Discussiones Mathematicae
Graph Theory 37 (2017) 5–12
doi:10.7151/dmgt.1905

ALL TIGHT DESCRIPTIONS OF 3-STARS IN 3-POLYTOPES WITH GIRTH 5

Oleg V. Borodin[1]

Institute of Mathematics Siberian Branch
Russian Academy of Sciences
Novosibirsk, 630090, Russia

e-mail: brdnoleg@math.nsc.ru

and

Anna O. Ivanova[2]

Ammosov North-Eastern Federal University
Yakutsk, 677000, Russia

e-mail: shmgnanna@mail.ru

Abstract

Lebesgue (1940) proved that every 3-polytope P_5 of girth 5 has a path of three vertices of degree 3. Madaras (2004) refined this by showing that every P_5 has a 3-vertex with two 3-neighbors and the third neighbor of degree at most 4. This description of 3-stars in P_5s is tight in the sense that no its parameter can be strengthened due to the dodecahedron combined with the existence of a P_5 in which every 3-vertex has a 4-neighbor.

We give another tight description of 3-stars in P_5s: there is a vertex of degree at most 4 having three 3-neighbors. Furthermore, we show that there are only these two tight descriptions of 3-stars in P_5s.

Also, we give a tight description of stars with at least three rays in P_5s and pose a problem of describing all such descriptions. Finally, we prove a structural theorem about P_5s that might be useful in further research.

Keywords: 3-polytope, planar graph, structure properties, k-star.

2010 Mathematics Subject Classification: 05C15.

[1]This work was supported by grants 16-01-00499 and 15-01-05867 of the Russian Foundation for Basic Research and President Grant for Government Support of the Leading Scientific Schools of the Russian Federation NSh-1939.2014.1.

[2]The author's work was performed as a part of government work "Organizing research" and supported by grant 15-01-05867 of the Russian Foundation for Basic Research.

1. Introduction

Let δ be the minimum vertex degree, and g be the girth (the length of a shortest cycle) in a given 3-polytope. We recall that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to Steinitz's famous theorem [22].

The degree of a vertex v or a face f, which is the number of edges incident with v or f in a 3-polytope, is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc.

Let $\mathbf{P}_5$ be the set of (finite) 3-polytopes with $g = 5$, and $\mathbf{P}_5^*$ be the 3-polytopes with $\delta = 5$. We note that $\mathbf{P}_5$ and $\mathbf{P}_5^*$ are in 1–1 correspondence due to the vertex-face duality, so structural results on $\mathbf{P}_5$ are easily translated to the language of $\mathbf{P}_5^*$ and vice versa.

The early interest of researchers to the structure of $\mathbf{P}_5^*$ was motivated by the Four Color Problem. Already in 1904, Wernicke [23] proved that every graph in $\mathbf{P}_5^*$ contains a 5-vertex adjacent to a 6^--vertex, and Franklin [12] in 1922 strengthened this to the existence of at least two 6^--neighbors. Franklin's result is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

In 1940, Lebesgue [17] gave, in particular, an approximate description of the neighborhoods of 5-vertices in $\mathbf{P}_5^*$ and proved that every 3-polytope in $\mathbf{P}_5$ has a 5-face incident with four 3-vertices and the fifth 5^--vertex, which face includes a path of three 3-vertices. In 2004, Madaras [19] refined the last mentioned result by Lebesgue as follows.

Theorem 1 (Madaras [19])**.** *Every 3-polytope with girth 5 has a 3-vertex adjacent to two 3-vertices and another vertex of degree at most 4, which is tight.*

In dual terms, Theorem 1 reads equivalently as follows.

Theorem 2 (Madaras [19])**.** *Every 3-polytope with minimum degree 5 has a 3-face adjacent to two 3-faces and another face of degree at most 4, which is tight.*

Nowadays, a lot of structural results on $\mathbf{P}_5$ and $\mathbf{P}_5^*$ can be found in the literature; for example, see [1–9, 13–16, 18–21].

We need a few definitions. A *k-star* $S_k(v; v_1, \ldots, v_k)$ in a 3-polytope consists of the central vertex v and its neighbor vertices $v_1, \ldots, v_k$, in no particular order. A *k^+-star* has at least k rays. In this note, we deal with 3^+-stars in $\mathbf{P}_5$.

We say that $S_k(v; v_1, \ldots, v_k)$ is an *$(a; b_1, \ldots, b_k)$-star*, or a star of *type* $(a; b_1, \ldots, b_k)$, where $b_1 \geq \cdots \geq b_k$, if $d(v) = a$ and $d(v_i) = b_i$ whenever $1 \leq i \leq k$.

A set $D = \{T_1, \ldots, T_n\}$ of star-types is a *description* for $\mathbf{P}_5$ if every graph in $\mathbf{P}_5$ has a star of one of the types from D. A description D is *tight* if all descriptions $D - T_i$ with $1 \leq i \leq n$ are invalid, which means that for every i there is a graph in $\mathbf{P}_5$ that has no stars of types from $D - T_i$, but it has a star of type T_i.

Madaras [19] constructed a polytope in $\mathbf{P}_5$ in which every 3-vertex has at least one 4^+-neighbor (see Figure 1). In what follows, this construction is called M_{04}. The tightness of the description of 3-stars given in Theorem 1 is implied by the dodecahedron (which has no $(3;4,3,3)$-stars) together with M_{04}.

One of the purposes of our paper is to augment Theorem 1 by giving another tight description of 3-stars in $\mathbf{P}_5$.

Theorem 3. *Every 3-polytope with girth 5 has a vertex of degree at most 4 having three 3-neighbors, which is a tight description of 3-stars in* $\mathbf{P}_5$.

Here, the tightness follows from the facts that the dodecahedron has no stars of type $(4;3,3,3)$, while M_{04} avoids the $(3;3,3,3)$-star.

Our next result is that there are only two tight descriptions of 3-stars in $\mathbf{P}_5$.

Theorem 4. *In* $\mathbf{P}_5$*, there are precisely two tight descriptions of 3-stars:*

(a) $D_{04} = \{(3;3,3,3),\ (3;4,3,3)\}$*, given by Theorem* 1 (Madaras [19])*, and*

(b) $D_{15} = \{(3;3,3,3),\ (4;3,3,3)\}$*, given by Theorem* 3.

A 3-vertex is *weak* if it has two 3-neighbors and a 4-neighbor. For further attempts to find tight descriptions of 3^+-stars in $\mathbf{P}_5$, the following structural result seems useful.

Theorem 5. *Every 3-polytope of girth 5 has one of the following configurations* (*see Figure* 2):

(a) *a 3-vertex with three 3-neighbors;*

(b) *a 4-vertex with four 3-neighbors, at least one of which is weak;*

(c) *a 4-vertex with a 4-neighbor and three 3-neighbors, at least two of which are weak.*

It is easy to see that Theorem 5 implies Theorems 1 and 3, as well as the next fact.

Corollary 6. *The following tight descriptions of* 3^+*-stars in* $\mathbf{P}_5$ *hold:*

(i) $D_{04} = \{(3;3,3,3),\ (3;4,3,3)\}$ (Madaras [19])*;*

(ii) $D_{15} = \{(3;3,3,3),\ (4;3,3,3)\}$*;*

(iii) $D_1 = \{(3;3,3,3),\ (4;3,3,3,3),\ (4;4,3,3,3)\}$.

The tightness of D_{04} and D_{15} follows from the dodecahedron combined with M_{04}. In Figure 3, we see a half of a graph H_1 whose every vertex has a 4-neighbor. (Note that in M_{04} only each 3-vertex has a 4-neighbor.) So the type $(4;4,3,3,3)$ cannot be dropped from D_1, while the first and second types cannot be dropped due to the dodecahedron and M_{04}, respectively.

It looks like the following tempting problem is hard.

Problem 7. Find all tight descriptions of 3^+-stars in $\mathbf{P}_5$.

In fact, we were not able to solve even the following two much more modest problems.

Problem 8. Is it true that $D_2 = \{(3;3,3,3), (3;4,4,4), (4;3,3,3,3)\}$ is a tight description of 3^+-stars in $\mathbf{P}_5$?

We note that if D_2 is a description, then it is tight due to the dodecahedron, H_1, and M_{04}.

Problem 9. Is it true that $\{(3;3,3,3), (3;4,4,3), (3;4,4,4), (4;3,3,3,3), (4;4,4,3,3), (4;4,4,4,3)\}$ is a description of 3^+-stars in $\mathbf{P}_5$?

In Section 2, we illustrate the constructions M_{04} and H_1 and configurations in Theorem 5. Section 3 contains proofs of Theorems 5 and 4. Note that Theorem 5 implies Theorem 1 and is proved shorter than Theorem 1 in Madaras [19].

2. Constructions M_{04} and H_1 and Configurations in Theorem 5

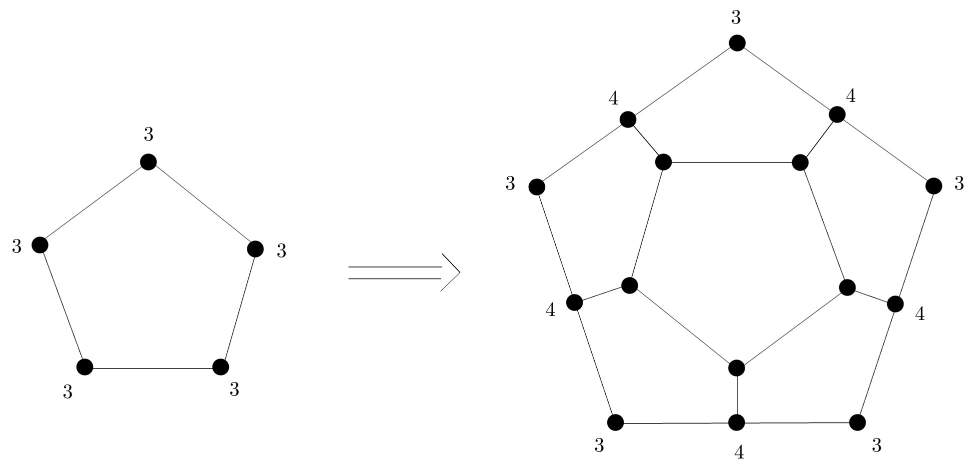

Figure 1. (Madaras [19]) M_{04}: every 3-vertex has a 4-neighbor.

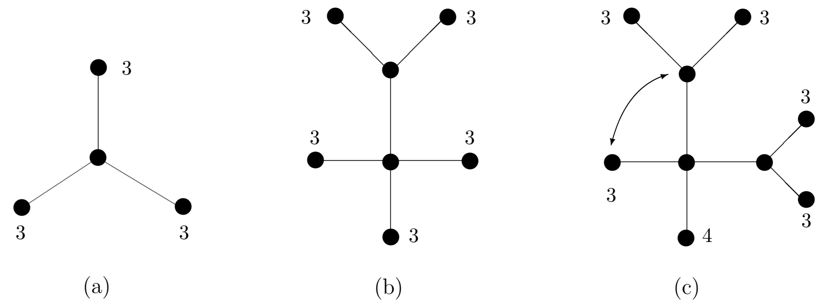

Figure 2. Configurations in Theorem 5.

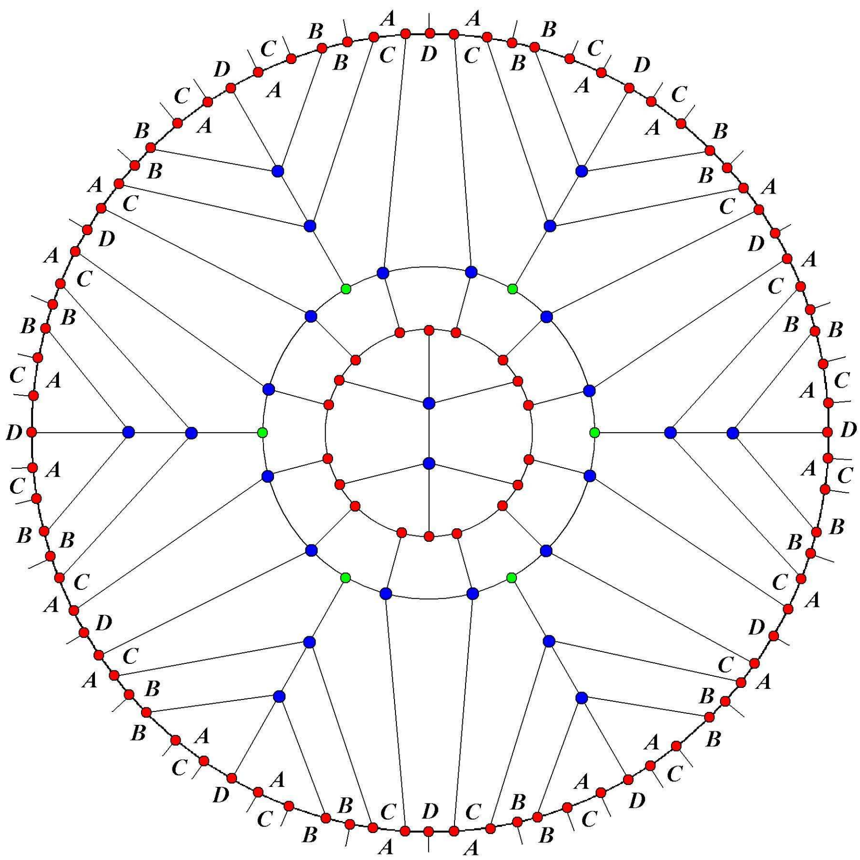

Figure 3. (A half of) H_1: every vertex has a 4-neighbor.

3. Proofs

3.1. Proving Theorem 5

Suppose that P is a counterexample to Theorem 5. Euler's formula $|V| - |E| + |F| = 2$ for P may be written as

$$\sum_{v\in V}\left(\frac{3d(v)}{2}-5\right) \le \sum_{v\in V}\left(\frac{3d(v)}{2}-5\right) + \sum_{f\in F}\Big(d(f)-5\Big) = -10, \tag{1}$$

where V, E, and F are the sets of vertices, edges and faces of P, respectively.

Let us assign a *charge* $\mu(v) = \frac{3d(v)}{2} - 5$ to every vertex v in V, so that the charge of vertices, depending on theirs degree, is $-\frac{1}{2}$, 1, $\frac{5}{2}$, and so on. Using the properties of P as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is at most -10, according to (1). Our rules of discharging are:

R1. *Every* 3*-vertex* v *receives from every adjacent* 4^+*-vertex either* $\frac{1}{2}$ *if* v *is weak or* $\frac{1}{4}$ *otherwise.*

R2. *Every* 4*-vertex receives* $\frac{1}{2}$ *from every adjacent* 5^+*-vertex.*

To complete the proof of Theorem 5, we first observe that every 3-vertex v receives from its 4^+-neighbors either $\frac{1}{2}$ if v is weak, or at least $2 \times \frac{1}{4}$ otherwise, so $\mu'(v) \ge -\frac{1}{2} + \frac{1}{2} = 0$.

Now if $d(v) = 4$, then $\mu(v) = 1$, and we are easily done unless v has at least three 3-neighbors but no 5^+-neighbors. If so, then v either gives $4 \times \frac{1}{4}$ to its four 3-neighbors, or at most $\frac{1}{2} + 2 \times \frac{1}{4}$ to its three 3-neighbors, which yields $\mu'(v) \ge 1 - 1 = 0$.

Finally, for $d(v) \ge 5$ we have $\mu'(v) \ge \frac{3d(v)}{2} - 5 - d(v) \times \frac{1}{2} = d(v) - 5 \ge 0$, as desired.

3.2. Proving Theorem 4

Suppose that D is a tight description of 3-stars for $\mathbf{P}_5$. Since the dodecahedron has 3-stars only of the type $(3;3,3,3)$, it follows that $(3;3,3,3) \in D$. Now we look at the graph M_{04}; it has 3-stars only of the types $(3;4,3,3)$ and $(4;3,3,3)$. As M_{04} obeys D, at least one of these types should appear in D.

Case 1. $(3;4,3,3) \in D$. Since $D' = \{(3;3,3,3),(3;4,3,3)\}$ is a tight description by Theorem 1, we have $D = \{(3;3,3,3),(3;4,3,3)\}$ due to the minimality of D.

Case 2. $(4;3,3,3) \in D$. Since $D' = \{(3;3,3,3),(4;3,3,3)\}$ is a tight description by Theorem 3, we have $D = \{(3;3,3,3),(4;3,3,3)\}$, as desired.

References

[1] O.V. Borodin, *Solution of Kotzig's and Grünbaum's problems on the separability of a cycle in a planar graph*, Mat. Zametki **46** (1989) 9–12, in Russian.

[2] O.V. Borodin, *Structural properties of planar maps with the minimal degree* 5, Math. Nachr. **158** (1992) 109–117.
doi:10.1002/mana.19921580108

[3] O.V. Borodin and D. Sanders, *On light edges and triangles in planar graphs of minimal degree five*, Math. Nachr. **170** (1994) 19–24.
doi:10.1002/mana.19941700103

[4] O.V. Borodin and D.R. Woodall, *Short cycles of low weight in normal plane maps with minimum degree* 5, Discuss. Math. Graph Theory **18** (1998) 159–164.
doi:0.7151/dmgt.1071

[5] O.V. Borodin and A.O. Ivanova, *Describing* 4*-stars at* 5*-vertices in normal plane maps with minimum degree* 5, Discrete Math. **313** (2013) 1710–1714.
doi:10.1016/j.disc.2013.04.025

[6] O.V. Borodin, A.O. Ivanova and A.V. Kostochka, *Every* 3*-polytope with minimum degree* 5 *has a* 6*-cycle with maximum degree at most* 11, Discrete Math. **315–316** (2014) 128–134.
doi:10.1016/j.disc.2013.10.021

[7] O.V. Borodin, A.O. Ivanova and D.R. Woodall, *Light* C_4 *and* C_5 *in* 3*-polytopes with minimum degree* 5, Discrete Math. **334** (2014) 63–69.
doi:10.1016/j.disc.2014.06.024

[8] O.V. Borodin, A.O. Ivanova and T.R. Jensen, 5*-stars of low weight in normal plane maps with minimum degree* 5, Discuss. Math. Graph Theory **34** (2014) 539–546.
doi:10.7151/dmgt.1748

[9] O.V. Borodin and A.O. Ivanova, *Every* 3*-polytope with minimum degree* 5 *has a* 7*-cycle with maximum degree at most* 15, Sibirsk. Mat. Zh. **56** (2015) 775–789, in Russian.

[10] B. Ferencová and T. Madaras, *On the structure of polyhedral graphs with prescribed edge and dual edge weight*, Acta Univ. M. Belii Ser. Math. **12** (2005) 13–18.

[11] B. Ferencová and T. Madaras, *Light graph in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight*, Discrete Math. **310** (2010) 1661–1675.
doi:10.1016/j.disc.2009.11.027

[12] Ph. Franklin, *The four-color problem*, Amer. J. Math. **44** (1922) 225–236.
doi:10.2307/2370527

[13] P. Hudák and T. Madaras, *On doubly light triangles in plane graphs*, Discrete Math. **313** (2013) 1978–1988.
doi:10.1016/j.disc.2012.11.018

[14] S. Jendrol' and T. Madaras,*On light subgraphs in plane graphs with minimum degree five*, Discuss. Math. Graph Theory **16** (1996) 207–217.
doi:10.7151/dmgt.1035

[15] S. Jendrol', T. Madaras, R. Soták and Zs. Tuza, *On light cycles in plane triangulations*, Discrete Math. **197**–**198** (1999) 453–467.
doi:10.1016/S0012-365X(98)00254-4

[16] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in the plane—a survey*, Discrete Math. **313** (2013) 406–421.
doi:10.1016/j.disc.2012.11.007

[17] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. **19** (1940) 27–43.

[18] T. Madaras and R. Soták, *The* 10*-cycle* C_{10} *is light in the family of all plane triangulations with minimum degree five*, Tatra Mt. Math. Publ. **18** (1999) 35–56.

[19] T. Madaras, *On the structure of plane graphs of minimum face size* 5, Discuss. Math. Graph Theory **24** (2004) 403–411.
doi:10.7151/dmgt.1239

[20] T. Madaras, *Two variations of Franklin's theorem*, Tatra Mt. Math. Publ. **36** (2007) 61–70.

[21] T. Madaras, R. Škrekovski and H.-J. Voss, *The* 7*-cycle* C_7 *is light in the family of planar graphs with minimum degree* 5, Discrete Math. **307** (2007) 1430–1435.
doi:10.1016/j.disc.2005.11.080

[22] E. Steinitz, *Polyeder und Raumeinteilungen*, Enzykl. Math. Wiss. **3** (1922) 1–139.

[23] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann. **58** (1904) 413–426.
doi:10.1007/BF01444968

Received 22 June 2015
Revised 16 January 2016
Accepted 16 January 2016