Discussiones Mathematicae
Graph Theory 36 (2016) 805–817
doi:10.7151/dmgt.1908

ON THE EDGE-HYPER-HAMILTONIAN LACEABILITY OF BALANCED HYPERCUBES

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Abstract

The balanced hypercube BH_n, defined by Wu and Huang, is a variant of the hypercube network Q_n, and has been proved to have better properties than Q_n with the same number of links and processors. For a bipartite graph $G = (V_0 \cup V_1, E)$, we say G is edge-hyper-Hamiltonian laceable if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, there is a Hamiltonian path containing e in $G - v$ between any two vertices of V_{1-i}. In this paper, we prove that BH_n is edge-hyper-Hamiltonian laceable.

Keywords: balanced hypercubes, hyper-Hamiltonian laceability, edge-hyper-Hamiltonian laceability.

2010 Mathematics Subject Classification: 05C38, 94C15.

1. Introduction

An interconnection network is usually represented by an undirected graph, where the vertices represent the processors and the edges represent the communication links between processors. Let $G = (V, E)$ be a simple undirected graph with

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vertex set V and edge set E. A graph $G = (V, E)$ is *bipartite* if $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$ such that the two ends of any edge come from different set. For $e = (u, v)$ (or alternatively $e = uv$), u (resp. v) is said to be *incident* with e, and e is said to be *incident* with u and v. A path $P[v_0, v_m] = \langle v_0, v_1, \ldots, v_m \rangle$ is a sequence of distinct vertices from v_0 to v_m such that two consecutive vertices are adjacent. A *Hamiltonian path* (resp. *Hamiltonian cycle*) of G is a path (resp. cycle) that traverses each vertex of G exactly once. For $x, y \in V$, a Hamiltonian path between x and y in G is called an (x, y)-*Hamiltonian path*. A graph G is *Hamiltonian* if it has a Hamiltonian cycle. In a Hamiltonian bipartite graph G, there exists no Hamiltonian path between two vertices in the same partite set. Simmons [9] introduced the notation of Hamiltonian laceability of a bipartite graph. A bipartite graph $G = (V_0 \cup V_1, E)$ is *Hamiltonian laceable* if there is a Hamiltonian path between any two vertices x and y with $x \in V_0$ and $y \in V_1$. Hsieh *et al.* [3] extended this concept to strongly Hamiltonian laceable. A Hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is *strongly Hamiltonian laceable* if there is a path of length $|V_0 \cup V_1| - 2$ between any two distinct vertices of the same partite set. Lewinter and Widulski [6] further proposed the concept of hyper-Hamiltonian laceability. A graph G is *hyper-Hamiltonian laceable* if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in V_{1-i}. A graph G is *edge-hyper-Hamiltonian laceable* if it is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G-v)$, there is a Hamiltonian path containing e in $G - v$ between any pair of vertices in V_{1-i}. G is a *vertex transitive graph* (resp. *edge transitive graph*), if for any two vertices x and y (resp. edges e_1 and e_2) of G, there is an automorphism T of G such that $T(x) = y$ (resp., $T(e_1) = e_2$). Some other definitions and notations not given in this paper are referred to [1, 10, 12].

Interconnection networks play an important role in parallel and distributed systems. The hypercube network has proved to be one of the most popular interconnection networks. The balanced hypercube BH_n, proposed by Huang and Wu [5], is a variant of the hypercube. Like hypercubes, the balanced hypercubes are bipartite, vertex-transitive and edge transitive [4, 11, 15]. The balanced hypercubes are superior to the hypercube in having smaller diameter, supporting an efficient reconfiguration without changing the adjacent relationship among tasks [11]. Plenty of properties of balanced hypercubes have been studied extensively [2, 4, 5, 8, 14]. Xu *et al.* [13] showed that the balanced hypercube is edge-bipancyclic and Hamiltonian laceable. Lv and Zhang [7] obtained that BH_n is hyper-Hamiltonian laceable. This means that BH_n is Hamiltonian laceable, and for any vertex $v \in V_i, i \in \{0, 1\}$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in V_{1-i}. So it is natural to propose the following problem.

For any vertex $v \in V_i, i \in \{0, 1\}$, any edge $e \in E(G - v)$, does there exist a Hamiltonian path containing e in $G - v$ between any pair of vertices in V_{1-i}?

This is the main motivation of this paper, and our answer is yes.

This paper is organized as follows. Section 2 introduces some definitions of balanced hypercubes and their basic properties. The proof of our main result is presented in Section 3. In Section 4, we draw a conclusion of this paper.

2. Preliminaries

In the following $'+'$ is an operation with modular 4.

Definition [5]**.** An *n-dimensional balanced hypercube*, denoted by BH_n, is defined as follows. For $n \geq 1$, BH_n has 4^n vertices with addresses $(a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for each $0 \leq i \leq n-1$. For $1 \leq i \leq n-1$, an arbitrary vertex $(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ in BH_n has the following $2n$ neighbors:

$$(a_0 \pm 1, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \text{ and}$$

$$(a_0 \pm 1, a_1, \ldots, a_{i-1}, a_i + (-1)^{a_0}, a_{i+1}, \ldots, a_{n-1}).$$

BH_1 is a cycle of length 4 (see Figure 1), since BH_n is vertex transitive and edge transitive, there are some automorphisms of BH_n, Figure 2 shows four graphs that are isomorphic to BH_2.

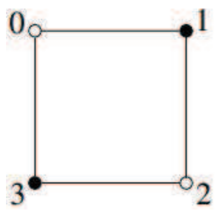

Figure 1. BH_1.

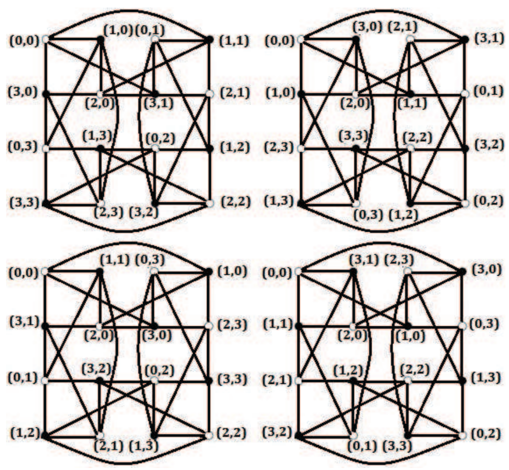

Figure 2. Four automorphisms of BH_2.

The balanced hypercube BH_n can be recursively defined.

Definition. BH_n is constructed recursively as follows:

(1) BH_1 is a cycle with 4 vertices labeled as $0, 1, 2, 3$, respectively.

(2) For $n \geq 2$, BH_n consists of four copies of BH_{n-1}, denoted by BH_{n-1}^i, for each integer i with $0 \leq i \leq 3$. The vertex in BH_{n-1}^i corresponding to a vertex $(a_0, a_1, \ldots, a_{n-2})$ in BH_{n-1} is denoted by $(a_0, a_1, \ldots, a_{n-2}, i)$, where $a_j \in \{0, 1, 2, 3\}$ for every $0 \leq j \leq n-2$. Each vertex $(a_0, a_1, \ldots, a_{n-2}, i)$ of BH_{n-1}^i has the following two extra neighbors:

$(a_0 \pm 1, a_1, \ldots, a_{n-2}, i+1)$ in BH_{n-1}^{i+1} if a_0 is even.

$(a_0 \pm 1, a_1, \ldots, a_{n-2}, i-1)$ in BH_{n-1}^{i-1} if a_0 is odd.

The first coordinate a_0 of vertex $(a_0, a_1, \ldots, a_j, \ldots, a_{n-1})$ is named *inner index*, and the other coordinates a_j $(1 \leq j \leq n-1)$ are named *j-dimension index*. In [11], it is seen that the balanced hypercube is bipartite. The vertex sets

$$V_0 = \{x = (a_0, a_1, \ldots, a_{n-1}) \mid x \in V(BH_n), \text{and } a_0 \text{ is odd}\}$$

and

$$V_1 = \{x = (a_0, a_1, \ldots, a_{n-1}) \mid x \in V(BH_n), \text{and } a_0 \text{ is even}\}$$

give the desired partition.

In the following, we use black vertices to denote the vertices in V_0 and white vertices to denote the vertices in V_1. Now we classify the edges of BH_n. If two adjacent vertices u, v differ only in the inner index, the edge (u, v) is said to be a 0-*dimension edge* and v is a 0-*dimension neighbor* of u. If two adjacent vertices u, v not only differ in the inner index, but also differ in some i-dimension index $(1 \leq i \leq n-1)$, then the edge (u, v) is said to be an *i-dimension edge* and v is an *i-dimension neighbor* of u. Let E_i denote the set of all edges of i-dimension edges for $i \in \{0, 1, 2, \ldots, n-1\}$. Then $E(BH_n) = \bigcup_{i=0}^{n-1} E_i$. For $i \in \{0, 1, 2, 3\}$ and $1 \leq j \leq n-1$, we use $BH_{n-1}^{j,i}$ to denote the $(n-1)$-dimension sub-balanced hypercubes of the BH_n induced by all vertices labeled by $(a_0, a_1, \ldots, a_{j-1}, i, a_{j+1}, \ldots, a_{n-1})$. Obviously, $BH_n - E_j = \bigcup_{i=0}^{3} BH_{n-1}^{j,i}$ and $BH_{n-1}^{j,i} \cong BH_{n-1}$. If $j = n-1$, $BH_{n-1}^{j,i}$ and E_j are denoted by BH_{n-1}^i and E_c, respectively, where $i \in \{0, 1, 2, 3\}$. One has $BH_n - E_c = \bigcup_{i=0}^{3} BH_{n-1}^i$.

In the rest of this paper we often use w_{ij} and b_{ij} to denote white and black vertices in BH_n, respectively, where $i \in N$ and $j \in \{0, 1, 2, 3\}$, and j means that the corresponding vertex lies in BH_{n-1}^j. By definition, any white vertex in BH_{n-1}^j has two black $(n-1)$-dimension neighbors in BH_{n-1}^{j+1}, $j \in \{0, 1, 2, 3\}$.

Some basic properties of BH_n are given below and will be used in the sequel.

Lemma 1 [11]**.** *The balanced hypercube BH_n is bipartite and vertex transitive.*

Lemma 2 [15]**.** *The balanced hypercube BH_n is edge transitive.*

Lemma 3 [13]**.** *Let (u, v) be an edge of BH_n. Then (u, v) is contained in a cycle C of length 8 in BH_n such that $|E(C) \cap E(BH_{n-1}^i)| = 1$, where $i \in \{0, 1, 2, 3\}$.*

Lemma 4 [11]**.** *The vertices* $(a_0, a_1, \ldots, a_{n-1})$ *and* $(a_0+2, a_1, \ldots, a_{n-1})$ *of* BH_n *have the same neighborhood.*

Lemma 5 [13]**.** *The balanced hypercube* BH_n *is Hamiltonian laceable for* $n \geq 1$.

Lemma 6 [7]**.** *The balanced hypercube* BH_n *is hyper-Hamiltonian laceable for* $n \geq 1$.

3. Edge-Hyper-Hamiltonian Lacebility of Balanced Hypercube

Lemma 7. *The balanced hypercube* BH_2 *is edge-hyper-Hamiltonian laceable.*

Proof. Let u be any vertex of BH_2. From Lemma 1, BH_2 is vertex transitive, therefore we may suppose $u = (0, 0)$ without loss of generality. For any $e \in BH_2 - u$ and any two black vertices x and y, we can prove there exists an (x, y)-Hamiltonian path containing e in $BH_2 - u$. Here we only prove the result for $e = (1, 0)(2, 0)$ without loss of generality. By the relative positions of e, x and y in BH_2, we distinguish into the following two cases.

Case 1. x and y are in the same BH_1^i, $i \in \{0, 1, 2, 3\}$. Since there are only two black vertices in BH_1^i, $i \in \{0, 1, 2, 3\}$, so if x and y are in the same BH_1^i, then one is x, the other is y. Let $x, y \in BH_1^0$, $x = (1, 0), y = (3, 0)$. Then $\langle(1, 0), (2, 0), (1, 1), (0, 1), (3, 1), (2, 1), (1, 2), (2, 2), (3, 2), (0, 2), (3, 3), (2, 3), (1, 3), (0, 3), (3, 0)\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_2 - u$.

Case 2. x and y are not in the same BH_1^i, $i \in \{0, 1, 2, 3\}$. Let $x \in BH_1^0$, $y \in BH_1^1$. Suppose $x = (1, 0), y = (1, 1)$. Then $\langle(1, 0), (2, 0), (3, 1), (0, 1), (1, 2), (2, 2), (3, 3), (2, 3), (3, 0), (0, 3), (1, 3), (0, 2), (3, 2), (2, 1), (1, 1)\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_2 - u$.

For other cases, we can similarly find an (x, y)-Hamiltonian path containing e in $BH_2 - u$, their proofs are omitted here. ■

Theorem 8. *The balanced hypercube* BH_n *is edge-hyper-Hamiltonian laceable for* $n \geq 2$.

Proof. We prove the result by induction on $n \geq 2$. The case $n = 2$ follows from Lemma 7. Next we consider $n \geq 3$. Suppose that the theorem is true for BH_k, $3 \leq k \leq n-1$. We suppose that $u \in V_0$ is a white vertex. For any edge $e \in BH_n - u$, then $e \in E(BH_{n-1}^i)$, $i \in \{0, 1, 2, 3\}$, or e is an $(n-1)$-dimension edge. If e is an $(n-1)$-dimension edge, since BH_n is vertex and edge transitive, there is an automorphism of BH_n so that $e \in E(BH_{n-1}^i)$, $i \in \{0, 1, 2, 3\}$, therefore in the following we only consider $e \in BH_{n-1}^i, i \in \{0, 1, 2, 3\}$. For any pair of vertices in V_1, say x and y, they are black vertices. By the relative positions of e, x and y in BH_n, we distinguish into the following cases.

Case 1. $e \in E(BH^0_{n-1})$.

Subcase 1.1. x and y are in the same $BH^i_{n-1}, i \in \{0,1,2,3\}$.

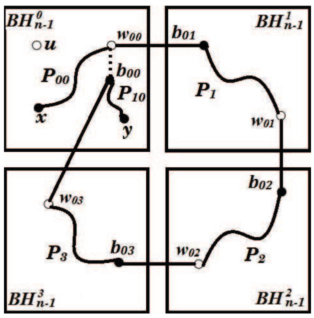

Figure 3. Subcase 1.1.1.

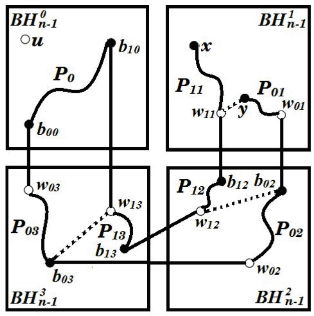

Figure 4. Subcase 1.1.2.

Subcase 1.1.1. $x, y \in BH^0_{n-1}$ (see Figure 3). By the induction hypothesis, the graph $BH^0_{n-1} - u$ contains a Hamiltonian path P_0 from x to y containing e. Since $n \geq 3$, there exists an edge different from e, say $w_{00}b_{00} \in E(P_0)$, such that the removal of the edge $w_{00}b_{00}$ decomposes P_0 into two sections $P_{00}[x, w_{00}]$ and $P_{10}[b_{00}, y]$. Let $b_{01} \in BH^1_{n-1}$ (resp. $w_{03} \in BH^3_{n-1}$) be $(n-1)$-dimension neighbors of w_{00} (resp. b_{00}). By Lemma 3, $w_{00}b_{00}$ is contained in a cycle C of length 8 in BH_n such that $|E(C) \cap E(BH^i_{n-1})| = 1$, $i \in \{0,1,2,3\}$. We denote the cycle $C = (w_{00}, b_{01}, w_{01}, b_{02}, w_{02}, b_{03}, w_{03}, b_{00}, w_{00})$. By Lemma 5, BH_n is Hamiltonian laceable for $n \geq 1$, so BH^i_{n-1} contains a (b_{0i}, w_{0i})-Hamiltonian path P_i, $i \in \{1,2,3\}$. Then $\langle x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_3, w_{03}, b_{00}, P_{10}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.1.2. $x, y \in BH^1_{n-1}$ (see Figure 4). Choose an arbitrary white vertex, say w_{01}, in BH^1_{n-1}. By Lemma 5, there exists an (x, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then there exists an edge, say $w_{11}y \in E(P_1)$, whose removal divides P_1 into two sections $P_{11}[x, w_{11}]$ and $P_{01}[y, w_{01}]$. Let b_{02} and b_{12} be $(n-1)$-dimension neighbors of w_{01} and w_{11} in BH^2_{n-1} respectively. Since every white vertex of BH^1_{n-1} has two $(n-1)$-dimension neighbors in BH^2_{n-1}, so we can always choose two vertices b_{02} and b_{12} such that $b_{02} \neq b_{12}$. Similarly, choose an arbitrary white vertex, say w_{02}, in BH^2_{n-1}. By Lemma 5, there exists a (b_{12}, w_{02})-Hamiltonian path P_2 in BH^2_{n-1} via the edge $b_{02}w_{12}$. We delete $b_{02}w_{12}$, then P_2 is divided to two sections $P_{02}[w_{02}, b_{02}]$ and $P_{12}[w_{12}, b_{12}]$. Let b_{03} and b_{13} be $(n-1)$-dimension neighbors of w_{02} and w_{12} in BH^3_{n-1} respectively, and $b_{03} \neq b_{13}$. Choose any white vertex w_{03} in BH^3_{n-1}; by Lemma 5, there is a (b_{13}, w_{03})-Hamiltonian path P_3 in BH^3_{n-1}. Therefore there exists an edge, say $b_{03}w_{13}$, in P_3 whose deletion divides P_3 into two parts $P_{03}[w_{03}, b_{03}]$ and $P_{13}[w_{13}, b_{13}]$. Let b_{00} and b_{10} be $(n-1)$-dimension neighbors of w_{03} and w_{13} in BH^1_{n-1} respectively, and $b_{00} \neq b_{10}$. By the induction hypothesis, there exists a

(b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then $\langle x,\ P_{11},\ w_{11},\ b_{12},\ P_{12},\ w_{12},\ b_{13},\ P_{13},\ w_{13},\ b_{10}, P_0, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, P_{02}, b_{02}, w_{01}, P_{01}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

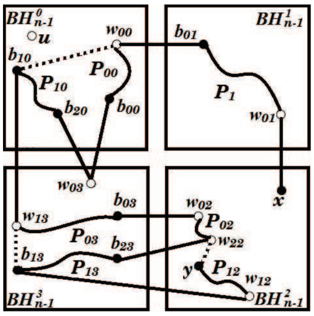

Figure 5. Subcase 1.1.3.

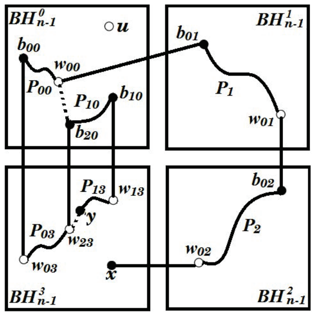

Figure 6. Subcase 1.1.4.

Subcase 1.1.3. $x, y \in V(BH^2_{n-1})$ (see Figure 5). Choose an arbitrary white vertex, say w_{03}, in BH^3_{n-1}. Let b_{00} and b_{20} be two $(n-1)$-dimension neighbors (different from u) of w_{03} in BH^0_{n-1}. By the induction hypothesis, for any $e \in BH^0_{n-1}$, the graph $BH^0_{n-1} - u$ contains a (b_{00}, b_{20})-Hamiltonian path P_0 containing e. Now we choose two arbitrary white vertices in BH^2_{n-1} with different $(n-1)$-dimension neighbors in BH^3_{n-1}, say w_{02} and w_{12}. By Lemma 6, there is a (w_{02}, w_{12})-Hamiltonian path P_2 in $BH^2_{n-1} - x$. There exists an edge, say $yw_{22} \in E(P_2)$, whose deletion divides P_2 into two sections $P_{12}[w_{12}, y]$ and $P_{02}[w_{22}, w_{02}]$. Let b_{03}, b_{13} and b_{23} (they are different from each other) be $(n-1)$-dimension neighbors of w_{02}, w_{12} and w_{22} in BH^3_{n-1}, respectively. By Lemma 6, there exists a (b_{03}, b_{23})-Hamiltonian path P_3 in $BH^3_{n-1} - w_{03}$. Then there exists an edge $w_{13}b_{13} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{03}[b_{03}, w_{13}]$ and $P_{13}[b_{13}, b_{23}]$. Let b_{10} be an $(n-1)$-dimension neighbor of w_{13} in BH^0_{n-1} such that b_{10} is not incident with e. Then there exists an edge $b_{10}w_{00} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{10}, b_{20}]$. Let b_{01} and w_{01} be $(n-1)$-dimension neighbors of w_{00} and x in BH^1_{n-1}, respectively. By Lemma 5, there exists a (w_{01}, b_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then $\langle x, w_{01},\ P_1,\ b_{01},\ w_{00},\ P_{00}, b_{00}, w_{03}, b_{20}, P_{10}, b_{10}, w_{13}, P_{03}, b_{03}, w_{02}, P_{02}, w_{22}, b_{23}, P_{13},\ b_{13},\ w_{12},\ P_{12}, y\rangle$ is an (x, y) -Hamiltonian path containing e in $BH_n - u$.

Subcase 1.1.4. $x, y \in V(BH^3_{n-1})$ (see Figure 6). Let w_{03} and w_{13} be any two white vertices in BH^3_{n-1}, with different $(n-1)$-dimension neighbors in BH^0_{n-1}. By the induction hypothesis, for any edge, say $yw_{23} \in E(BH^3_{n-1} - x)$, there exists a (w_{03}, w_{13})-Hamiltonian path P_3 in $BH^3_{n-1} - x$. We choose such an edge yw_{23} so that b_{20} is not incident with e and $b_{20} \neq b_{00}$, where b_{00} and b_{20} are $(n-1)$-dimension neighbors of w_{03} and w_{23} in BH^0_{n-1}, respectively. Now $P_3 = \langle w_{03}, P_{03}[w_{03}, w_{23}], w_{23}, y, P_{13}[y, w_{13}], w_{13}\rangle$. Let b_{10} be an $(n-1)$-dimension neighbor of w_{13} in BH^0_{n-1}. By the induction hypothesis, there exists a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then there exists an edge, say $w_{00}b_{20} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{20}, b_{10}]$. Since w_{03} and w_{13} are arbitrary vertices, so we can always choose two such vertices so that $w_{00}b_{20} \neq e$. Let b_{01} and w_{02} be $(n-1)$-dimension neighbors of w_{00} and x respectively. Furthermore, let b_{02} be any black vertex in BH^2_{n-1} and w_{01} be an $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there exists a (b_{0i}, w_{0i})-Hamiltonian path P_i in BH^i_{n-1}, $i \in \{1, 2\}$. Then $\langle x, w_{02}, P_2, b_{02}, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{00}, w_{03}, P_{03}, w_{23}, b_{20}, P_{10}, b_{10}, w_{13}, P_{13}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2. $x \in V(BH^i_{n-1})$, $y \in V(BH^j_{n-1})$, $0 \leq i < j \leq 3$.

Subcase 1.2.1. $x \in V(BH^0_{n-1})$, $y \in V(BH^1_{n-1})$. We choose any black vertex in BH^0_{n-1}, say b_{00}. By the induction hypothesis, there exists an (x, b_{00})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Let w_{03} be an $(n-1)$-dimension neighbor of b_{00} in BH^3_{n-1}. By Lemma 5, for any black vertex $b_{03} \in BH^3_{n-1}$, there exists a (w_{03}, b_{03})-Hamiltonian path P_3 in BH^3_{n-1}. Similarly, for any white vertex $w_{01} \in BH^1_{n-1}$, there exists a (y, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Let b_{02} and w_{02} be $(n-1)$-dimension neighbors of w_{01} and b_{03} in BH^2_{n-1}, respectively. By Lemma 5, there exists a (b_{02}, w_{02})-Hamiltonian path P_2 in BH^2_{n-1}. Then $\langle x, P_0, b_{00}, w_{03}, P_3, b_{03}, w_{02}, P_2, b_{02}, w_{01}, P_1, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

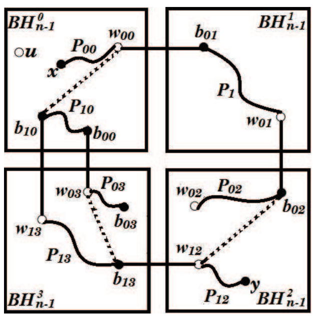

Figure 7. Subcase 1.2.2.

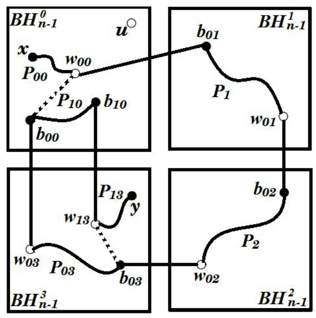

Figure 8. Subcase 1.2.3.

Subcase 1.2.2. $x \in V(BH^0_{n-1})$, $y \in V(BH^2_{n-1})$ (see Figure 7). Choose an arbitrary black vertex, say b_{00} in BH^0_{n-1}. By the induction hypothesis, the graph $BH^0_{n-1} - u$ contains an (x, b_{00})-Hamiltonian path P_0 containing e. Then there exists an edge, say $w_{00}b_{10} \in E(P_0)$ so that $w_{00}b_{10} \neq e$ and $P_0 = \langle x, P_{00}[x, w_{00}], w_{00}, b_{10}, P_{10}[b_{10}, b_{00}], b_{00}\rangle$. Let $b_{01} \in BH^1_{n-1}, w_{03} \in BH^3_{n-1}$ and $w_{13} \in BH^3_{n-1}$ be $(n-1)$-dimension neighbors of w_{00}, b_{00} and b_{10}, respectively. Let b_{03} be an arbitrary black vertex in BH^3_{n-1}. By Lemma 5, there exists a (b_{03}, w_{13})-Hamiltonian path P_3. Hence there exists an edge in P_3, say $w_{03}b_{13}$, whose deletion divides P_3 into two sections $P_{03}[b_{03}, w_{03}]$ and $P_{13}[b_{13}, w_{13}]$. Let w_{02} and w_{12} be $(n-1)$-dimension neighbors of b_{03} and b_{13} in BH^2_{n-1} respectively, and $w_{02} \neq w_{12}$. Since y is black vertex in BH^2_{n-1}, in view of Lemma 5, there exists a (w_{02}, y)-Hamiltonian path P_2 in BH^2_{n-1}. Then there exists an edge in P_2, say $b_{02}w_{12}$, whose deletion divides P_2 into two sections $P_{02}[w_{02}, b_{02}]$ and $P_{12}[w_{12}, y]$. Let w_{01} be $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there exists a (b_{01}, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then $\langle x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2.3. $x \in V(BH^0_{n-1})$, $y \in V(BH^3_{n-1})$ (see Figure 8). Assume b_{10} is an arbitrary black vertex in BH^0_{n-1}. By the induction hypothesis, $BH^0_{n-1} - u$ contains an (x, b_{10})-Hamiltonian path P_0 containing e. Then there exists an edge different from e, say $w_{00}b_{00} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[x, w_{00}]$ and $P_{10}[b_{00}, b_{10}]$. Let b_{01} be an $(n-1)$-dimension neighbor of w_{00} in BH^1_{n-1}, w_{03} and w_{13} be $(n-1)$-dimension neighbors of b_{00} and b_{10} in BH^3_{n-1}, respectively. By Lemma 5, there exists a (w_{03}, y)-Hamiltonian path P_3 in BH^3_{n-1}. Then there exists an edge, say $w_{13}b_{03} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{03}[w_{03}, b_{03}]$ and $P_{13}[w_{13}, y]$. Let w_{02} be an $(n-1)$-dimension neighbor of b_{03} in BH^2_{n-1}. For any black vertex in BH^2_{n-1}, say b_{02}, we assume w_{01} is an $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there is a (b_{0i}, w_{0i})-Hamiltonian path P_i in BH^i_{n-1}, $i \in \{1, 2\}$. Then $\langle x, P_{00}, w_{00}, b_{01}, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_{10}, b_{10}, w_{13}, P_{13}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

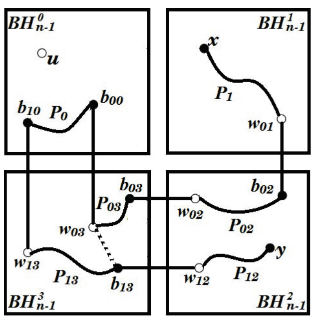

Figure 9. Subcase 1.2.4.

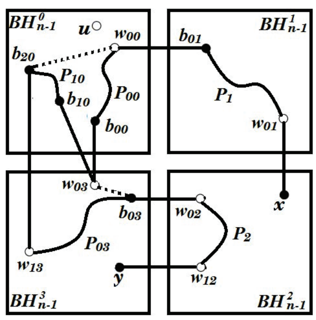

Figure 10. Subcase 1.2.6.

Subcase 1.2.4. $x \in V(BH^1_{n-1})$, $y \in V(BH^2_{n-1})$ (see Figure 9). Assume $w_{01} \in BH^1_{n-1}$ and $w_{02} \in BH^2_{n-1}$ are arbitrary white vertices. By Lemma 5, there exists an (x, w_{01})-Hamiltonian path P_1 in BH^1_{n-1} and a (y, w_{02})-Hamiltonian path P_2 in BH^2_{n-1}. Let $b_{02} \neq y$ be an $(n-1)$-dimension neighbor of w_{01} in BH^2_{n-1}. Then there exists an edge, say $b_{02}w_{12} \in E(P_2)$, whose deletion divides P_2 into two sections $P_{02}[b_{02}, w_{02}]$ and $P_{12}[y, w_{12}]$. Let b_{03} and b_{13} be $(n-1)$-dimension neighbors of w_{02} and w_{12}, respectively. $b_{03} \neq b_{13}$ since every white vertex in BH^2_{n-1} has two black $(n-1)$-dimension neighbors in BH^3_{n-1}. Let w_{13} be any white vertex in BH^3_{n-1}. By Lemma 5, there exists a (b_{03}, w_{13})-Hamiltonian path P_3 in BH^3_{n-1}. Then there exists an edge, say $w_{03}b_{13} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{03}[b_{03}, w_{03}]$ and $P_{13}[b_{13}, w_{13}]$. Let b_{00} and b_{10} be an $(n-1)$-dimension neighbors of w_{03} and w_{13} in BH^0_{n-1}, respectively and $b_{00} \neq b_{10}$. By the induction hypothesis, there is a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then $\langle x, P_1, w_{01}, b_{02}, P_{02}, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_0, b_{10}, w_{13}, P_{13}, b_{13}, w_{12}, P_{12}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2.5. $x \in V(BH^1_{n-1})$, $y \in V(BH^3_{n-1})$. Assume that b_{00} and b_{10} are any two black vertices in BH^0_{n-1}. By the induction hypothesis, there exists a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Let w_{03} and w_{13} be an $(n-1)$-dimension neighbors of b_{00} and b_{10} in BH^3_{n-1} respectively, and $w_{03} \neq w_{13}$. By Lemma 5, there is a (y, w_{03})-Hamiltonian path P_3 in BH^3_{n-1}. Hence there exists an edge, say $w_{13}b_{03} \in E(P_3)$, whose deletion divides P_3 into two sections $P_{13}[y, w_{13}]$ and $P_{03}[b_{03}, w_{03}]$. Let w_{02} be an $(n-1)$-dimension neighbor of b_{03} in BH^2_{n-1}, b_{02} be any black vertex in BH^2_{n-1}, w_{01} be an $(n-1)$-dimension neighbor of b_{02} in BH^1_{n-1}. By Lemma 5, there exists an (x, w_{01})-Hamiltonian path P_1 in BH^1_{n-1} and (b_{02}, w_{02})-Hamiltonian path P_2 in BH^2_{n-1}. Then $\langle x, P_1, w_{01}, b_{02}, P_2, w_{02}, b_{03}, P_{03}, w_{03}, b_{00}, P_0, b_{10}, w_{13}, P_{13}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Subcase 1.2.6. $x \in V(BH^2_{n-1})$, $y \in V(BH^3_{n-1})$ (see Figure 10). Let w_{03} be an arbitrary white vertex in BH^3_{n-1}, b_{00} and b_{10} be two $(n-1)$-dimension neighbors of w_{03} in BH^0_{n-1}. By the induction hypothesis, for any e, there exists a (b_{00}, b_{10})-Hamiltonian path P_0 containing e in $BH^0_{n-1} - u$. Then there exists an edge in P_0, say $w_{00}b_{20}$ and $w_{00}b_{20} \neq e$, whose deletion divides P_0 into two sections $P_{00}[b_{00}, w_{00}]$ and $P_{10}[b_{20}, b_{10}]$. Let $b_{01} \in BH^1_{n-1}$ (resp. $w_{13} \in BH^3_{n-1}$) be an $(n-1)$-dimension neighbor of w_{00} (resp. b_{20}). Here we choose $w_{13} \neq w_{03}$ since every black vertex in BH^0_{n-1} has two white $(n-1)$-dimension neighbors in BH^3_{n-1}. By Lemma 6, there exists a (w_{03}, w_{13})-Hamiltonian path P_3 in $BH^3_{n-1} - y$. Hence there exists an edge, say $w_{03}b_{03} \in E(P_3)$, so that $P_3 = \langle w_{03}, b_{03}, P_{03}[b_{03}, w_{13}], w_{13}\rangle$. Let w_{02} and w_{12} be $(n-1)$-dimension neighbors of b_{03} and y in BH^2_{n-1} respectively, and $w_{02} \neq w_{12}$. By Lemma 6, there exists a (w_{02}, w_{12})-Hamiltonian path P_2 in $BH^2_{n-1} - x$. Let $w_{01} \in BH^1_{n-1}$ be an $(n-1)$-dimension neighbor of x. By Lemma 5, there exists a (b_{01}, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Thus it follows that $\langle x, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{00}, w_{03}, b_{10}, P_{10}, b_{20}, w_{13}, P_{03}, b_{03}, w_{02}, P_2, w_{12}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Case 2. $e \in BH^1_{n-1}$. The proof of this case is similar to that of Case 1, and is omitted.

Case 3. $e \in BH^2_{n-1}$. If $x, y \in V(BH^2_{n-1})$, then since $xy \neq e$, at most one of x and y can be incident with e. Hence the proof is similar to that of Subcase 1.1.3 since we can exchange x and y in this case when y is incident with e.

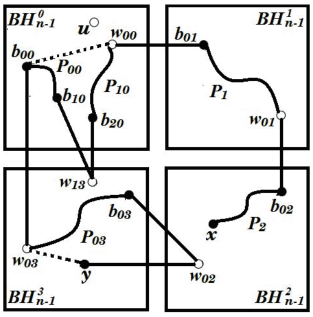

Figure 11

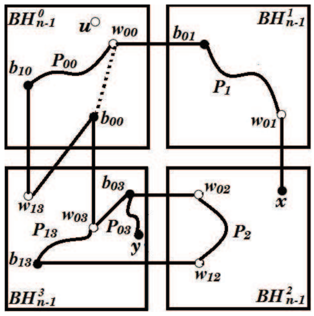

Figure 12. Case 4.

For $x \in V(BH^2_{n-1}), y \in V(BH^3_{n-1})$ (see Figure 11), if x is not incident with e, then the proof is similar to Subcase 1.2.6. If x is incident with e, we may assume w_{02} is an $(n-1)$-dimension neighbor of y in BH^2_{n-1} and b_{03} is another $(n-1)$-dimension neighbor of w_{02} in BH^3_{n-1}. Choose any white vertex, say w_{13} in BH^3_{n-1} and let b_{10} and b_{20} be two $(n-1)$-dimension neighbors of w_{13} in BH^0_{n-1}. By Lemma 6, there exists a (b_{03}, y)-Hamiltonian path P_3 in $BH^3_{n-1} - w_{13}$. Then there exists an edge, say $yw_{03} \in E(P_3)$, and $P_3 = \langle y, w_{03}, P_{03}[w_{03}, b_{03}], b_{03}\rangle$. Let b_{00} be the $(n-1)$-dimension neighbor of w_{03} in BH^0_{n-1} and $b_{00} \neq b_{20}$ as w_{03} has two $(n-1)$-dimension neighbors in BH^0_{n-1}. Similarly, by Lemma 6, there exists a (b_{10}, b_{20})-Hamiltonian path P_0 in $BH^0_{n-1} - u$. Then there exists an edge, say $b_{00}w_{00} \in E(P_0)$, whose deletion divides P_0 into two sections $P_{00}[b_{10}, b_{00}]$ and $P_{10}[w_{00}, b_{20}]$. Let b_{01} be one $(n-1)$-dimension neighbor of w_{00} in BH^1_{n-1}. By Lemma 5, for any white vertex w_{01} in BH^1_{n-1}, there exists a (b_{01}, w_{01})-Hamiltonian path P_1. Let $b_{02} \in BH^2_{n-1}$ be one $(n-1)$-dimension neighbor of w_{01}. By the induction hypothesis, for any e, there exists an (x, b_{02})-Hamiltonian path P_2 containing e in $BH^2_{n-1} - w_{02}$. Then $\langle x, P_2, b_{02}, w_{01}, P_1, b_{01}, w_{00}, P_{10}, b_{20}, w_{13}, b_{10}, P_{00}, b_{00}, w_{03}, P_{03}, b_{03}, w_{02}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Since the other subcases are similar to the corresponding subcases of Case 1, we omit their proofs.

Case 4. $e \in BH^3_{n-1}$. We just prove the subcase that $x \in V(BH^2_{n-1}), y \in V(BH^3_{n-1})$, the other cases are similar to the corresponding subcases of Case 1.

Subcase 4.1. $x \in V(BH^2_{n-1})$, $y \in V(BH^3_{n-1})$ (see Figure 12). For any $e \in E(BH^3_{n-1}))$, if y is not incident with e, then in the proof of Subcase 1.2.6, we choose any white vertex w_{03} which is not incident with e, then the remainder of the proof is similar to that of Subcase 1.2.6. If y is incident with e, as $n \geq 3$, there exist two white vertices in BH^3_{n-1}, say w_{03} and w_{13}, who are not adjacent to y and have a common $(n-1)$-dimension neighbor, say $b_{00} \in BH^0_{n-1}$. For any black vertex in BH^3_{n-1}, say b_{13}, by induction hypothesis, the graph $BH^3_{n-1} - w_{13}$ contains a (y, b_{13})-Hamiltonian path P_3 containing e. Thus there exists an edge, say $w_{03}b_{03} \in E(P_3)$ and $w_{03}b_{03} \neq e$ since w_{03} is not adjacent to y and y is incident with e. Now $P_3 = \langle y, P_{03}[y, b_{03}], b_{03}, w_{03}, P_{13}[w_{03}, b_{13}], b_{13}\rangle$. Let b_{10} be another $(n-1)$-dimension neighbor of w_{13} in BH^0_{n-1}, w_{02} and w_{12} be $(n-1)$-dimension neighbors of b_{03} and b_{13} in BH^2_{n-1} respectively, and $w_{02} \neq w_{12}$ since every black vertex in BH^3_{n-1} has two $(n-1)$-dimension neighbors in BH^2_{n-1}. By Lemma 6, there exists one (w_{02}, w_{12})-Hamiltonian path P_2 in $BH^2_{n-1} - x$, and one (b_{00}, b_{10})-Hamiltonian path P_0 in $BH^0_{n-1} - u$. Then there exists an edge $w_{00}b_{00} \in E(P_0)$ such that $P_0 = \langle b_{10}, P_{00}[b_{10}, w_{00}], w_{00}, b_{00}\rangle$. Let w_{01} and b_{01} be $(n-1)$-dimension neighbors of x and w_{00} in BH^1_{n-1} respectively. By Lemma 5, there exists one (b_{01}, w_{01})-Hamiltonian path P_1 in BH^1_{n-1}. Then $\langle x, w_{01}, P_1, b_{01}, w_{00}, P_{00}, b_{10}, w_{13}, b_{00}, w_{03}, P_{13}, b_{13}, w_{12}, P_2, w_{02}, b_{03}, P_{03}, y\rangle$ is an (x, y)-Hamiltonian path containing e in $BH_n - u$.

Combining the above cases, the proof of this theorem is completed. ■

4. Conclusion

The balance hypercube BH_n, proposed by Huang and Wu [5], is a variant of the hypercube that gives better performance with the same number of edges and vertices. It has been shown that the balanced hypercube BH_n is Hamiltonian laceable and hyper-Hamiltonian laceable for $n \geq 1$. In this paper, we show that, for any vertex $v \in V_i, i \in \{0, 1\}$, and any $e \in E(BH_n - v)$, there exists a Hamiltonian path containing e in $G - v$ between any pair of vertices in V_{1-i}.

Acknowledgements

Cao and Shi were supported by Engineering Planning Project for Communication University of China (3132015XNG1526 and XNG1504) and The Comprehensive Reform Project of Computer Science and Technology (ZL140103 and ZL1503). Feng was supported by NSFC (Nos. 11271208, 11301302), Mathematics and Interdisciplinary Sciences Project of Central South University.

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Received 24 June 2015
Revised 5 December 2015
Accepted 5 December 2015