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ON THE EDGE-HYPER-HAMILTONIAN LACEABILITY OF BALANCED HYPERCUBES

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Abstract

The balanced hypercube BH_n, defined by Wu and Huang, is a variant of the hypercube network Q_n, and has been proved to have better properties than Q_n with the same number of links and processors. For a bipartite graph $G = (V_0 \cup V_1, E)$, we say G is edge-hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, there is a Hamiltonian path containing e in $G - v$ between any two vertices of V_{1-i}. In this paper, we prove that BH_n is edge-hyper-Hamiltonian laceable.

Keywords: balanced hypercubes, hyper-Hamiltonian laceability, edge-hyper-Hamiltonian laceability.

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1. Introduction

An interconnection network is usually represented by an undirected graph, where the vertices represent the processors and the edges represent the communication links between processors. Let $G = (V, E)$ be a simple undirected graph with

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vertex set V and edge set E. A graph $G = (V, E)$ is *bipartite* if $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$ such that the two ends of any edge come from different set. For $e = (u, v)$ (or alternatively $e = uv$), u (resp. v) is said to be *incident* with e, and e is said to be *incident* with u and v. A path $P[v_0, v_m] = \langle v_0, v_1, \ldots, v_m\rangle$ is a sequence of distinct vertices from v_0 to v_m such that two consecutive vertices are adjacent. A *Hamiltonian path* (resp. *Hamiltonian cycle*) of G is a path (resp. cycle) that traverses each vertex of G exactly once. For $x, y \in V$, a Hamiltonian path between x and y in G is called an (x, y)-*Hamiltonian path*. A graph G is *Hamiltonian* if it has a Hamiltonian cycle. In a Hamiltonian bipartite graph G, there exists no Hamiltonian path between two vertices in the same partite set. Simmons [9] introduced the notation of Hamiltonian laceability of a bipartite graph. A bipartite graph $G = (V_0 \cup V_1, E)$ is *Hamiltonian laceable* if there is a Hamiltonian path between any two vertices x and y with $x \in V_0$ and $y \in V_1$. Hsieh *et al.* [3] extended this concept to strongly Hamiltonian laceable. A Hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is *strongly Hamiltonian laceable* if there is a path of length $|V_0 \cup V_1| - 2$ between any two distinct vertices of the same partite set. Lewinter and Widulski [6] further proposed the concept of hyper-Hamiltonian laceability. A graph G is *hyper-Hamiltonian laceable* if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in V_{1-i}. A graph G is *edge-hyper-Hamiltonian laceable* if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G-v)$, there is a Hamiltonian path containing e in $G - v$ between any pair of vertices in V_{1-i}. G is a *vertex transitive graph* (resp. *edge transitive graph*), if for any two vertices x and y (resp. edges e_1 and e_2) of G, there is an automorphism T of G such that $T(x) = y$ (resp., $T(e_1) = e_2$). Some other definitions and notations not given in this paper are referred to [1, 10, 12].

Interconnection networks play an important role in parallel and distributed systems. The hypercube network has proved to be one of the most popular interconnection networks. The balanced hypercube BH_n, proposed by Huang and Wu [5], is a variant of the hypercube. Like hypercubes, the balanced hypercubes are bipartite, vertex-transitive and edge transitive [4, 11, 15]. The balanced hypercubes are superior to the hypercube in having smaller diameter, supporting an efficient reconfiguration without changing the adjacent relationship among tasks [11]. Plenty of properties of balanced hypercubes have been studied extensively [2, 4, 5, 8, 14]. Xu *et al.* [13] showed that the balanced hypercube is edge-bipancyclic and Hamiltonian laceable. Lv and Zhang [7] obtained that BH_n is hyper-Hamiltonian laceable. This means that BH_n is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in V_{1-i}. So it is natural to propose the following problem.

For any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, does there exist a Hamiltonian path containing e in $G - v$ between any pair of vertices in V_{1-i}?

This is the main motivation of this paper, and our answer is yes.

This paper is organized as follows. Section 2 introduces some definitions of balanced hypercubes and their basic properties. The proof of our main result is presented in Section 3. In Section 4, we draw a conclusion of this paper.

2. Preliminaries

In the following $'+'$ is an operation with modular 4.

Definition [5]**.** An *n-dimensional balanced hypercube*, denoted by BH_n, is defined as follows. For $n \geq 1$, BH_n has 4^n vertices with addresses $(a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for each $0 \leq i \leq n-1$. For $1 \leq i \leq n-1$, an arbitrary vertex $(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ in BH_n has the following $2n$ neighbors:

$$(a_0 \pm 1, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \text{ and}$$

$$(a_0 \pm 1, a_1, \ldots, a_{i-1}, a_i + (-1)^{a_0}, a_{i+1}, \ldots, a_{n-1}).$$

BH_1 is a cycle of length 4 (see Figure 1), since BH_n is vertex transitive and edge transitive, there are some automorphisms of BH_n, Figure 2 shows four graphs that are isomorphic to BH_2.

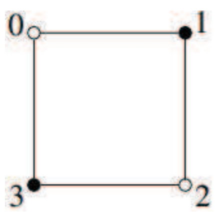

Figure 1. BH_1.

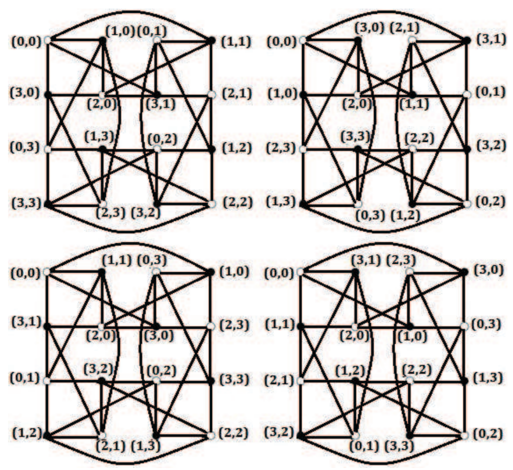

Figure 2. Four automorphisms of BH_2.

The balanced hypercube BH_n can be recursively defined.

Definition. BH_n is constructed recursively as follows:

(1) BH_1 is a cycle with 4 vertices labeled as $0, 1, 2, 3$, respectively.

(2) For $n \geq 2$, BH_n consists of four copies of BH_{n-1}, denoted by BH_{n-1}^i, for each integer i with $0 \leq i \leq 3$. The vertex in BH_{n-1}^i corresponding to a vertex $(a_0, a_1, \ldots, a_{n-2})$ in BH_{n-1} is denoted by $(a_0, a_1, \ldots, a_{n-2}, i)$, where $a_j \in \{0, 1, 2, 3\}$ for every $0 \leq j \leq n-2$. Each vertex $(a_0, a_1, \ldots, a_{n-2}, i)$ of BH_{n-1}^i has the following two extra neighbors:

$(a_0 \pm 1, a_1, \ldots, a_{n-2}, i+1)$ in BH_{n-1}^{i+1} if a_0 is even.

$(a_0 \pm 1, a_1, \ldots, a_{n-2}, i-1)$ in BH_{n-1}^{i-1} if a_0 is odd.

The first coordinate a_0 of vertex $(a_0, a_1, \ldots, a_j, \ldots, a_{n-1})$ is named *inner index*, and the other coordinates a_j $(1 \leq j \leq n-1)$ are named *j-dimension index*. In [11], it is seen that the balanced hypercube is bipartite. The vertex sets

$$V_0 = \{x = (a_0, a_1, \ldots, a_{n-1}) \,|\, x \in V(BH_n), \text{and } a_0 \text{ is odd}\}$$

and

$$V_1 = \{x = (a_0, a_1, \ldots, a_{n-1}) \,|\, x \in V(BH_n), \text{and } a_0 \text{ is even}\}$$

give the desired partition.

In the following, we use black vertices to denote the vertices in V_0 and white vertices to denote the vertices in V_1. Now we classify the edges of BH_n. If two adjacent vertices u, v differ only in the inner index, the edge (u, v) is said to be a 0-*dimension edge* and v is a 0-*dimension neighbor* of u. If two adjacent vertices u, v not only differ in the inner index, but also differ in some i-dimension index $(1 \leq i \leq n-1)$, then the edge (u, v) is said to be an *i-dimension edge* and v is an *i-dimension neighbor* of u. Let E_i denote the set of all edges of i-dimension edges for $i \in \{0, 1, 2, \ldots, n-1\}$. Then $E(BH_n) = \bigcup_{i=0}^{n-1} E_i$. For $i \in \{0, 1, 2, 3\}$ and $1 \leq j \leq n-1$, we use $BH_{n-1}^{j,i}$ to denote the $(n-1)$-dimension sub-balanced hypercubes of the BH_n induced by all vertices labeled by $(a_0, a_1, \ldots, a_{j-1}, i, a_{j+1}, \ldots, a_{n-1})$. Obviously, $BH_n - E_j = \bigcup_{i=0}^{3} BH_{n-1}^{j,i}$ and $BH_{n-1}^{j,i} \cong BH_{n-1}$. If $j = n-1$, $BH_{n-1}^{j,i}$ and E_j are denoted by BH_{n-1}^i and E_c, respectively, where $i \in \{0, 1, 2, 3\}$. One has $BH_n - E_c = \bigcup_{i=0}^{3} BH_{n-1}^i$.

In the rest of this paper we often use w_{ij} and b_{ij} to denote white and black vertices in BH_n, respectively, where $i \in N$ and $j \in \{0, 1, 2, 3\}$, and j means that the corresponding vertex lies in BH_{n-1}^j. By definition, any white vertex in BH_{n-1}^j has two black $(n-1)$-dimension neighbors in BH_{n-1}^{j+1}, $j \in \{0, 1, 2, 3\}$.

Some basic properties of BH_n are given below and will be used in the sequel.

Lemma 1 [11]**.** *The balanced hypercube BH_n is bipartite and vertex transitive.*

Lemma 2 [15]**.** *The balanced hypercube BH_n is edge transitive.*

Lemma 3 [13]**.** *Let (u, v) be an edge of BH_n. Then (u, v) is contained in a cycle C of length 8 in BH_n such that $|E(C) \cap E(BH_{n-1}^i)| = 1$, where $i \in \{0, 1, 2, 3\}$.*

Lemma 4 [11]**.** *The vertices* $(a_0, a_1, \ldots, a_{n-1})$ *and* $(a_0+2, a_1, \ldots, a_{n-1})$ *of* BH_n *have the same neighborhood.*

Lemma 5 [13]**.** *The balanced hypercube* BH_n *is Hamiltonian laceable for* $n \geq 1$.

Lemma 6 [7]**.** *The balanced hypercube* BH_n *is hyper-Hamiltonian laceable for* $n \geq 1$.

3. Edge-Hyper-Hamiltonian Lacebility of Balanced Hypercube

Lemma 7. *The balanced hypercube* BH_2 *is edge-hyper-Hamiltonian laceable.*

Proof. Let u be any vertex of BH_2. From Lemma 1, BH_2 is vertex transitive, therefore we may suppose $u = (0, 0)$ without loss of generality. For any $e \in BH_2 - u$ and any two black vertices x and y, we can prove there exists an (x, y)-Hamiltonian path containing e in $BH_2 - u$. Here we only prove the result for $e = (1, 0)(2, 0)$ without loss of generality. By the relative positions of e, x and y in BH_2, we distinguish into the following two cases.

Case 1. x and y are in the same BH_1^i, $i \in \{0, 1, 2, 3\}$. Since there are only two black vertices in BH_1^i, $i \in \{0, 1, 2, 3\}$, so if x and y are in the same BH_1^i, then one is x, the other is y. Let $x, y \in BH_1^0$, $x = (1, 0), y = (3, 0)$. Then $\langle(1, 0), (2, 0), (1, 1), (0, 1), (3, 1), (2, 1), (1, 2), (2, 2), (3, 2), (0, 2), (3, 3), (2, 3), (1, 3), (0, 3), (3, 0)\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_2 - u$.

Case 2. x and y are not in the same BH_1^i, $i \in \{0, 1, 2, 3\}$. Let $x \in BH_1^0$, $y \in BH_1^1$. Suppose $x = (1, 0), y = (1, 1)$. Then $\langle(1, 0), (2, 0), (3, 1), (0, 1), (1, 2), (2, 2), (3, 3), (2, 3), (3, 0), (0, 3), (1, 3), (0, 2), (3, 2), (2, 1), (1, 1)\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_2 - u$.

For other cases, we can similarly find an (x, y)-Hamiltonian path containing e in $BH_2 - u$, their proofs are omitted here. ■

Theorem 8. *The balanced hypercube* BH_n *is edge-hyper-Hamiltonian laceable for* $n \geq 2$.

Proof. We prove the result by induction on $n \geq 2$. The case $n = 2$ follows from Lemma 7. Next we consider $n \geq 3$. Suppose that the theorem is true for BH_k, $3 \leq k \leq n-1$. We suppose that $u \in V_0$ is a white vertex. For any edge $e \in BH_n - u$, then $e \in E(BH_{n-1}^i)$, $i \in \{0, 1, 2, 3\}$, or e is an $(n-1)$-dimension edge. If e is an $(n-1)$-dimension edge, since BH_n is vertex and edge transitive, there is an automorphism of BH_n so that $e \in E(BH_{n-1}^i)$, $i \in \{0, 1, 2, 3\}$, therefore in the following we only consider $e \in BH_{n-1}^i, i \in \{0, 1, 2, 3\}$. For any pair of vertices in V_1, say x and y, they are black vertices. By the relative positions of e, x and y in BH_n, we distinguish into the following cases.

Case 1. $e \in E(BH^0_{n-1})$.

Subcase 1.1. x and y are in the same $BH^i_{n-1}, i \in \{0,1,2,3\}$.

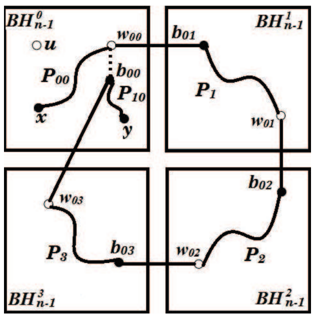

Figure 3. Subcase 1.1.1.

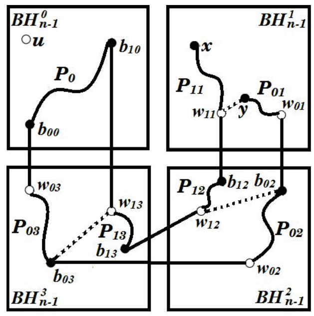

Figure 4. Subcase 1.1.2.

Subcase 1.1.1. $x, y \in BH^0_{n-1}$ (see Figure 3). By the induction hypothesis, the graph $BH^0_{n-1} - u$ contains a Hamiltonian path P_0 from x to y containing e. Since $n \geq 3$, there exists an edge different from e, say $w_{00}b_{00} \in E(P_0)$, such that the removal of the edge $w_{00}b_{00}$ decomposes P_0 into two sections $P_{00}[x, w_{00}]$ and $P_{10}[b_{00}, y]$. Let $b_{01} \in BH^1_{n-1}$ (resp. $w_{03} \in BH^3_{n-1}$) be $(n-1)$-dimension neighbors of w_{00} (resp. b_{00}). By Lemma 3, $w_{00}b_{00}$ is contained in a cycle C of length 8 in BH_n such that $|E(C) \cap E(BH^i_{n-1})| = 1$, $i \in \{0,1,2,3\}$. We denote the cycle $C = (w_{00}, b_{01}, w_{01}, b_{02}, w_{02}, b_{03}, w_{03}, b_{00}, w_{00})$. By Lemma 5, BH_n is Hamiltonian laceable for $n \geq 1$, so BH^i_{n-1} contains a (b_{0i}, w_{0i})-Hamiltonian path P_i, $i \in \{1,2,3\}$. Then $\langle x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_3, w_{03}, b_{00}, P_{10}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.1.2. $x, y \in BH^1_{n-1}$ (see Figure 4). Choose an arbitrary white vertex, say w_{01}, in BH^1_{n-1}. By Lemma 5, there exists an (x, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then there exists an edge, say $w_{11}y \in E(P_1)$, whose removal divides P_1 into two sections $P_{11}[x, w_{11}]$ and $P_{01}[y, w_{01}]$. Let b_{02} and b_{12} be $(n-1)$-dimension neighbors of w_{01} and w_{11} in BH^2_{n-1} respectively. Since every white vertex of BH^1_{n-1} has two $(n-1)$-dimension neighbors in BH^2_{n-1}, so we can always choose two vertices b_{02} and b_{12} such that $b_{02} \neq b_{12}$. Similarly, choose an arbitrary white vertex, say w_{02}, in BH^2_{n-1}. By Lemma 5, there exists a (b_{12}, w_{02})-Hamiltonian path P_2 in BH^2_{n-1} via the edge $b_{02}w_{12}$. We delete $b_{02}w_{12}$, then P_2 is divided to two sections $P_{02}[w_{02}, b_{02}]$ and $P_{12}[w_{12}, b_{12}]$. Let b_{03} and b_{13} be $(n-1)$-dimension neighbors of w_{02} and w_{12} in BH^3_{n-1} respectively, and $b_{03} \neq b_{13}$. Choose any white vertex w_{03} in BH^3_{n-1}; by Lemma 5, there is a (b_{13}, w_{03})-Hamiltonian path P_3 in BH^3_{n-1}. Therefore there exists an edge, say $b_{03}w_{13}$, in P_3 whose deletion divides P_3 into two parts $P_{03}[w_{03}, b_{03}]$ and $P_{13}[w_{13}, b_{13}]$. Let b_{00} and b_{10} be $(n-1)$-dimension neighbors of w_{03} and w_{13} in BH^1_{n-1} respectively, and $b_{00} \neq b_{10}$. By the induction hypothesis, there exists a

(b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then $\langle x,\ P_{11},\ w_{11},\ b_{12},\ P_{12},\ w_{12},\ b_{13},\ P_{13},\ w_{13},\ b_{10}, P_0, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, P_{02}, b_{02}, w_{01}, P_{01}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

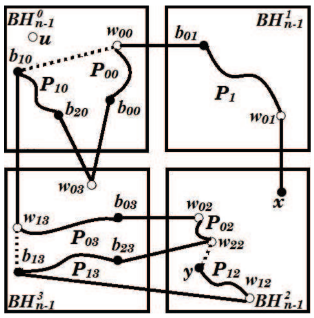

Figure 5. Subcase 1.1.3.

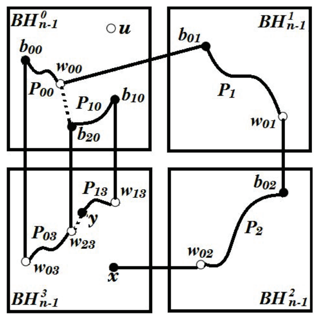

Figure 6. Subcase 1.1.4.

Subcase 1.1.3. $x, y \in V(BH^2_{n-1})$ (see Figure 5). Choose an arbitrary white vertex, say w_{03}, in BH^3_{n-1}. Let b_{00} and b_{20} be two $(n-1)$-dimension neighbors (different from u) of w_{03} in BH^0_{n-1}. By the induction hypothesis, for any $e \in BH^0_{n-1}$, the graph $BH^0_{n-1} - u$ contains a (b_{00}, b_{20})-Hamiltonian path P_0 containing e. Now we choose two arbitrary white vertices in BH^2_{n-1} with different $(n-1)$-dimension neighbors in BH^3_{n-1}, say w_{02} and w_{12}. By Lemma 6, there is a (w_{02}, w_{12})-Hamiltonian path P_2 in $BH^2_{n-1} - x$. There exists an edge, say $yw_{22} \in E(P_2)$, whose deletion divides P_2 into two sections $P_{12}[w_{12}, y]$ and $P_{02}[w_{22}, w_{02}]$. Let b_{03}, b_{13} and b_{23} (they are different from each other) be $(n-1)$-dimension neighbors of w_{02}, w_{12} and w_{22} in BH^3_{n-1}, respectively. By Lemma 6, there exists a (b_{03}, b_{23})-Hamiltonian path P_3 in $BH^3_{n-1} - w_{03}$. Then there exists an edge $w_{13}b_{13} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{03}[b_{03}, w_{13}]$ and $P_{13}[b_{13}, b_{23}]$. Let b_{10} be an $(n-1)$-dimension neighbor of w_{13} in BH^0_{n-1} such that b_{10} is not incident with e. Then there exists an edge $b_{10}w_{00} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{10}, b_{20}]$. Let b_{01} and w_{01} be $(n-1)$-dimension neighbors of w_{00} and x in BH^1_{n-1}, respectively. By Lemma 5, there exists a (w_{01}, b_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then $\langle x, w_{01},\ P_1,\ b_{01},\ w_{00},\ P_{00}, b_{00}, w_{03}, b_{20}, P_{10}, b_{10}, w_{13}, P_{03}, b_{03}, w_{02}, P_{02}, w_{22}, b_{23}, P_{13},\ b_{13},\ w_{12},\ P_{12}, y\rangle$ is an (x, y) -Hamiltonian path containing e in $BH_n - u$.

Subcase 1.1.4. $x, y \in V(BH^3_{n-1})$ (see Figure 6). Let w_{03} and w_{13} be any two white vertices in BH^3_{n-1}, with different $(n-1)$-dimension neighbors in BH^0_{n-1}. By the induction hypothesis, for any edge, say $yw_{23} \in E(BH^3_{n-1} - x)$, there exists a (w_{03}, w_{13})-Hamiltonian path P_3 in $BH^3_{n-1} - x$. We choose such an edge yw_{23} so that b_{20} is not incident with e and $b_{20} \neq b_{00}$, where b_{00} and b_{20} are $(n-1)$-dimension neighbors of w_{03} and w_{23} in BH^0_{n-1}, respectively. Now $P_3 = \langle w_{03}, P_{03}[w_{03}, w_{23}], w_{23}, y, P_{13}[y, w_{13}], w_{13}\rangle$. Let b_{10} be an $(n-1)$-dimension neighbor of w_{13} in BH^0_{n-1}. By the induction hypothesis, there exists a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then there exists an edge, say $w_{00}b_{20} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{20}, b_{10}]$. Since w_{03} and w_{13} are arbitrary vertices, so we can always choose two such vertices so that $w_{00}b_{20} \neq e$. Let b_{01} and w_{02} be $(n-1)$-dimension neighbors of w_{00} and x respectively. Furthermore, let b_{02} be any black vertex in BH^2_{n-1} and w_{01} be an $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there exists a (b_{0i}, w_{0i})-Hamiltonian path P_i in BH^i_{n-1}, $i \in \{1, 2\}$. Then $\langle x, w_{02}, P_2, b_{02}, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{00}, w_{03}, P_{03}, w_{23}, b_{20}, P_{10}, b_{10}, w_{13}, P_{13}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2. $x \in V(BH^i_{n-1})$, $y \in V(BH^j_{n-1})$, $0 \leq i < j \leq 3$.

Subcase 1.2.1. $x \in V(BH^0_{n-1})$, $y \in V(BH^1_{n-1})$. We choose any black vertex in BH^0_{n-1}, say b_{00}. By the induction hypothesis, there exists an (x, b_{00})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Let w_{03} be an $(n-1)$-dimension neighbor of b_{00} in BH^3_{n-1}. By Lemma 5, for any black vertex $b_{03} \in BH^3_{n-1}$, there exists a (w_{03}, b_{03})-Hamiltonian path P_3 in BH^3_{n-1}. Similarly, for any white vertex $w_{01} \in BH^1_{n-1}$, there exists a (y, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Let b_{02} and w_{02} be $(n-1)$-dimension neighbors of w_{01} and b_{03} in BH^2_{n-1}, respectively. By Lemma 5, there exists a (b_{02}, w_{02})-Hamiltonian path P_2 in BH^2_{n-1}. Then $\langle x, P_0, b_{00}, w_{03}, P_3, b_{03}, w_{02}, P_2, b_{02}, w_{01}, P_1, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

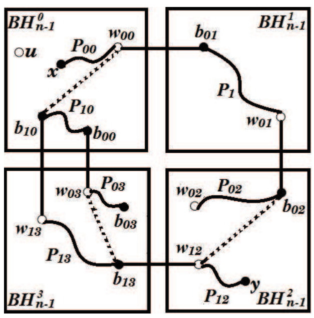

Figure 7. Subcase 1.2.2.

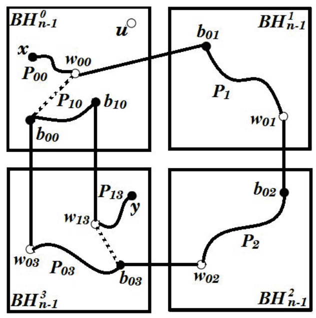

Figure 8. Subcase 1.2.3.

Subcase 1.2.2. $x \in V(BH^0_{n-1})$, $y \in V(BH^2_{n-1})$ (see Figure 7). Choose an arbitrary black vertex, say b_{00} in BH^0_{n-1}. By the induction hypothesis, the graph $BH^0_{n-1} - u$ contains an (x, b_{00})-Hamiltonian path P_0 containing e. Then there exists an edge, say $w_{00}b_{10} \in E(P_0)$ so that $w_{00}b_{10} \neq e$ and $P_0 = \langle x, P_{00}[x, w_{00}], w_{00}, b_{10}, P_{10}[b_{10}, b_{00}], b_{00}\rangle$. Let $b_{01} \in BH^1_{n-1}, w_{03} \in BH^3_{n-1}$ and $w_{13} \in BH^3_{n-1}$ be $(n-1)$-dimension neighbors of w_{00}, b_{00} and b_{10}, respectively. Let b_{03} be an arbitrary black vertex in BH^3_{n-1}. By Lemma 5, there exists a (b_{03}, w_{13})-Hamiltonian path P_3. Hence there exists an edge in P_3, say $w_{03}b_{13}$, whose deletion divides P_3 into two sections $P_{03}[b_{03}, w_{03}]$ and $P_{13}[b_{13}, w_{13}]$. Let w_{02} and w_{12} be $(n-1)$-dimension neighbors of b_{03} and b_{13} in BH^2_{n-1} respectively, and $w_{02} \neq w_{12}$. Since y is black vertex in BH^2_{n-1}, in view of Lemma 5, there exists a (w_{02}, y)-Hamiltonian path P_2 in BH^2_{n-1}. Then there exists an edge in P_2, say $b_{02}w_{12}$, whose deletion divides P_2 into two sections $P_{02}[w_{02}, b_{02}]$ and $P_{12}[w_{12}, y]$. Let w_{01} be $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there exists a (b_{01}, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then $\langle x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2.3. $x \in V(BH^0_{n-1})$, $y \in V(BH^3_{n-1})$ (see Figure 8). Assume b_{10} is an arbitrary black vertex in BH^0_{n-1}. By the induction hypothesis, $BH^0_{n-1} - u$ contains an (x, b_{10})-Hamiltonian path P_0 containing e. Then there exists an edge different from e, say $w_{00}b_{00} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[x, w_{00}]$ and $P_{10}[b_{00}, b_{10}]$. Let b_{01} be an $(n-1)$-dimension neighbor of w_{00} in BH^1_{n-1}, w_{03} and w_{13} be $(n-1)$-dimension neighbors of b_{00} and b_{10} in BH^3_{n-1}, respectively. By Lemma 5, there exists a (w_{03}, y)-Hamiltonian path P_3 in BH^3_{n-1}. Then there exists an edge, say $w_{13}b_{03} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{03}[w_{03}, b_{03}]$ and $P_{13}[w_{13}, y]$. Let w_{02} be an $(n-1)$-dimension neighbor of b_{03} in BH^2_{n-1}. For any black vertex in BH^2_{n-1}, say b_{02}, we assume w_{01} is an $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there is a (b_{0i}, w_{0i})-Hamiltonian path P_i in BH^i_{n-1}, $i \in \{1, 2\}$. Then $\langle x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

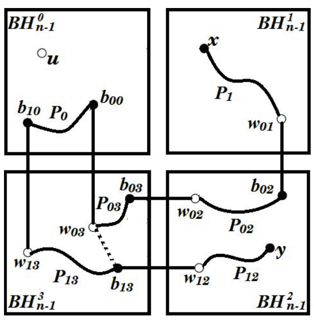

Figure 9. Subcase 1.2.4.

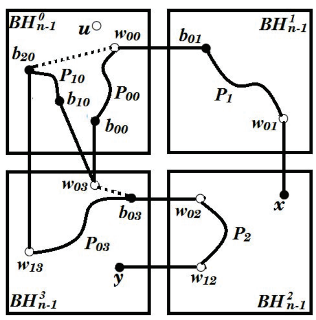

Figure 10. Subcase 1.2.6.

Subcase 1.2.4. $x \in V(BH^1_{n-1})$, $y \in V(BH^2_{n-1})$ (see Figure 9). Assume $w_{01} \in BH^1_{n-1}$ and $w_{02} \in BH^2_{n-1}$ are arbitrary white vertices. By Lemma 5, there exists an (x, w_{01})-Hamiltonian path P_1 in BH^1_{n-1} and a (y, w_{02})-Hamiltonian path P_2 in BH^2_{n-1}. Let $b_{02} \neq y$ be an $(n-1)$-dimension neighbor of w_{01} in BH^2_{n-1}. Then there exists an edge, say $b_{02}w_{12} \in E(P_2)$, whose deletion divides P_2 into two sections $P_{02}[b_{02}, w_{02}]$ and $P_{12}[y, w_{12}]$. Let b_{03} and b_{13} be $(n-1)$-dimension neighbors of w_{02} and w_{12}, respectively. $b_{03} \neq b_{13}$ since every white vertex in BH^2_{n-1} has two black $(n-1)$-dimension neighbors in BH^3_{n-1}. Let w_{13} be any white vertex in BH^3_{n-1}. By Lemma 5, there exists a (b_{03}, w_{13})-Hamiltonian path P_3 in BH^3_{n-1}. Then there exists an edge, say $w_{03}b_{13} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{03}[b_{03}, w_{03}]$ and $P_{13}[b_{13}, w_{13}]$. Let b_{00} and b_{10} be an $(n-1)$-dimension neighbors of w_{03} and w_{13} in BH^0_{n-1}, respectively and $b_{00} \neq b_{10}$. By the induction hypothesis, there is a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then $\langle x, P_1, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_0, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2.5. $x \in V(BH^1_{n-1})$, $y \in V(BH^3_{n-1})$. Assume that b_{00} and b_{10} are any two black vertices in BH^0_{n-1}. By the induction hypothesis, there exists a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Let w_{03} and w_{13} be an $(n-1)$-dimension neighbors of b_{00} and b_{10} in BH^3_{n-1} respectively, and $w_{03} \neq w_{13}$. By Lemma 5, there is a (y, w_{03})-Hamiltonian path P_3 in BH^3_{n-1}. Hence there exists an edge, say $w_{13}b_{03} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{13}[y, w_{13}]$ and $P_{03}[b_{03}, w_{03}]$. Let w_{02} be an $(n-1)$-dimension neighbor of b_{03} in BH^2_{n-1}, b_{02} be any black vertex in BH^2_{n-1}, w_{01} be an $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there exists an (x, w_{01})-Hamiltonian path P_1 in BH^1_{n-1} and (b_{02}, w_{02})-Hamiltonian path P_2 in BH^2_{n-1}. Then $\langle x, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_0, b_{10}, w_{13}, P_{13}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2.6. $x \in V(BH^2_{n-1})$, $y \in V(BH^3_{n-1})$ (see Figure 10). Let w_{03} be an arbitrary white vertex in BH^3_{n-1}, b_{00} and b_{10} be two $(n-1)$-dimension neighbors of w_{03} in BH^0_{n-1}. By the induction hypothesis, for any e, there exists a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then there exists an edge in P_0, say $w_{00}b_{20}$ and $w_{00}b_{20} \neq e$, whose deletion divides P_0 into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{20}, b_{10}]$. Let $b_{01} \in BH^1_{n-1}$ (resp. $w_{13} \in BH^3_{n-1}$) be an $(n-1)$-dimension neighbor of w_{00} (resp. b_{20}). Here we choose $w_{13} \neq w_{03}$ since every black vertex in BH^0_{n-1} has two white $(n-1)$-dimension neighbors in BH^3_{n-1}. By Lemma 6, there exists a (w_{03}, w_{13})-Hamiltonian path P_3 in $BH^3_{n-1} - y$. Hence there exists an edge, say $w_{03}b_{03} \in E(P_3)$, so that $P_3 = \langle w_{03}, b_{03}, P_{03}[b_{03}, w_{13}], w_{13}\rangle$. Let w_{02} and w_{12} be $(n-1)$-dimension neighbors of b_{03} and y in BH^2_{n-1} respectively, and $w_{02} \neq w_{12}$. By Lemma 6, there exists a (w_{02}, w_{12})-Hamiltonian path P_2 in $BH^2_{n-1} - x$. Let $w_{01} \in BH^1_{n-1}$ be an $(n-1)$-dimension neighbor of x. By Lemma 5, there exists a (b_{01}, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Thus it follows that $\langle x, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{00}, w_{03}, b_{10}, P_{10}, b_{20}, w_{13}, P_{03}, b_{03}, w_{02}, P_2, w_{12}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Case 2. $e \in BH^1_{n-1}$. The proof of this case is similar to that of Case 1, and is omitted.

Case 3. $e \in BH^2_{n-1}$. If $x, y \in V(BH^2_{n-1})$, then since $xy \neq e$, at most one of x and y can be incident with e. Hence the proof is similar to that of Subcase 1.1.3 since we can exchange x and y in this case when y is incident with e.

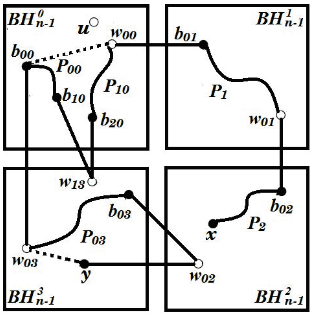

Figure 11

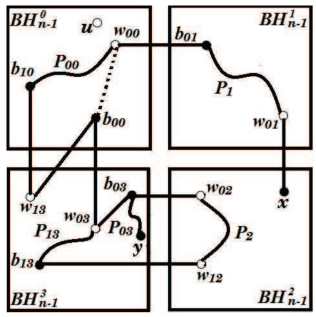

Figure 12. Case 4.

For $x \in V(BH^2_{n-1}), y \in V(BH^3_{n-1})$ (see Figure 11), if x is not incident with e, then the proof is similar to Subcase 1.2.6. If x is incident with e, we may assume w_{02} is an $(n-1)$-dimension neighbor of y in BH^2_{n-1} and b_{03} is another $(n-1)$-dimension neighbor of w_{02} in BH^3_{n-1}. Choose any white vertex, say w_{13} in BH^3_{n-1} and let b_{10} and b_{20} be two $(n-1)$-dimension neighbors of w_{13} in BH^0_{n-1}. By Lemma 6, there exists a (b_{03}, y)-Hamiltonian path P_3 in $BH^3_{n-1} - w_{13}$. Then there exists an edge, say $yw_{03} \in E(P_3)$, and $P_3 = \langle y, w_{03}, P_{03}[w_{03}, b_{03}], b_{03}\rangle$. Let b_{00} be the $(n-1)$-dimension neighbor of w_{03} in BH^0_{n-1} and $b_{00} \neq b_{20}$ as w_{03} has two $(n-1)$-dimension neighbors in BH^0_{n-1}. Similarly, by Lemma 6, there exists a (b_{10}, b_{20})-Hamiltonian path P_0 in $BH^0_{n-1} - u$. Then there exists an edge, say $b_{00}w_{00} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[b_{10}, b_{00}]$ and $P_{10}[w_{00}, b_{20}]$. Let b_{01} be one $(n-1)$-dimension neighbor of w_{00} in BH^1_{n-1}. By Lemma 5, for any white vertex w_{01} in BH^1_{n-1}, there exists a (b_{01}, w_{01})-Hamiltonian path P_1. Let $b_{02} \in BH^2_{n-1}$ be one $(n-1)$-dimension neighbor of w_{01}. By the induction hypothesis, for any e, there exists an (x, b_{02})-Hamiltonian path P_2 containing e in $BH^2_{n-1} - w_{02}$. Then $\langle x, P_2, b_{02}, w_{01}, P_1, b_{01}, w_{00}, P_{10}, b_{20}, w_{13}, b_{10}, P_{00}, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Since the other subcases are similar to the corresponding subcases of Case 1, we omit their proofs.

Case 4. $e \in BH^3_{n-1}$. We just prove the subcase that $x \in V(BH^2_{n-1}), y \in V(BH^3_{n-1})$, the other cases are similar to the corresponding subcases of Case 1.

Subcase 4.1. $x \in V(BH^2_{n-1})$, $y \in V(BH^3_{n-1})$ (see Figure 12). For any $e \in E(BH^3_{n-1}))$, if y is not incident with e, then in the proof of Subcase 1.2.6, we choose any white vertex w_{03} which is not incident with e, then the remainder of the proof is similar to that of Subcase 1.2.6. If y is incident with e, as $n \geq 3$, there exist two white vertices in BH^3_{n-1}, say w_{03} and w_{13}, who are not adjacent to y and have a common $(n-1)$-dimension neighbor, say $b_{00} \in BH^0_{n-1}$. For any black vertex in BH^3_{n-1}, say b_{13}, by induction hypothesis, the graph $BH^3_{n-1} - w_{13}$ contains a (y, b_{13})-Hamiltonian path P_3 containing e. Thus there exists an edge, say $w_{03}b_{03} \in E(P_3)$ and $w_{03}b_{03} \neq e$ since w_{03} is not adjacent to y and y is incident with e. Now $P_3 = \langle y, P_{03}[y, b_{03}], b_{03}, w_{03}, P_{13}[w_{03}, b_{13}], b_{13}\rangle$. Let b_{10} be another $(n-1)$-dimension neighbor of w_{13} in BH^0_{n-1}, w_{02} and w_{12} be $(n-1)$-dimension neighbors of b_{03} and b_{13} in BH^2_{n-1} respectively, and $w_{02} \neq w_{12}$ since every black vertex in BH^3_{n-1} has two $(n-1)$-dimension neighbors in BH^2_{n-1}. By Lemma 6, there exists one (w_{02}, w_{12})-Hamiltonian path P_2 in $BH^2_{n-1} - x$, and one (b_{00}, b_{10})-Hamiltonian path P_0 in $BH^0_{n-1} - u$. Then there exists an edge $w_{00}b_{00} \in E(P_0)$ such that $P_0 = \langle b_{10}, P_{00}[b_{10}, w_{00}], w_{00}, b_{00}\rangle$. Let w_{01} and b_{01} be $(n-1)$-dimension neighbors of x and w_{00} in BH^1_{n-1} respectively. By Lemma 5, there exists one (b_{01}, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then $\langle x, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{10}, w_{13}, b_{00}, w_{03}, P_{13}, b_{13}, w_{12}, P_2, w_{02}, b_{03}, P_{03}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Combining the above cases, the proof of this theorem is completed. ■

4. Conclusion

The balance hypercube BH_n, proposed by Huang and Wu [5], is a variant of the hypercube that gives better performance with the same number of edges and vertices. It has been shown that the balanced hypercube BH_n is Hamiltonian laceable and hyper-Hamiltonian laceable for $n \geq 1$. In this paper, we show that, for any vertex $v \in V_i, i \in \{0, 1\}$, and any $e \in E(BH_n - v)$, there exists a Hamiltonian path containing e in $G - v$ between any pair of vertices in V_{1-i}.

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