_{20}$ are $(n-1)$-dimension neighbors of $w_{03}$ and $w_{23}$ in $B H_{n-1}^{0}$, respectively. Now $P_{3}=$ $\left\langle w_{03}, P_{03}\left[w_{03}, w_{23}\right], w_{23}, y, P_{13}\left[y, w_{13}\right], w_{13}\right\rangle$. Let $b_{10}$ be an $(n-1)$-dimension neigh-
bor of $w_{13}$ in $B H_{n-1}^{0}$. By the induction hypothesis, there exists a $\left(b_{00}, b_{10}\right)$ Hamiltonian path $P_{0}$ containing $e$ in $B H_{n-1}^{0}-u$. Then there exists an edge, say $w_{00} b_{20} \in E\left(P_{0}\right)$, whose deletion divides $P_{0}$ into two sections $P_{00}\left[b_{00}, w_{00}\right]$ and $P_{10}\left[b_{20}, b_{10}\right]$. Since $w_{03}$ and $w_{13}$ are arbitrary vertices, so we can always choose two such vertices so that $w_{00} b_{20} \neq e$. Let $b_{01}$ and $w_{02}$ be $(n-1)$-dimension neighbors of $w_{00}$ and $x$ respectively. Furthermore, let $b_{02}$ be any black vertex in $B H_{n-1}^{2}$ and $w_{01}$ be an $(n-1)$-dimension neighbor of $b_{02}$ in $B H_{n-1}^{1}$. By Lemma 5 , there exists a $\left(b_{0 i}, w_{0 i}\right)$-Hamiltonian path $P_{i}$ in $B H_{n-1}^{i}, i \in\{1,2\}$. Then $\left\langle x, w_{02}, P_{2}, b_{02}, w_{01}, P_{1}, b_{01}, w_{00}, P_{00}, b_{00}, w_{03}, P_{03}, w_{23}, b_{20}, P_{10}, b_{10}, w_{13}, P_{13}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Subcase 1.2. $x \in V\left(B H_{n-1}^{i}\right), y \in V\left(B H_{n-1}^{j}\right), 0 \leq i<j \leq 3$.
Subcase 1.2.1. $x \in V\left(B H_{n-1}^{0}\right), y \in V\left(B H_{n-1}^{1}\right)$. We choose any black vertex in $B H_{n-1}^{0}$, say $b_{00}$. By the induction hypothesis, there exists an $\left(x, b_{00}\right)$-Hamiltonian path $P_{0}$ containing $e$ in $B H_{n-1}^{0}-u$. Let $w_{03}$ be an $(n-1)$-dimension neighbor of $b_{00}$ in $B H_{n-1}^{3}$. By Lemma 5, for any black vertex $b_{03} \in B H_{n-1}^{3}$, there exists a $\left(w_{03}, b_{03}\right)$-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}$. Similarly, for any white vertex $w_{01} \in B H_{n-1}^{1}$, there exists a $\left(y, w_{01}\right)$-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$. Let $b_{02}$ and $w_{02}$ be $(n-1)$-dimension neighbors of $w_{01}$ and $b_{03}$ in $B H_{n-1}^{2}$, respectively. By Lemma 5, there exists a $\left(b_{02}, w_{02}\right)$-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}$. Then $\left\langle x, P_{0}, b_{00}, w_{03}, P_{3}, b_{03}, w_{02}, P_{2}, b_{02}, w_{01}, P_{1}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.


Figure 7. Subcase 1.2.2.


Figure 8. Subcase 1.2.3.

Subcase 1.2.2. $x \in V\left(B H_{n-1}^{0}\right), y \in V\left(B H_{n-1}^{2}\right)$ (see Figure 7). Choose an arbitrary black vertex, say $b_{00}$ in $B H_{n-1}^{0}$. By the induction hypothesis, the graph $B H_{n-1}^{0}-u$ contains an $\left(x, b_{00}\right)$-Hamiltonian path $P_{0}$ containing $e$. Then there exists an edge, say $w_{00} b_{10} \in E\left(P_{0}\right)$ so that $w_{00} b_{10} \neq e$ and $P_{0}=\left\langle x, P_{00}\left[x, w_{00}\right]\right.$, $\left.w_{00}, b_{10}, P_{10}\left[b_{10}, b_{00}\right], b_{00}\right\rangle$. Let $b_{01} \in B H_{n-1}^{1}, w_{03} \in B H_{n-1}^{3}$ and $w_{13} \in B H_{n-1}^{3}$ be $(n-1)$-dimension neighbors of $w_{00}, b_{00}$ and $b_{10}$, respectively. Let $b_{03}$ be an arbitrary black vertex in $B H_{n-1}^{3}$. By Lemma 5 , there exists a $\left(b_{03}, w_{13}\right)$-Hamiltonian
path $P_{3}$. Hence there exists an edge in $P_{3}$, say $w_{03} b_{13}$, whose deletion divides $P_{3}$ into two sections $P_{03}\left[b_{03}, w_{03}\right]$ and $P_{13}\left[b_{13}, w_{13}\right]$. Let $w_{02}$ and $w_{12}$ be $(n-1)$ dimension neighbors of $b_{03}$ and $b_{13}$ in $B H_{n-1}^{2}$ respectively, and $w_{02} \neq w_{12}$. Since $y$ is black vertex in $B H_{n-1}^{2}$, in view of Lemma 5, there exists a ( $w_{02}, y$ )-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}$. Then there exists an edge in $P_{2}$, say $b_{02} w_{12}$, whose deletion divides $P_{2}$ into two sections $P_{02}\left[w_{02}, b_{02}\right]$ and $P_{12}\left[w_{12}, y\right]$. Let $w_{01}$ be $(n-1)$ dimension neighbor of $b_{02}$ in $B H_{n-1}^{1}$. By Lemma 5, there exists a $\left(b_{01}, w_{01}\right)$ Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$. Then $\left\langle x, P_{00}, w_{00}, b_{01}, P_{1}, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}\right.$, $\left.P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Subcase 1.2.3. $x \in V\left(B H_{n-1}^{0}\right), y \in V\left(B H_{n-1}^{3}\right)$ (see Figure 8). Assume $b_{10}$ is an arbitrary black vertex in $B H_{n-1}^{0}$. By the induction hypothesis, $B H_{n-1}^{0}-u$ contains an $\left(x, b_{10}\right)$-Hamiltonian path $P_{0}$ containing $e$. Then there exists an edge different from $e$, say $w_{00} b_{00} \in E\left(P_{0}\right)$, whose deletion divides $P_{0}$ into two sections $P_{00}\left[x, w_{00}\right]$ and $P_{10}\left[b_{00}, b_{10}\right]$. Let $b_{01}$ be an ( $n-1$ )-dimension neighbor of $w_{00}$ in $B H_{n-1}^{1}, w_{03}$ and $w_{13}$ be $(n-1)$-dimension neighbors of $b_{00}$ and $b_{10}$ in $B H_{n-1}^{3}$, respectively. By Lemma 5, there exists a ( $w_{03}, y$ )-Hamiltonian path $P_{3}$ in $\mathrm{BH}_{n-1}^{3}$. Then there exists an edge, say $w_{13} b_{03} \in E\left(P_{3}\right)$, whose deletion divides $P_{3}$ into two sections $P_{03}\left[w_{03}, b_{03}\right]$ and $P_{13}\left[w_{13}, y\right]$. Let $w_{02}$ be an $(n-1)$-dimension neighbor of $b_{03}$ in $B H_{n-1}^{2}$. For any black vertex in $B H_{n-1}^{2}$, say $b_{02}$, we assume $w_{01}$ is an $(n-1)$-dimension neighbor of $b_{02}$ in $B H_{n-1}^{1}$. By Lemma 5, there is a $\left(b_{0 i}, w_{0 i}\right)$-Hamiltonian path $P_{i}$ in $B H_{n-1}^{i}, i \in\{1,2\}$. Then $\left\langle x, P_{00}, w_{00}, b_{01}, P_{1}, w_{01}, b_{02}, P_{2}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, y\right\rangle$ is an ( $x, y$ )-Hamiltonian path containing $e$ in $B H_{n}-u$.


Figure 9. Subcase 1.2.4.


Figure 10. Subcase 1.2.6.

Subcase 1.2.4. $x \in V\left(B H_{n-1}^{1}\right), y \in V\left(B H_{n-1}^{2}\right)$ (see Figure 9). Assume $w_{01} \in$ $B H_{n-1}^{1}$ and $w_{02} \in B H_{n-1}^{2}$ are arbitrary white vertices. By Lemma 5 , there exists an $\left(x, w_{01}\right)$-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$ and a ( $y, w_{02}$ )-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}$. Let $b_{02} \neq y$ be an $(n-1)$-dimension neighbor of $w_{01}$ in $B H_{n-1}^{2}$. Then there exists an edge, say $b_{02} w_{12} \in E\left(P_{2}\right)$, whose deletion divides $P_{2}$ into two sec-
tions $P_{02}\left[b_{02}, w_{02}\right]$ and $P_{12}\left[y, w_{12}\right]$. Let $b_{03}$ and $b_{13}$ be $(n-1)$-dimension neighbors of $w_{02}$ and $w_{12}$, respectively. $b_{03} \neq b_{13}$ since every white vertex in $B H_{n-1}^{2}$ has two black ( $n-1$ )-dimension neighbors in $B H_{n-1}^{3}$. Let $w_{13}$ be any white vertex in $B H_{n-1}^{3}$. By Lemma 5, there exists a $\left(b_{03}, w_{13}\right)$-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}$. Then there exists an edge, say $w_{03} b_{13} \in E\left(P_{3}\right)$, whose deletion divides $P_{3}$ into two sections $P_{03}\left[b_{03}, w_{03}\right]$ and $P_{13}\left[b_{13}, w_{13}\right]$. Let $b_{00}$ and $b_{10}$ be an $(n-1)$-dimension neighbors of $w_{03}$ and $w_{13}$ in $B H_{n-1}^{0}$, respectively and $b_{00} \neq b_{10}$. By the induction hypothesis, there is a $\left(b_{00}, b_{10}\right)$-Hamiltonian path $P_{0}$ containing $e$ in $B H_{n-1}^{0}-u$. Then $\left\langle x, P_{1}, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{0}, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y\right\rangle$ is an ( $x, y$ )-Hamiltonian path containing $e$ in $B H_{n}-u$.

Subcase 1.2.5. $x \in V\left(B H_{n-1}^{1}\right), y \in V\left(B H_{n-1}^{3}\right)$. Assume that $b_{00}$ and $b_{10}$ are any two black vertices in $B H_{n-1}^{0}$. By the induction hypothesis, there exists a $\left(b_{00}, b_{10}\right)$-Hamiltonian path $P_{0}$ containing $e$ in $B H_{n-1}^{0}-u$. Let $w_{03}$ and $w_{13}$ be an $(n-1)$-dimension neighbors of $b_{00}$ and $b_{10}$ in $B H_{n-1}^{3}$ respectively, and $w_{03} \neq w_{13}$. By Lemma 5, there is a ( $y, w_{03}$ )-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}$. Hence there exists an edge, say $w_{13} b_{03} \in E\left(P_{3}\right)$, whose deletion divides $P_{3}$ into two sections $P_{13}\left[y, w_{13}\right]$ and $P_{03}\left[b_{03}, w_{03}\right]$. Let $w_{02}$ be an $(n-1)$ dimension neighbor of $b_{03}$ in $B H_{n-1}^{2}, b_{02}$ be any black vertex in $B H_{n-1}^{2}, w_{01}$ be an $(n-1)$-dimension neighbor of $b_{02}$ in $B H_{n-1}^{1}$. By Lemma 5 , there exists an $\left(x, w_{01}\right)$-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$ and ( $b_{02}, w_{02}$ )-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}$. Then $\left\langle x, P_{1}, w_{01}, b_{02}, P_{2}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{0}, b_{10}, w_{13}, P_{13}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Subcase 1.2.6. $x \in V\left(B H_{n-1}^{2}\right), y \in V\left(B H_{n-1}^{3}\right)$ (see Figure 10). Let $w_{03}$ be an arbitrary white vertex in $B H_{n-1}^{3}, b_{00}$ and $b_{10}$ be two ( $n-1$ )-dimension neighbors of $w_{03}$ in $B H_{n-1}^{0}$. By the induction hypothesis, for any $e$, there exists a ( $b_{00}, b_{10}$ )-Hamiltonian path $P_{0}$ containing $e$ in $B H_{n-1}^{0}-u$. Then there exists an edge in $P_{0}$, say $w_{00} b_{20}$ and $w_{00} b_{20} \neq e$, whose deletion divides $P_{0}$ into two sections $P_{00}\left[b_{00}, w_{00}\right]$ and $P_{10}\left[b_{20}, b_{10}\right]$. Let $b_{01} \in B H_{n-1}^{1}\left(\right.$ resp. $\left.w_{13} \in B H_{n-1}^{3}\right)$ be an $(n-1)$-dimension neighbor of $w_{00}$ (resp. $b_{20}$ ). Here we choose $w_{13} \neq w_{03}$ since every black vertex in $B H_{n-1}^{0}$ has two white $(n-1)$-dimension neighbors in $B H_{n-1}^{3}$. By Lemma 6, there exists a $\left(w_{03}, w_{13}\right)$-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}-y$. Hence there exists an edge, say $w_{03} b_{03} \in E\left(P_{3}\right)$, so that $P_{3}=\left\langle w_{03}, b_{03}\right.$, $\left.P_{03}\left[b_{03}, w_{13}\right], w_{13}\right\rangle$. Let $w_{02}$ and $w_{12}$ be $(n-1)$-dimension neighbors of $b_{03}$ and $y$ in $B H_{n-1}^{2}$ respectively, and $w_{02} \neq w_{12}$. By Lemma 6 , there exists a $\left(w_{02}, w_{12}\right)$ Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}-x$. Let $w_{01} \in B H_{n-1}^{1}$ be an $(n-1)$-dimension neighbor of $x$. By Lemma 5, there exists a ( $b_{01}, w_{01}$ )-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$. Thus it follows that $\left\langle x, w_{01}, P_{1}, b_{01}, w_{00}, P_{00}, b_{00}, w_{03}, b_{10}, P_{10}, b_{20}, w_{13}\right.$, $\left.P_{03}, b_{03}, w_{02}, P_{2}, w_{12}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Case 2. $e \in B H_{n-1}^{1}$. The proof of this case is similar to that of Case 1, and is omitted.

Case 3. $e \in B H_{n-1}^{2}$. If $x, y \in V\left(B H_{n-1}^{2}\right)$, then since $x y \neq e$, at most one of $x$ and $y$ can be incident with $e$. Hence the proof is similar to that of Subcase 1.1.3 since we can exchange $x$ and $y$ in this case when $y$ is incident with $e$.


Figure 11


Figure 12. Case 4.

For $x \in V\left(B H_{n-1}^{2}\right), y \in V\left(B H_{n-1}^{3}\right)$ (see Figure 11), if $x$ is not incident with $e$, then the proof is similar to Subcase 1.2.6. If $x$ is incident with $e$, we may assume $w_{02}$ is an $(n-1)$-dimension neighbor of $y$ in $B H_{n-1}^{2}$ and $b_{03}$ is another $(n-1)$-dimension neighbor of $w_{02}$ in $B H_{n-1}^{3}$. Choose any white vertex, say $w_{13}$ in $B H_{n-1}^{3}$ and let $b_{10}$ and $b_{20}$ be two ( $n-1$ )-dimension neighbors of $w_{13}$ in $B H_{n-1}^{0}$. By Lemma 6, there exists a $\left(b_{03}, y\right)$-Hamiltonian path $P_{3}$ in $B H_{n-1}^{3}-w_{13}$. Then there exists an edge, say $y w_{03} \in E\left(P_{3}\right)$, and $P_{3}=$ $\left\langle y, w_{03}, P_{03}\left[w_{03}, b_{03}\right], b_{03}\right\rangle$. Let $b_{00}$ be the $(n-1)$-dimension neighbor of $w_{03}$ in $B H_{n-1}^{0}$ and $b_{00} \neq b_{20}$ as $w_{03}$ has two ( $n-1$ )-dimension neighbors in $B H_{n-1}^{0}$. Similarly, by Lemma 6 , there exists a $\left(b_{10}, b_{20}\right)$-Hamiltonian path $P_{0}$ in $B H_{n-1}^{0}-u$. Then there exists an edge, say $b_{00} w_{00} \in E\left(P_{0}\right)$, whose deletion divides $P_{0}$ into two sections $P_{00}\left[b_{10}, b_{00}\right]$ and $P_{10}\left[w_{00}, b_{20}\right]$. Let $b_{01}$ be one ( $n-1$ )-dimension neighbor of $w_{00}$ in $B H_{n-1}^{1}$. By Lemma 5, for any white vertex $w_{01}$ in $B H_{n-1}^{1}$, there exists a $\left(b_{01}, w_{01}\right)$-Hamiltonian path $P_{1}$. Let $b_{02} \in B H_{n-1}^{2}$ be one $(n-1)$ dimension neighbor of $w_{01}$. By the induction hypothesis, for any $e$, there exists an $\left(x, b_{02}\right)$-Hamiltonian path $P_{2}$ containing $e$ in $B H_{n-1}^{2}-w_{02}$. Then $\left\langle x, P_{2}, b_{02}\right.$, $\left.w_{01}, P_{1}, b_{01}, w_{00}, P_{10}, b_{20}, w_{13}, b_{10}, P_{00}, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Since the other subcases are similar to the corresponding subcases of Case 1, we omit their proofs.

Case 4. $e \in B H_{n-1}^{3}$. We just prove the subcase that $x \in V\left(B H_{n-1}^{2}\right), y \in$ $V\left(B H_{n-1}^{3}\right)$, the other cases are similar to the corresponding subcases of Case 1.

Subcase 4.1. $x \in V\left(B H_{n-1}^{2}\right), y \in V\left(B H_{n-1}^{3}\right)$ (see Figure 12). For any $e \in$ $E\left(B H_{n-1}^{3}\right)$ ), if $y$ is not incident with $e$, then in the proof of Subcase 1.2.6, we choose any white vertex $w_{03}$ which is not incident with $e$, then the remainder of the proof is similar to that of Subcase 1.2.6. If $y$ is incident with $e$, as $n \geq 3$,
there exist two white vertices in $B H_{n-1}^{3}$, say $w_{03}$ and $w_{13}$, who are not adjacent to $y$ and have a common $(n-1)$-dimension neighbor, say $b_{00} \in B H_{n-1}^{0}$. For any black vertex in $B H_{n-1}^{3}$, say $b_{13}$, by induction hypothesis, the graph $B H_{n-1}^{3}-w_{13}$ contains a ( $y, b_{13}$ )-Hamiltonian path $P_{3}$ containing $e$. Thus there exists an edge, say $w_{03} b_{03} \in E\left(P_{3}\right)$ and $w_{03} b_{03} \neq e$ since $w_{03}$ is not adjacent to $y$ and $y$ is incident with $e$. Now $P_{3}=\left\langle y, P_{03}\left[y, b_{03}\right], b_{03}, w_{03}, P_{13}\left[w_{03}, b_{13}\right], b_{13}\right\rangle$. Let $b_{10}$ be another $(n-1)$-dimension neighbor of $w_{13}$ in $B H_{n-1}^{0}, w_{02}$ and $w_{12}$ be $(n-1)$-dimension neighbors of $b_{03}$ and $b_{13}$ in $B H_{n-1}^{2}$ respectively, and $w_{02} \neq w_{12}$ since every black vertex in $B H_{n-1}^{3}$ has two ( $n-1$ )-dimension neighbors in $B H_{n-1}^{2}$. By Lemma 6, there exists one ( $w_{02}, w_{12}$ )-Hamiltonian path $P_{2}$ in $B H_{n-1}^{2}-x$, and one $\left(b_{00}, b_{10}\right)$ Hamiltonian path $P_{0}$ in $B H_{n-1}^{0}-u$. Then there exists an edge $w_{00} b_{00} \in E\left(P_{0}\right)$ such that $P_{0}=\left\langle b_{10}, P_{00}\left[b_{10}, w_{00}\right], w_{00}, b_{00}\right\rangle$. Let $w_{01}$ and $b_{01}$ be $(n-1)$-dimension neighbors of $x$ and $w_{00}$ in $B H_{n-1}^{1}$ respectively. By Lemma 5 , there exists one $\left(b_{01}, w_{01}\right)$-Hamiltonian path $P_{1}$ in $B H_{n-1}^{1}$. Then $\left\langle x, w_{01}, P_{1}, b_{01}, w_{00}, P_{00}, b_{10}\right.$, $\left.w_{13}, b_{00}, w_{03}, P_{13}, b_{13}, w_{12}, P_{2}, w_{02}, b_{03}, P_{03}, y\right\rangle$ is an $(x, y)$-Hamiltonian path containing $e$ in $B H_{n}-u$.

Combining the above cases, the proof of this theorem is completed.

## 4. Conclusion

The balance hypercube $B H_{n}$, proposed by Huang and Wu [5], is a variant of the hypercube that gives better performance with the same number of edges and vertices. It has been shown that the balanced hypercube $B H_{n}$ is Hamiltonian laceable and hyper-Hamiltonian laceable for $n \geq 1$. In this paper, we show that, for any vertex $v \in V_{i}, i \in\{0,1\}$, and any $e \in E\left(B H_{n}-v\right)$, there exists a Hamiltonian path containing $e$ in $G-v$ between any pair of vertices in $V_{1-i}$.

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