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AN EXTENSION OF KOTZIG'S THEOREM

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Abstract

In 1955, Kotzig proved that every 3-connected planar graph has an edge with the degree sum of its end vertices at most 13, which is tight. An edge uv is of type (i, j) if $d(u) \leq i$ and $d(v) \leq j$. Borodin (1991) proved that every normal plane map contains an edge of one of the types $(3, 10)$, $(4, 7)$, or $(5, 6)$, which is tight. Cole, Kowalik, and Škrekovski (2007) deduced from this result by Borodin that Kotzig's bound of 13 is valid for all planar graphs with minimum degree δ at least 2 in which every d-vertex, $d \geq 12$, has at most $d - 11$ neighbors of degree 2.

We give a common extension of the three above results by proving for any integer $t \geq 1$ that every plane graph with $\delta \geq 2$ and no d-vertex, $d \geq 11 + t$, having more than $d - 11$ neighbors of degree 2 has an edge of one of the following types: $(2, 10 + t)$, $(3, 10)$, $(4, 7)$, or $(5, 6)$, where all parameters are tight.

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1. Introduction

A plane map is a plane pseudograph (loops and multiple edges are allowed). A *normal plane map* (NPM) is a plane pseudograph in which each vertex and face is incident with at least three edges.

The degree of a vertex or face x in a plane pseudograph, that is the number of edges incident with x, is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-face f satisfies $d(f) \geq k$, etc.

An edge uv is an *(i,j)-edge* or *an edge of type* (i,j) if $d(u) \leq i$ and $d(v) \leq j$. The *weight* $w_2(e)$ of an edge e in a normal plane map (NPM) is the degree-sum of its end-vertices. Let $\delta(G)$ be the minimum degree, and $w_2(G)$ be the minimum weight of an edge in a plane pseudograph G.

By $\mathbf{M_q}$, $\mathbf{G_q}$, and $\mathbf{P_q}$ denote the classes of NPMs, (simple) plane graphs, and 3-connected planar graphs, respectively, with $\delta \geq q$.

Back in 1904, Wernicke [32] proved that every $P \in \mathbf{P_5}$ satisfies $w_2(P) \leq 11$, which is tight. It follows from Lebesgue's results in [29] that each $P \in \mathbf{P_3}$ has an edge of weight at most 14 incident with a 3-vertex, or an edge of weight at most 11, where 11 is sharp. In 1955, Kotzig [27] proved a tight result: $w_2 \leq 13$ in $\mathbf{P_3}$.

Theorem 1 (Kotzig [27])**.** *Every* 3*-connected planar graph has an edge of weight at most* 13*, which is tight.*

In 1972, Erdős (see [21]) conjectured that Kotzig's bound $w_2 \leq 13$ holds also in $\mathbf{G_3}$. Barnette (see [21]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [3].

More generally, Borodin [4–6] proved that every NPM contains an edge of one of the types $(3, 10)$, $(4, 7)$, and $(5, 6)$ (as easy corollaries of some stronger structural facts having applications to coloring of plane graphs, see [14]).

Theorem 2 (Borodin [4–6])**.** *Every normal plane map contains an edge of one of the types* $(3, 10)$*,* $(4, 7)$*, and* $(5, 6)$*, which is tight.*

Note that $\delta(K_{2,t}) = 2$ and $w_2(K_{2,t}) = t + 2$, so w_2 is unbounded in $\mathbf{G_2}$. In addition to forbidding certain collections of cycle lengths, another way to find subclasses of $\mathbf{G_2}$ with bounded w_2 is to impose restrictions on the set of 2-vertices in a graph. An example is forbidding 2*-alternating cycles*, which are cycles $v_1 \cdots v_{2k}$ with $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. This notion, along with its more sophisticated analogues, turns out to be useful for the study of graph coloring, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse plane graphs, see Borodin [14]). Its first application was to show that the total chromatic number of planar graphs with maximum degree Δ at least 14 equals $\Delta + 1$ (Borodin [3]). In some coloring applications, it is important to find a light edge incident with one or two 5^{-}-faces.

Some other results concerning the structure of edge neighborhoods in plane graphs can be found in [1, 2, 4–6, 8–12, 15–19, 22, 24, 25, 30] and recent surveys of Borodin [14] and Jendrol' and Voss [26].

In particular, Theorems 1 and 2 have been refined or extended in several directions, and the following natural extension of Kotzig's Theorem can be proved by a nice short reduction to Theorem 2, as shown in [17].

Theorem 3 (Cole, Kowalik, Škrekovski [17])**.** *Every planar graph with $\delta \geq 2$ in which every d-vertex, $d \geq 12$, has at most $d - 11$ neighbors of degree 2 contains an edge of weight at most 13.*

The purpose of our paper is to prove the following fact, which, in particular, absorbs Theorems 1–3 by putting $t = 1$.

Theorem 4. *For any integer $t \geq 1$, every plane graph with $\delta \geq 2$ such that every d-vertex, $d \geq 11+t$, has at most $d-11$ neighbors of degree 2 contains an edge of one of the following types: $(2, 10+t)$, $(3, 10)$, $(4, 7)$, or $(5, 6)$, where all parameters are tight.*

Note that if we join vertices a and b by $t \geq 2$ multiple edges and paths ax_ib, where $d(x_i) = 2$ whenever $1 \leq i \leq t$, so that to obtain a triangulation, then $d(a) = d(b) = 2t$. In particular, each 2-vertex is adjacent to two $2t$-vertices, which produces all edges of weight much greater than declared in Theorem 4.

This observation leads us to the following definition, which makes it possible to produce a more general and easier proved version of Theorem 4.

A *normal plane quasi-map M* is a connected plane pseudograph with $\delta(M) \geq 2$ such that deleting all 2-vertices at once does not create faces of degree at most 2. We note that all normal plane maps and connected plane graphs with $\delta = 2$ are special cases of normal quasi-maps.

So we are going to prove the following fact.

Theorem 5. *For any integer $t \geq 1$, every normal plane quasi-map with $\delta \geq 2$ such that every d-vertex, $d \geq 11 + t$, has at most $d - 11$ neighbors of degree 2 contains an edge of one of the following types: $(2, 10+t)$, $(3, 10)$, $(4, 7)$, or $(5, 6)$, where all parameters are tight.*

2. Proof of Theorem 5

2.1. The tightness of Theorem 5

We first define a graph G_p, $p \geq 1$, by taking a perfect matching $\{e_1, \ldots, e_6\}$ of the icosahedron, putting a 3-vertex inside every face, and joining the end vertices of every e_i, $1 \leq i \leq 6$, by p independent paths of length 2 going through 2-vertices. It is easy to see that G_p has only $(10+p)$-vertices and 3^--vertices, and the 3^--vertices are pairwise non-adjacent.

We next show that the hypothesis "no d-vertex, $d \geq 11+t$, having at most $d-11$ neighbors of degree 2" cannot be relaxed. Indeed, suppose we relax it at d_0, $d_0 \geq 11+t$, by allowing d_0-vertices to have at least d_0-10 neighbors of degree 2, while for other d-vertices, $d \geq 11+t$, the requirement to have at most $d-11$ neighbors of degree 2 is preserved. Then the theorem strengthened this way becomes wrong due to G_{d_0-10}. Indeed, every d_0-vertex has d_0-10 rather than d_0-11 neighbors of degree 2. However, $d_0 > 10+t$, so G_{d_0-10} does not contain an edge of any of the types in the conclusion of Theorem 5, a contradiction.

Now to show the tightness of the term $(2, 10+t)$ with $t \geq 1$, it suffices to take G_t. The tightness of $(3, 10)$ and $(5, 6)$ follows from the Archimedean solids $(3, 10, 10)$ and $(5, 6, 6)$, respectively.

Finally, to justify the term $(4, 7)$, we take the $(3, 4, 4, 4)$-Archimedean solid, which is a 4-regular plane graph such that each vertex is incident with a 3-face and three 4-faces, and put a 4-vertex into each 4-face.

2.2. Structural properties of a counterexample to Theorem 5

Suppose that a normal plane quasi-map M is a counterexample to the main statement of Theorem 5, and M is maximal with respect to the addition of edges.

The *underlying map* $U(M)$ *of* M is obtained from M by deleting all 2-vertices at once. We note that $U(M)$ is connected, since the $(11+t)^+$-neighbors a and c of each 2-vertex b either coincide or are adjacent due to the maximality of M. (Indeed, otherwise adding the non-loop edge ac "close" to the path abc would create a "denser" counterexample to Theorem 5.)

Lemma 6. *$U(M)$ is a triangulation.*

Proof. Suppose there is an internal 4^+-face f with boundary $\partial(f) = v_1 v_2 \cdots v_{d(f)}$ in $U(M)$. By definition, we have $d(v_i) \geq 3$ in M whenever $1 \leq i \leq d(f)$.

First note that no 2-vertex b of M joins two non-consecutive vertices in $\partial(f)$, for otherwise we could create another counterexample by adding an edge close to b inside f. This implies that it is possible to draw an edge $v_{i-1}v_{i+1}$ (addition modulo $d(f)$) inside f.

If $d(v_2) \leq 4$, then $d(v_1)$ and $d(v_3)$ are large enough, depending on $d(v_2)$, since M is a counterexample, so adding an edge $v_1 v_3$ inside f yields another counterexample to Theorem 5, a contradiction.

Finally, if $d(v_i) \geq 5$ whenever $1 \leq i \leq d(v)$, then adding an edge $v_1 v_3$ creates a $(6^+, 6^+)$-edge, which does not appear in the statement of Theorem 5, a contradiction. ∎

We now look more attentively at the 2-vertices of M lying inside an internal 3-face $f = uvw$ of $U(M)$. Suppose we have paths ux_iv, vy_jw, and wz_ku, where $d(x_i) = d(y_j) = d(z_k) = 2$ whenever $0 \le i \le l_{uv}$, $0 \le j \le l_{vw}$, and $0 \le k \le l_{wu}$, respectively. Thus l_{uv} vertices of degree 2 are attached to the edge uv, etc. Also suppose that the indices of 2-vertices attached to each of the three edges of $\partial(f)$ grow as we proceed from $\partial(f)$ inwards f.

If $l_{uv} \ge 1$, then there is a *coastal* 3-face ux_1v of M. If $l_{uv} \ge 2$, then there are also $l_{uv} - 1$ *intermediate* 4-faces ux_ivx_{i+1} of M, where $1 \le i \le l_{uv} - 1$. Similar notation is used for the coastal and intermediate 4^--faces of M associated with the edges vw and wu.

Note that the *central face* f_c of M, lying inside f and being neither coastal nor intermediate, satisfies $d(f_c) = 3 + \min\{1, l_{uv}\} + \min\{1, l_{vw}\} + \min\{1, l_{wu}\}$. In other words, $d(f_c)$ equals 3 plus the number of non-zero elements in $\{l_{uv}, l_{vw}, l_{wu}\}$.

Let $F(f)$ be the set of the central, coastal and intermediate faces of M that partition the interior of the internal face f of $U(M)$. Similar notions and notation can be used for the external 3-face of $U(M)$.

Lemma 7. *If $f = uvw$ is a face of $U(M)$, then*

$$l_{uv} + l_{vw} + l_{wu} = \sum_{f' \in F(f)} (d(f') - 3).$$

Proof. This fact is proved by a straightforward induction on $l_{uv} + l_{vw} + l_{wu}$, which is left to the reader. ■

2.3. Discharging

Euler's formula $|V| - |E| + |F| = 2$ for the counterexample M may be written as

$$\sum_{v \in V(M)} (d(v) - 6) + \sum_{f^* \in F(M)} (2d(f^*) - 6) = -12, \tag{1}$$

where $V(M)$, $E(M)$, and $F(M)$ are sets of vertices, edges, and faces of M, respectively. By $F(U(M))$ denote the sets of faces of $U(M)$.

As we remember, every $f \in F(U(M))$ is partitioned into the central face plus possibly coastal and intermediate faces, and each 2-vertex as well as each face of M belongs to the partition of precisely one $f \in F(U(M))$.

Every vertex $v \in V(M)$ contributes the *initial charge* $\mu(v) = d(v) - 6$ to (1). The *initial charge* of the face $f \in F(U(M))$ is defined to be $\mu(f) = \sum_{f' \in F(f)} (2d(f') - 6)$. Now (1) can be written as follows.

$$\sum_{v \in V(M)} \mu(v) + \sum_{f \in F(U(M))} \mu(f) = -12. \tag{2}$$

Using the properties of M as a counterexample, we define a local redistribution of the charge preserving its total value such that the *new charge* μ' is non-negative for all $v \in V$ and $f \in F(U(M))$. This will contradict the fact that the sum of the new charges is, by (2), equal to -12.

For $f \in F(U(M))$, by $V_2(f)$ denote the set of 2-vertices that belong to f. Our rules of discharging are as follows.

R1. *Each face f of the underlying quasi-map $U(M)$ of M gives 2 to each 2-vertex from $V_2(f)$.*

R2. *Every 5^--vertex v receives along every edge vw from w*

(a) 1 *if $d(v) = 2$, or*

(b) $\frac{6-d(v)}{d(v)}$ *otherwise.*

Now we check that $\mu'(v) \geq 0$ for $d(v) \in V$ and $\mu'(f) \geq 0$ for $f \in F(U(M))$. If $f \in F(U(M))$, then

$$\mu'(f) = \mu(f) - 2 \sum_{v \in V_2(f)} 1 = \sum_{f' \in F(f)} (2d(f') - 6)) - 2 \sum_{v \in V_2(f)} 1 = 0$$

by R1 combined with Lemma 7, as desired.

Now suppose $d(v) \in V$.

Case 1. $d(v) = 2$. We have $\mu'(v) = 2 - 6 + 2 + 2 \cdot 1 = 0$ by R1 and R2(a).

Case 2. $3 \leq d(v) \leq 5$. Here, $\mu'(v) = d(v) - 6 + d(v) \cdot \frac{6-d(v)}{d(v)} = 0$ by R2(b).

Case 3. $d(v) = 6$. Note that v has no 5^--neighbors, so $\mu'(v) = \mu(v) = 0$.

Remark 8. Note that no vertex in M can have two 5^--neighbors adjacent to each other due to the absence of $(5,5)$-edges. Since $U(M)$ is a triangulation by Lemma 6, it follows that v has at most $\left\lfloor \frac{d(v)-d_2(v)}{2} \right\rfloor$ neighbors of degree from 3 to 5, where $d_2(v)$ is the number of 2-neighbors.

Case 4. $d(v) = 7$. Since v can only give $\frac{1}{5}$ along at most $\lfloor \frac{7}{2} \rfloor$ edges leading to 5-vertices by R2(b) due to the absence of $(4,7)$-edges combined with Remark 8, we have $\mu'(v) \geq 1 - 3 \cdot \frac{1}{5} > 0$.

Case 5. $8 \leq d(v) \leq 10$. Now v has no 3^--neighbors, so it can give at most $\frac{1}{2}$ along each of at most $\left\lfloor \frac{d(v)}{2} \right\rfloor$ edges by Remark 8, hence $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} \cdot \frac{1}{2} = \frac{3(d(v)-8)}{4} \geq 0$.

Case 6. $11 \leq d(v) \leq 10 + t$. Since $d_2(v) = 0$ due to the absence of $(2, 10+t)$-edges, it similarly follows that $\mu'(v) \geq d(v) - 6 - \left\lfloor \frac{d(v)}{2} \right\rfloor \geq \left\lfloor \frac{d(v)-11}{2} \right\rfloor \geq 0$ by Remark 8.

Case 7. $d(v) \geq 11 + t$. By the assumption of Theorem 5, we have $d(v) - d_2(v) \geq 11$. Therefore, $\mu'(v) \geq d(v) - 6 - d_2(v) \cdot 1 - \left\lfloor \frac{d(v)-d_2(v)}{2} \right\rfloor \cdot 1 \geq \left\lceil \frac{d(v)-d_2(v)}{2} \right\rceil - 6 \geq 0$ by R2 in view of Remark 8.

This completes the proof of Theorem 5.

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