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EDGE-TRANSITIVITY OF CAYLEY GRAPHS GENERATED BY TRANSPOSITIONS

Ashwin Ganesan

53 Deonar House
Deonar Village Road, Deonar
Mumbai 400088, Maharashtra, India

e-mail: ashwin.ganesan@gmail.com

Abstract

Let S be a set of transpositions generating the symmetric group S_n $(n \geq 5)$. The transposition graph of S is defined to be the graph with vertex set $\{1, \ldots, n\}$, and with vertices i and j being adjacent in $T(S)$ whenever $(i, j) \in S$. In the present note, it is proved that two transposition graphs are isomorphic if and only if the corresponding two Cayley graphs are isomorphic. It is also proved that the transposition graph $T(S)$ is edge-transitive if and only if the Cayley graph $\mathrm{Cay}(S_n, S)$ is edge-transitive.

Keywords: Cayley graphs, transpositions, automorphisms of graphs, edge-transitive graphs, line graphs, Whitney's isomorphism theorem.

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1. Introduction

Let $X = (V, E)$ be a simple, undirected graph. An automorphism of X is a permutation of the vertex set that preserves adjacency. The automorphism group of X, denoted by $\mathrm{Aut}(X)$, is the set of all automorphisms of the graph X, that is, $\mathrm{Aut}(X) := \{g \in \mathrm{Sym}(V) : E^g = E\}$. A graph X is said to be vertex-transitive if for any two vertices $u, v \in V(X)$, there exists an automorphism $g \in \mathrm{Aut}(X)$ that takes u to v. A graph X is said to be edge-transitive if for any two edges $\{u, v\}, \{x, y\} \in E(X)$, there exists an automorphism $g \in \mathrm{Aut}(X)$ such that $\{u^g, v^g\} = \{x, y\}$. In other words, X is edge-transitive iff the action of $\mathrm{Aut}(X)$ on the edge set $E(X)$ has a single orbit. A graph X is said to be arc-transitive if for any two ordered pairs $(u, v), (x, y)$ of adjacent vertices, there is an automorphism $g \in \mathrm{Aut}(X)$ such that $u^g = x$ and $v^g = y$.

Given a group H and a subset S of H such that $1 \notin S$ and $S = S^{-1}$, the Cayley graph of H with respect to S, denoted by $\mathrm{Cay}(H, S)$, is the graph with vertex set H and edge set $\{\{h, sh\} : h \in H, s \in S\}$. Let r_h be the bijection $x \mapsto xh$ from H to itself. The right regular representation of H, denoted by $R(H)$, is the set $\{r_h : h \in H\}$ of permutations of H. The automorphism group of a Cayley graph $\mathrm{Cay}(H, S)$ contains the right regular representation $R(H)$ as a subgroup, whence all Cayley graphs are vertex-transitive (cf. [2]). Let S be a set of transpositions in the symmetric group S_n. The transposition graph of S, denoted by $T(S)$, is the graph with vertex set $[n] = \{1, \ldots, n\}$, and with vertices i and j being adjacent in $T(S)$ whenever $(i, j) \in S$. A set S of transpositions in S_n generate S_n if and only if the transposition graph $T(S)$ is connected (cf. [7]).

If S is a set of transpositions in S_n, then the Cayley graph $\mathrm{Cay}(S_n, S)$ is called a Cayley graph generated by transpositions. The family of Cayley graphs generated by transpositions has been well-studied because it is a suitable topology for consideration in interconnection networks (cf. [8, 10, 12] for surveys). This family of graphs has better degree-diameter properties than the hypercube [1]. The automorphism group of Cayley graphs generated by transpositions has also been determined in some cases (cf. [4, 5, 6, 20]). In the present note, we further study the symmetry properties of $\mathrm{Cay}(S_n, S)$, especially with regard to how symmetry properties of $\mathrm{Cay}(S_n, S)$ depend on the properties of the generating set S.

The main result of this note is the following.

Theorem 1. *Let $n \geq 5$.*

(a) *Let S, S' be sets of transpositions generating S_n. Then, the Cayley graphs $\mathrm{Cay}(S_n, S)$ and $\mathrm{Cay}(S_n, S')$ are isomorphic if and only if the transposition graphs $T(S)$ and $T(S')$ are isomorphic.*

(b) *Let S be a set of transpositions generating S_n. Then, the Cayley graph $\mathrm{Cay}(S_n, S)$ is edge-transitive if and only if the transposition graph $T(S)$ is edge-transitive.*

Remark 2. Three comments and corollaries of Theorem 1.

1. The reverse implication of Theorem 1(a) is proved in [12, Theorem 4.5]. Parts of Theorem 1 are stated in Heydemann *et al.* [9] and Heydemann [8] without a proof; they attribute the result to unpublished reports. We could not find a proof of Theorem 1 in the literature.

2. If the transposition graph $T(S)$ is the path graph on n vertices, then the Cayley graph $\mathrm{Cay}(S_n, S)$ is called the bubble-sort graph of dimension n. This Cayley graph is called the bubble-sort graph because of its relation to the (inefficient) bubble-sort algorithm for sorting an array. Given a permutation π in the form of a linear arrangement $[\pi(1), \pi(2), \ldots, \pi(n)]$, the bubble-sort algorithm sorts the array by swapping elements in consecutive positions of the array. The

minimum number of swaps of consecutive elements needed to sort an array π is exactly the distance in the bubble-sort graph $\mathrm{Cay}(S_n, S)$ between the vertex π and the identity vertex e. If the transposition graph $T(S)$ is the n-cycle graph, then the Cayley graph $\mathrm{Cay}(S_n, S)$ is called the modified bubble-sort graph. Thus, the modified bubble-sort graph is obtained from the bubble-sort graph by adding one more generator (and hence by adding extra edges to the bubble-sort graph).

Some of the literature (cf. [11, 13, 14]) incorrectly assumes the bubble-sort graph is edge-transitive. Since the path graph is not edge-transitive, Theorem 1(b) implies that the bubble-sort graph is not edge-transitive.

On the other hand, if $T(S)$ is the complete graph K_n, the cycle C_n or the star $K_{1,n-1}$, then the corresponding Cayley graphs $\mathrm{Cay}(S_n, S)$, which are referred to as the complete transposition graph, the modified bubble-sort graph and the star graph, respectively, are edge-transitive because K_n, C_n and $K_{1,n-1}$ are edge-transitive.

3. The vertex-connectivity of a connected graph X, denoted by $\kappa(X)$, is the minimal number of vertices whose removal disconnects the graph (cf. [3]). Clearly, $\kappa(X)$ is at most the minimum degree $\delta(X)$. By Menger's Theorem [16], graphs with high connectivity have a large number of parallel paths between any two nodes, making communication in such interconnection networks efficient and fault-tolerant. Latifi and Srimani [13, 14] proved that the complete transposition graphs have vertex-connectivity equal to the minimum degree.

Watkins [18] proved that the vertex-connectivity of a connected edge-transitive graph is maximum possible. Thus, Theorem 1(b) (in conjunction with the theorem of Watkins [18]) gives another proof that many families of graphs, including the complete transposition graphs, modified bubble-sort graphs and the star graphs, have vertex-connectivity that is maximum possible.

Incidentally, Mader [15] showed that if X is a connected vertex-transitive graph that does not contain a K_4, then X has vertex-connectivity equal to its minimum degree. Since all Cayley graphs generated by transpositions are bipartite, they do not contain a K_4, and so all connected Cayley graphs generated by transpositions have vertex-connectivity maximum possible. This gives an independent proof of the optimal vertex-connectivity of connected Cayley graphs generated by transpositions.

We use the following notation. Let $X = (V, E)$ be a simple, undirected graph, and let $v \in V$. Then $X_i(v)$ denotes the set of vertices of X whose distance to the vertex v is exactly i. In other words, $X_i(v)$ is the ith layer in the distance partition of X with respect to the vertex v. The identity element of the symmetric group S_n and the corresponding vertex of a Cayley graph of S_n are both denoted by e. Thus, if S is a set of transpositions generating S_n ($n \geq 5$) and X is the Cayley graph $\mathrm{Cay}(S_n, S)$, then $X_0(e) = \{e\}$ and $X_1(e) = S$.

2. Preliminaries

Let $X = (V, E)$ be a graph. The line graph of X, denoted by $L(X)$, is the graph with vertex set E, and $e, f \in E(X)$ are adjacent vertices in $L(X)$ iff e, f are incident edges in X. If two graphs are isomorphic, then clearly their line graphs are isomorphic. Whitney proved that if X and Y are connected graphs with isomorphic line graphs, then X and Y are also isomorphic unless one of X or Y is K_3 and the other is $K_{1,3}$. Every automorphism of a graph induces an automorphism of the line graph. Whitney [19] showed that we can go in the reverse direction: every automorphism of the line graph $L(T)$ is induced by a unique automorphism of T if T is a connected graph on 5 or more vertices.

Theorem 3 ([17, 19])**.** *Let T be a connected graph on 5 or more vertices. Then, every automorphism of the line graph $L(T)$ is induced by a unique automorphism of T, and the automorphism group of T and of $L(T)$ are isomorphic.*

Given a set S of transpositions in S_n, let $\mathrm{Aut}(S_n, S)$ denote the set of automorphisms of S_n that fixes S setwise. If $G = \mathrm{Aut}(\mathrm{Cay}(S_n, S))$, then $\mathrm{Aut}(S_n, S)$ is contained in G_e (cf. [2]). In the sequel, we shall refer to the following result due to Feng [4] and its proof (the proof sketch is given below).

Theorem 4 ([4, Theorem 2.1])**.** *Let S be a set of transpositions in S_n $(n \geq 3)$. Then, the group of automorphisms of S_n that fixes S setwise is isomorphic to the automorphism group of the transposition graph of S, i.e., $\mathrm{Aut}(S_n, S) \cong \mathrm{Aut}(T(S))$.*

Proof sketch. In the proof of this result, the bijective correspondence between $\mathrm{Aut}(S_n, S)$ and $\mathrm{Aut}(T(S))$ is as follows. If $g \in S_n$ is an automorphism of the transposition graph $T(S)$, then conjugation by g, denoted by c_g, is the corresponding element in $\mathrm{Aut}(S_n, S)$. In the other direction, every element in $\mathrm{Aut}(S_n, S)$ coincides with a conjugation c_g by some element $g \in S_n$, and it can be shown that if $c_g \in \mathrm{Aut}(S_n, S)$, then $g \in \mathrm{Aut}(T(S))$. ■

We shall also refer to the following result.

Proposition 5 ([6, Proposition 3.2])**.** *Let S be a set of transpositions generating S_n $(n \geq 5)$ and let G be the automorphism group of $X = \mathrm{Cay}(S_n, S)$. Let $g \in G_e$. Then, the restriction map $g|_S$ is an automorphism of the line graph of the transposition graph of S.*

3. Proof of Theorem 1

In this section, we prove both parts of Theorem 1.

***Proof of Theorem* 1(a).** Let $X = \mathrm{Cay}(S_n, S)$ and $X' = \mathrm{Cay}(S_n, S')$. Suppose f is an isomorphism from the transposition graph $T(S)$ to the transposition graph $T(S')$. We show that the Cayley graphs X and X' are isomorphic. Suppose f takes i to i', for $i \in [n]$. Since f preserves adjacency and nonadjacency, the transposition $(i, j) \in S$ iff $(i', j') \in S'$. Let σ be the map from S_n to itself obtained by conjugation by f. Denote the image of $g \in S_n$ under the action of σ by g'. Since f is an isomorphism, it takes the edge set of $T(S)$ to the edge set of $T(S')$. Hence, $S^\sigma = S$.

We show that $\sigma : V(X) \to V(X')$ is an isomorphism from X to X'. Suppose vertices g, h are adjacent in X. Then there exists an $s \in S$ such that $sg = h$. Applying σ to both sides, we get that $(sg)^\sigma = h^\sigma$, whence $s'g' = h'$. Note that $s' \in S'$. Hence, vertices g' and h' are adjacent in X'. By applying σ^{-1} to both sides, we get the converse that if g', h' are adjacent vertices in X', then g, h are adjacent vertices in X. We have shown that X and X' are isomorphic.

Now suppose the Cayley graphs X and X' are isomorphic, and let $f : V(X) \to V(X')$ be an isomorphism. Since X' admits the right regular representation $R(S_n)$ as a subgroup of automorphisms, if f takes the identity vertex $e \in V(X)$ to $h' \in V(X')$, then f composed with $r_{h'}^{-1} \in R(S_n)$ takes e to e. Therefore, we may assume without loss of generality that the isomorphism f maps the identity vertex of X to the identity vertex of X'. The neighbors of e in the Cayley graphs X and X' are S and S', respectively. Hence, f takes S to S'. Consider the restriction map $f|_S$. By the proof of Proposition 5, the restriction map is an isomorphism from the line graph of $T(S)$ to the line graph of $T(S')$. Denote these two transposition graphs $T(S), T(S')$ by T, T', respectively, and their line graphs by $L(T), L(T')$, respectively. We have just shown that the line graphs $L(T)$ and $L(T')$ are isomorphic.

Since S, S' generate S_n, their transposition graphs T, T', respectively, are connected. Because X and X' are isomorphic, $|E(T)| = |E(T')|$ and $|V(T)| = |V(T')|$. Therefore, it is not possible that one of T, T' is K_3 and the other $K_{1,3}$. Since the line graphs $L(T)$ and $L(T')$ are isomorphic, by Whitney's Theorem 3, the transposition graphs T and T' are isomorphic. ■

Given two subgroups H and K of G, HK denotes the set $\{hk : h \in H, k \in K\}$. We say G is the semidirect product of H by K, denoted by $G = H \rtimes K$, if $G = HK$, H is a normal subgroup of G, and $H \cap K = 1$.

Proposition 6. *Let S be a set of transpositions generating S_n ($n \geq 5$). Let G be the automorphism group of $X = \mathrm{Cay}(S_n, S)$ and let L_e denote the set of elements in G_e that fixes the vertex e and each of its neighbors. Then, $G_e = L_e \rtimes \mathrm{Aut}(S_n, S)$.*

Proof. Let $g \in G_e$. Then $g|_S$ is an automorphism of the line graph of $T(S)$ (cf. Proposition 5). By Whitney's Theorem 3, the automorphism $g|_S$ of the line graph of $T(S)$ is induced by an automorphism h of $T(S)$. Conjugation by h, denoted by c_h, which is an element of $\mathrm{Aut}(S_n, S)$, has the same action on S as g, i.e., $g|_S = c_h|_S$. This implies that $gc_h^{-1} \in L_e$, whence $g \in L_e c_h$. It follows that G_e is contained in $L_e \,\mathrm{Aut}(S_n, S)$. Clearly $L_e \,\mathrm{Aut}(S_n, S)$ is contained in G_e. Hence, $G_e = L_e \,\mathrm{Aut}(S_n, S)$.

Since L_e is a normal subgroup of G_e (cf. [2]), it remains to show that $L_e \cap \mathrm{Aut}(S_n, S) = 1$. Each element in L_e fixes $X_1(e)$ pointwise. The only element in $\mathrm{Aut}(S_n, S)$ which fixes $X_1(e)$ pointwise is the trivial permutation of S_n because if $g \in \mathrm{Aut}(S_n, S)$ fixes $X_1(e)$ pointwise, then the restriction map $g|_S$ is a trivial automorphism of the line graph of $T(S)$, and hence is induced by the trivial automorphism h of $T(S)$. Since g is conjugation by h (cf. proof of Theorem 4), $g = 1$. We have shown that the only element in $\mathrm{Aut}(S_n, S)$ which fixes S pointwise is the trivial permutation of S_n. It follows that $L_e \cap \mathrm{Aut}(S_n, S) = 1$. ∎

***Proof of Theorem* 1(b).** Suppose the transposition graph $T(S)$ is edge-transitive. Let G be the automorphism group of $X = \mathrm{Cay}(S_n, S)$. To prove X is edge-transitive, it suffices to show that G_e acts transitively on $X_1(e)$. Let $t, k \in X_1(e) = S$. Note that t, k are edges of $T(S)$. By hypothesis, there exists an automorphism $g \in S_n$ of $T(S)$ that takes the edge t to the edge k. Conjugation by g, denoted by c_g, is an automorphism of S_n that takes permutation $t \in S_n$ to k. Also, $c_g \in \mathrm{Aut}(S_n, S)$ (cf. proof of Theorem 4). Since $\mathrm{Aut}(S_n, S) \leq G_e$, G_e contains an element c_g which takes t to k. It follows that G_e acts transitively on $X_1(e)$.

For the converse, suppose the Cayley graph $\mathrm{Cay}(S_n, S)$ is edge-transitive. Fix $t \in S$. Let r_t be the map from S_n to itself that takes x to xt. Observe that r_t takes the arc (e, t) to the arc (t, e), since $t^2 = e$. Hence, the Cayley graph $\mathrm{Cay}(S_n, S)$ is arc-transitive. This implies that G_e acts transitively on $X_1(e) = S$. Since G_e acts transitively on $X_1(e)$ and L_e fixes $X_1(e)$ pointwise, the formula $G_e = L_e \,\mathrm{Aut}(S_n, S)$ (cf. Proposition 6) implies that $\mathrm{Aut}(S_n, S)$ acts transitively on $X_1(e)$.

Let t, k be two edges of the transposition graph $T(S)$, so that $t, k \in X_1(e)$. By the argument in the previous paragraph, there exists an element $g \in \mathrm{Aut}(S_n, S)$ that takes the vertex t of X to the vertex k of X. By the bijective correspondence between $\mathrm{Aut}(S_n, S)$ and $\mathrm{Aut}(T(S))$ (cf. proof of Theorem 4), there exists an automorphism h of $T(S)$ such that $g = c_h$, where c_h denotes conjugation by h, and such that h takes the edge t of the transposition graph to the edge k. Thus, the set of permutations $\{h \in S_n : c_h \in \mathrm{Aut}(S_n, S)\}$ is contained in $\mathrm{Aut}(T(S))$ and acts transitively on the edges of $T(S)$. ∎

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