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HOW LONG CAN ONE BLUFF IN THE DOMINATION GAME?

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Abstract

The domination game is played on an arbitrary graph G by two players, Dominator and Staller. The game is called Game 1 when Dominator starts it, and Game 2 otherwise. In this paper bluff graphs are introduced as the graphs in which every vertex is an optimal start vertex in Game 1 as well as in Game 2. It is proved that every minus graph (a graph in which Game 2 finishes faster than Game 1) is a bluff graph. A non-trivial infinite family of minus (and hence bluff) graphs is established. minus graphs with game domination number equal to 3 are characterized. Double bluff graphs are also introduced and it is proved that Kneser graphs $K(n, 2)$, $n \geq 6$, are double bluff. The domination game is also studied on generalized Petersen graphs and on Hamming graphs. Several generalized Petersen graphs that are bluff graphs but not vertex-transitive are found. It is proved that Hamming graphs are not double bluff.

Keywords: domination game, game domination number, bluff graphs, minus graphs, generalized Petersen graphs, Kneser graphs, Cartesian product of graphs, Hamming graphs.

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1. Introduction

The domination game [5] is played on an arbitrary graph G by two players, *Dominator* and *Staller*. They are taking turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated. The game ends when no move is possible; then the score of the game is the total number of vertices chosen. Dominator wants to minimize the score, while Staller wants to maximize it. By *Game* 1 (*Game* 2, resp.) we mean a game in which Dominator (Staller, resp.) has the first move. Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$ (the *Staller-start game domination number* $\gamma_g'(G)$, resp.) of a graph G, denotes the score of Game 1 (Game 2, resp.).

The game already received considerable attention. One of the reasons for it is the 3/5-conjecture due to Kinnersley, West and Zamani [20]. Using a powerful greedy discharging-like method due to Bujtás [6, 7], the conjecture was successfully attacked [8, 15], so that the remaining open case is formed by the graphs with minimum degree 1. Extending [6, 7], Schmidt [23] determined a largest known class of trees for which the conjecture holds. In this respect we also mention that forests with the game domination number equal to the domination number were characterized in [22]. Bujtás's method also led to the 4/5-theorem for the total version of the domination game [16, 17] and was applied elsewhere [9]. For additional aspects of the domination game see [1, 4, 10, 21]. Finally, a closely related disjoint domination game was recently introduced in [11].

A fundamental result about the domination game is the following.

Theorem 1.1 [5, 20]**.** *For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.*

We say that a graph G *realizes* (k, ℓ) if $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. By Theorem 1.1, G realizes either $(k, k+1)$, (k, k), or $(k, k-1)$ (for some integer k) and is called a PLUS *graph*, an EQUAL *graph*, or a MINUS *graph*, respectively. We also say that G is either a $(k,+)$ *graph*, a $(k,=)$ *graph*, or a $(k,-)$ *graph*.

The MINUS graphs play a special role. It was proved in [20] that no partially dominated forest is a MINUS graph, in particular, no forest is a MINUS graph.

Further it was demonstrated in [12] that the game on disjoint unions of graphs whose partially dominated subgraphs are not MINUS is much easier than in general. Now, motivated by Observation 2.5 from [12] we realized that every MINUS graph has the following very special property: every vertex is an optimal start vertex in both games. Hence we say that a graph is a *bluff graph* if every vertex is an optimal start vertex for Dominator in Game 1 and every vertex is also an optimal start vertex for Staller in Game 2. The natural question appears whether there are also bluff graphs which are EQUAL or PLUS graphs.

We proceed as follows. In the rest of this section we introduce additional concepts and notation, and recall or prove results needed later. In the subsequent section we obtain several general results about bluff graphs. Among other results we prove that every MINUS graph is a bluff graph and establish a non-trivial infinite family of MINUS (and hence bluff) graphs. We follow with a characterization of $(3,-)$ graphs in Section 3. Then, in Section 4, we introduce the double bluff graphs as bluff graphs where after the first move, any legal answer is an optimal second move for any player. We prove that Kneser graphs $K(n,2)$, $n \geq 6$, are such. We find this fact quite surprising. Note that the Petersen graph, that is, the Kneser graph $K(5,2)$, is not double bluff, hence it seems that some intrinsic symmetry property stronger than vertex-transitivity might be the reason for a graph to be double-bluff. We conclude with a section in which the domination game is studied on generalized Petersen graphs and on Hamming graphs. In particular, several generalized Petersen graphs that are bluff graphs but not vertex-transitive are found.

Throughout the paper we use the convention that $d_1, d_2, \ldots$ denotes the sequence of vertices chosen by Dominator and $s_1, s_2, \ldots$ the sequence chosen by Staller in Game 1. Similarly, we use the convention that $s'_1, s'_2, \ldots$ denotes the sequence of vertices chosen by Staller and $d'_1, d'_2, \ldots$ the sequence chosen by Dominator in Game 2.

A *partially-dominated graph* is a graph together with a declaration that some vertices are already dominated, that is they need not be dominated in the rest of the game. For a vertex subset S of a graph G, let $G|S$ denote the partially dominated graph in which vertices from S are already dominated (note that S can be an arbitrary subset of $V(G)$, and not only a union of closed neighborhoods of some vertices). The notions MINUS, EQUAL, and PLUS extend naturally to partially dominated graphs. We use the following two earlier results.

Theorem 1.2 [20, Lemma 2.1 (Continuation Principle)]. *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

Lemma 1.3 [12, Observation 2.5]. *If a partially dominated graph $G|S$ is a $(k,-)$ graph, then for any legal move u in $G|S$, the graph $G|(S \cup N[u])$ is a $(k-2,+)$ graph.*

If u is a vertex of a graph G, then let $G\{u\}$ denote the graph with $V(G\{u\}) = V(G) \cup \{u'\}$ and $E(G\{u\}) = E(G) \cup \{u'v : v \in N[u]\}$. We say that u' is a *twin* of u.

Proposition 1.4. *Let u be a vertex of a graph G. Then $\gamma_g(G) = \gamma_g(G\{u\})$ and $\gamma'_g(G) = \gamma'_g(G\{u\})$.*

Proof. Let u' be the constructed twin of u in $G\{u\}$. Then $N[u] = N[u']$ holds. Hence in the course of the game u' is dominated if and only if u is dominated. Therefore any strategy in G uniquely corresponds to a strategy in $G\{u\}$ and vice versa. The result follows. ■

A graph is called *twin-free* if it contains no pair of vertices u and v such that $N[u] = N[v]$. It follows by Proposition 1.4 that it suffices to consider only twin-free graphs when studying graphs with respect to the MINUS, EQUAL, and PLUS properties.

Finally, the *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with vertex set $V(G \,\Box\, H) = V(G) \times V(H)$ in which (g, h) is adjacent to (g', h') if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$.

2. Bluff Graphs

In this section we first detect some rather obvious bluff graphs, including graphs obtained from bluff graphs by inserting twins. Then we prove the key property that every MINUS graph is a bluff graph and show that these two properties are equivalent as soon as the graph in question contains a pendant vertex. We conclude the section by establishing a non-trivial infinite family of MINUS (and hence bluff) graphs.

Observe first that it is possible that every vertex is an optimal start for Dominator but not for Staller (e.g. P_5) and vice versa (e.g. P_4). On the other hand, vertex-transitive graphs are bluff graphs. For instance, complete graphs and cocktail-party graphs are bluff graphs that realize $(1, 1)$ and $(2, 2)$, respectively. (Recall that a *cocktail-party graph* is a graph obtained from a complete graph of even order by removing a perfect matching.) The latter is also true for the smallest cocktail-party graph consisting of two isolated vertices. More generally, an arbitrary disjoint union of some complete graphs and some cocktail-party graphs is again a bluff graph. However, a disjoint union of two bluff graphs need not be such. For a small example consider the disjoint union of C_6 and K_1.

The following result is parallel to Proposition 1.4.

Proposition 2.1. *If u is a vertex of a bluff graph G, then $G\{u\}$ is also a bluff graph.*

Proof. Let u' be the constructed twin of u in $G\{u\}$. Since G is a bluff graph, u is an optimal start vertex in both games. Because $G|N[u] = G|N[u']$, also u' is an optimal start vertex in both games. Since in the course of the game u' is dominated if and only if u is dominated, any strategy in G uniquely corresponds to a strategy in $G\{u\}$ and vice versa. It follows that any vertex $x \neq u'$ is also an optimal start vertex in both games played on $G\{u\}$. ■

Hence while studying bluff graphs we may in principle restrict ourselves to twin-free graphs. Proposition 2.1 can of course be applied iteratively. In particular, applying it n-times to the same vertex u of G, the obtained graph can be described as the graph obtained from G by replacing u with a complete graph on $n+1$ vertices and preserving all the adjacencies of u.

Using Lemma 1.3 we can deduce the following fundamental result.

Theorem 2.2. *Every* MINUS *graph is a bluff graph. Moreover, if G is a connected graph with $\gamma_g(G) \geq 2$ and $\delta(G) = 1$, then G is a bluff graph if and only if G is a* MINUS *graph.*

Proof. Let G be a $(k,-)$ graph and let u be an arbitrary vertex of G. Then by Lemma 1.3, $\gamma_g'(G|N[u]) = k-1$, which implies that u is an optimal start vertex for Dominator. By the same lemma we have $\gamma_g(G|N[u]) = k-2$, hence u is also an optimal start vertex for Staller.

For the second assertion we only need to prove that the condition is necessary. Suppose that G is a bluff graph. Let u be a pendant vertex of G and let w be its (unique) neighbor. Since G is a bluff graph, Dominator may start Game 1 by selecting u. Suppose that Staller replied by playing w. This is a legal move because $\gamma_g(G) \geq 2$ and because G is connected. Since the move of Staller may not be optimal, we infer that $\gamma_g(G|N[w]) \leq \gamma_g(G) - 2$. On the other hand, using the assumption that G is a bluff graph, selecting w as the first move of Staller in Game 2 we get $\gamma_g(G|N[w]) = \gamma_g'(G) - 1$. It follows that $\gamma_g'(G) - 1 \leq \gamma_g(G) - 2$, that is, $\gamma_g'(G) \leq \gamma_g(G) - 1$. By Theorem 1.1 it follows that $\gamma_g'(G) = \gamma_g(G) - 1$, that is, G is a MINUS graph. ■

The list of known MINUS (and hence bluff), not vertex-transitive graphs includes complete bipartite graphs $K_{m,n}$, $m > n \geq 3$, the Cartesian product $P_4 \,\square\, P_2$ (see [20]), and two infinite families from [21]. Later we will add to this list several generalized Petersen graphs.

Note that the second assertion of Theorem 2.2 holds also when the condition $\delta = 1$ is replaced by the condition that there exists a vertex whose closed neighborhood is dominated by another vertex.

Corollary 2.3. *Graphs K_1 and K_2 are the only bluff graphs among trees.*

Proof. Let T be a tree. Suppose first that $\gamma_g(T) = 1$. Then T is a star and it is straightforward to verify that only K_1 and K_2 are bluff graphs among the stars.

Assume next that $\gamma_g(T) \geq 2$. Then combine the second assertion of Theorem 2.2 with the fact proved in [20, Theorem 4.6] that no tree is a MINUS graph. ■

We next show that the class of graphs characterized in the second assertion of Theorem 2.2 is not empty. Let G_k, $k \geq 1$, be the graph obtained from the disjoint union of C_{4k+2} and P_4 by identifying an end vertex of P_4 with a vertex of C_{4k+2}. See Figure 3 at the end of the paper for G_1. Then we have the following result by which we enlarge the short list of non-trivial families of graphs for which γ_g and γ'_g are known.

Theorem 2.4. *If $k \geq 1$, then G_k is a $(2k+3, -)$ graph. In particular, G_k is a bluff graph.*

Proof. By Theorem 1.1 it suffices to describe a strategy of Staller in Game 1 that ensures at least $2k+3$ moves, and a strategy of Dominator in Game 2 that ensures at most $2k+2$ moves. Let $v_1, \ldots, v_4$ be the vertices of P_4 where v_1 is also a vertex of the cycle.

Consider Game 1. Suppose first that $d_1 = v_3$ or $d_1 = v_4$. Then Staller responds by playing v_2. The remaining partially dominated graph is $C_{4k+2}|v_1$ which means that in the rest of the game $4k+1$ more vertices must be dominated. Clearly, in each move of Dominator at most three new vertices are dominated. On the other hand, Staller can always play in such a way that only one new vertex is dominated. Indeed, after the first move of Dominator on the cycle there are at least three consecutive dominated vertices, a property which remains valid until the end of the game. Then Staller can play any dominated vertex with only one undominated neighbor. It follows that in a pair of moves of Dominator and Staller at most four new vertices are dominated. Consequently on $C_{4k+2}|v_1$ at least $2k+1$ moves are needed to finish the game. Hence $\gamma_g(G_k) \geq 2k+3$ in this case. The case when $d_1 = v_2$ is reduced to the previous case by setting $s_1 = v_3$.

Assume next that the first move of the Dominator is made on the cycle. The strategy of Staller is to play on the cycle as long as Dominator plays there and she has a legal move. Suppose that all vertices of the cycle are dominated before any of v_2, v_3, v_4 was played. Then since $\gamma_g(C_{4k+2}) = 2k+1$ (see [19]), either exactly $2k+1$ moves were played or at least $2k+2$. In the first case it was Dominator who played the last move and hence Staller can play v_2 to force two more moves. In the second case at least one more move is needed to finish the game on G_k. Hence in any case $\gamma_g(G_k) \geq 2k+3$. Consider now that at some point before the whole cycle is dominated, Dominator plays v_3 (by the Continuation Principle, we do not need to consider Dominator's moves on v_2 or v_4). Then Staller has two consecutive moves on the cycle in which only one new vertex is dominated. Suppose that Dominator played on v_3 after 2ℓ, $\ell \geq 1$, moves. Then after the second consecutive move of Staller on the cycle, there are still at least $4k+2-(4\ell+1) = 4(k-\ell)+1$ undominated vertices. Because also in the rest of the game Staller can ensure that at most four new vertices are dominated in a pair of moves, at least $2(k-\ell)+1$ moves are needed to finish the game. Hence $\gamma_g(G_k) \geq 2\ell + 1 + 1 + 2(k-\ell) + 1 = 2k+3$.

Consider now Game 2. Recall that we only need to design a strategy of Dominator such that the game ends in no more than $2k+2$ moves. If $s_1' = v_3$ or $s_1' = v_4$, then setting $d_1' = v_1$ we actually start Game 1 on C_{4k+2}. Recalling again that $\gamma_g(C_{4k+2}) = 2k+1$ the game ends in at most $2k+2$ moves. If $s_1' = v_1$ or $s_1' = v_2$, then Dominator responds by playing v_4. By the Continuation Principle and having in mind that $\gamma_g'(C_{4k+2}) = 2k$ (see [19] again) we infer that the game will end in at most $2k+2$ moves. Assume finally that Staller starts the game on the cycle. Then Dominator follows her on the cycle as long as possible by using an optimal strategy according to the cycle. If the whole cycle is dominated in this way, than this was done in no more than $2k$ moves and at most two more moves are required on G_k. If, however, Staller played on the P_4 before the cycle is dominated, we consider two subcases. If Dominator has a legal move on P_4, then he plays it in order to finish the game on P_4. Then the game continues in the same ways on the cycle as before, yielding at most $2k+2$ moves in total. If on the other hand P_4 is completely dominated after the move of Staller, only one move will be played on the vertices v_2, v_3, v_4. Let S be the set of dominated vertices on the cycle and let ℓ be the number of moves played in the cycle up to this point. Since the vertices $v_1, \ldots, v_4$ are dominated, the game on $C_{4k+2}|S$ is the same as the game on $G_k|(S \cup \{v_1, \ldots, v_4\})$. By Theorem 1.1, $\gamma_g(C_{4k+2}|S) \leq \gamma_g'(C_{4k+2}|S) + 1$ holds, hence the number of remaining moves $\gamma_g(C_{4k+2}|S) - \ell$ in Game 1 is at most $\gamma_g'(C_{4k+2}|S) - \ell + 1 = 2k - \ell + 1$. It follows that the game will finish in no more than $2k+2$ moves. ■

3. A Characterization of $(3, -)$ Graphs

The family of $(3,-)$ graphs is quite rich which we justify by listing the following examples for which it is not difficult to verify that they are indeed $(3,-)$.

- C_5, the complement of the Petersen graph, and the graph G from Figure 1;
- C_6, and $C_6 + M$, where M is an independent set of edges that are not in C_6;
- $K_2 \,\square\, K_n$, $n \geq 3$;
- circulant graphs $C(n; \{1, \ldots, k\})$, where $n \geq 7$ and $\lceil \frac{n-2}{4} \rceil \leq k \leq \lfloor \frac{n-3}{2} \rfloor$.

Theorem 3.1. *A graph G is a $(3,-)$ graph if and only if* (i) *every vertex is non-adjacent to two vertices that are not twins, and* (ii) $\gamma(G) = 2$ *and every vertex is in a $\gamma(G)$-set.*

Proof. Let G be a $(3,-)$ graph. To prove (i), note that if u were a vertex of G adjacent to all vertices but twins, then Dominator playing u would enforce that

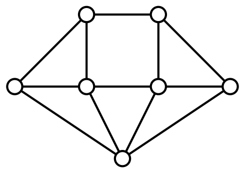

Figure 1. Graph G.

$\gamma_g(G) \le 2$ hold. It remains to prove (ii). Clearly, $\gamma(G) = 2$ since $\gamma_g'(G) = 2$ and the existence of a universal vertex would imply that $\gamma_g(G) = 1$. Suppose that there exists a vertex u that is not in a $\gamma(G)$-set. Then the move $s_1' = u$ forces Game 2 to last at least three moves, a contradiction with $\gamma_g'(G) = 2$.

Conversely, assume that G fulfills conditions (i) and (ii). The first condition provides that after any first move of Dominator in Game 1 Staller can choose a vertex that dominates only one of non-twin vertices in the corresponding partially dominated graph. It follows that $\gamma_g(G) \ge 3$. Hence, using (ii), we infer that $\gamma_g(G) = 3$. In addition, (ii) also implies that Dominator can finish Game 2 after any first move of Staller, that is, $\gamma_g'(G) = 2$. ■

It is interesting to note here that the graphs G with $\gamma(G) = 2$ such that every pair of vertices form a γ-set have been characterized by Jayaram [18]: they are precisely the cocktail-party graphs.

Suppose that G is twin-free. Then the condition (i) of Theorem 3.1, i.e., that every vertex is non-adjacent to two vertices that are not twins is equivalent to the fact that $\Delta(G) \le n-3$. Hence, in this case Theorem 3.1 simplifies as follows.

Corollary 3.2. *Let G be a twin-free graph of order n. Then G is a $(3,-)$ graph if and only if $\Delta(G) \le n-3$, $\gamma(G) = 2$, and every vertex is in a $\gamma(G)$-set.*

To see that in general the condition (i) of Theorem 3.1 cannot be replaced with the simpler condition on the maximum degree, consider the family of graphs obtained from the disjoint union of K_k ($k \ge 2$), a graph H without a universal vertex, and two additional vertices v and u by adding all possible edges between K_k and H, between H and v, and connecting v with u. See Figure 2 for this construction and for the smallest graph of the family and observe that any two vertices from K_k are twins.

Note first that any graph from this family fulfills both conditions of Corollary 3.2. On the other hand, if Dominator first plays v, then Staller is forced to finish the game in the next move. Hence these are not $(3,-)$ graphs.

We conclude the section with several simple, nice properties of $(3,-)$ graphs.

Proposition 3.3. *If G is a $(3,-)$ graph, then G is connected, $2 \le \operatorname{diam}(G) \le 3$, and $\delta(G) \ge 2$.*

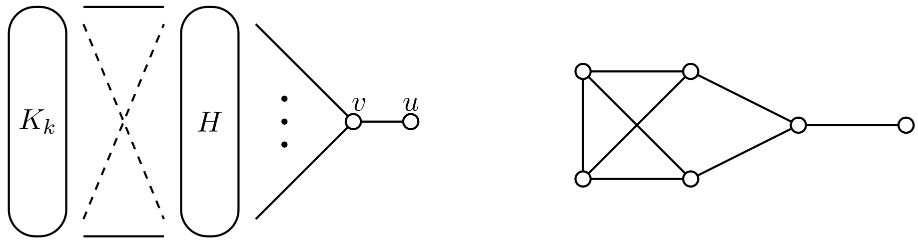

Figure 2. Graphs G with $\gamma_g(G) = \gamma_g'(G) = 2$ that fulfill the conditions of Corollary 3.2.

Proof. Let G be a $(3, -)$ graph. If G were not connected, then G would clearly consist of two connected components. Since $\gamma_g'(G) = 2$, this means that the components are complete graphs which in turn implies that $\gamma_g(G) = 2$, a contradiction.

The only graphs with diameter 1 are complete graphs, which proves the left inequality on the diameter. To prove the right one assume that $\mathrm{diam}(G) \geq 4$ and let $u_0, \ldots, u_4$ be a shortest path of G. Then setting $s_1 = u_2$ ensures that Game 2 will last at least three moves, because u_0 and u_4 cannot be dominated with a single move, a contradiction.

Suppose now that there is a degree one vertex u, with adjacent vertex v. By Theorem 3.1(i), we infer that the subgraph Q induced by $V(G) - N[v]$ has at least two vertices. Suppose two vertices x and y in Q are not adjacent. Then x does not lie in any $\gamma(G)$-set since u and y have no common neighbor, a contradiction with Theorem 3.1(ii). Hence Q must be a clique. Suppose that a neighbor z of v is adjacent to some vertex x in Q but not all, say not to y. Then for the same reason z is not in any $\gamma(G)$-set, a contradiction. This implies that every vertex adjacent to some vertex in Q is adjacent to all of them. Now observe that in Game 1 the move $d_1 = v$ forces Staller to finish the game with her first move, and thus $\gamma_g(G) = 2$, a final contradiction. ∎

4. Double Bluff Graphs

In this section, we explore the possible existence of a *double bluff graph*, that is a bluff graph where after the first move, any legal answer is an optimal second move for any player. This would mean that the first two moves are arbitrary, provided they are legal. We prove the existence of such graphs and explore their status further. Beforehand, we propose the following lemma parallel to Lemma 1.3.

Lemma 4.1. *Let G be a bluff graph. Then if G is a $(k, +)$ or $(k, =)$ graph, then after any first move u, $G|N[u]$ is a $(k, -)$ or a $(k-1, =)$ graph, respectively.*

Proof. Let G be a bluff graph and denote $\gamma_g(G) = k$ and $\gamma'_g(G) = \ell$. Let u be an arbitrary vertex of G. Since G is a bluff graph, u is an optimal move and $\gamma_g(G|N[u]) = \gamma'_g(G) - 1 = \ell - 1$ and $\gamma'_g(G|N[u]) = \gamma_g(G) - 1 = k - 1$. If $\ell = k+1$, then $G|N[u]$ thus realizes $(k, k-1)$, while in the case when $\ell = k$, $G|N[u]$ realizes $(k-1, k-1)$. ∎

Thanks to this lemma together with Theorem 2.2 we get:

Corollary 4.2. *If a graph G is a* PLUS *and a bluff graph, then G is a double bluff graph.*

At the first sight it seems unlikely that double bluff graphs exist. However, as we now prove, there are infinitely many graphs that are vertex-transitive and PLUS, therefore also double bluff.

We first prove that the Kneser graphs $K(n, 2)$, $n \geq 6$, are double bluff. Recall that the vertex set of $K(n, 2)$ consists of all two elements subset of an n-set, two subsets being adjacent if they are disjoint. In particular, $K(5, 2)$ is the Petersen graph. For a substantial information on domination in Kneser graphs we refer to [14].

Theorem 4.3. *For all $n \geq 6$, the Kneser graphs $K(n, 2)$ realize $(3, 4)$ and are double bluff.*

Proof. First consider Game 1. By vertex-transitivity we may assume that the first move of Dominator is $\{1, 2\}$. If Staller answers with a vertex non adjacent to $\{1, 2\}$, say $\{1, 3\}$, then Dominator can finish the game by playing $\{2, 3\}$. If Staller instead plays $\{3, 4\}$, then Dominator finishes the game playing $\{5, 6\}$. Thus $\gamma_g(K(n, 2)) = 3$, and this is independent of the two first moves.

Consider now Game 2. We may again assume that Staller starts the game by playing $\{1, 2\}$. If Dominator answers $\{1, 3\}$, then Staller can reply with $\{1, 4\}$ so that e.g. $\{1, 5\}$ is not dominated and a fourth move is needed to finish the game, possibly on $\{2, 3\}$. Now if Dominator answer with $\{3, 4\}$, Staller can play $\{1, 3\}$ so that e.g. $\{1, 4\}$ is not dominated and a fourth move is needed to finish the game, possibly $\{5, 6\}$. Thus $\gamma'_g(K(n, 2)) = 4$, and the two first moves can be played arbitrarily.

As $K(n, 2)$ is vertex-transitive, it is a bluff graphs, and since it realizes $(3, 4)$, Corollary 4.2 implies that it is double-bluff. ∎

We know of two other examples of double bluff graphs. The first one is the Cartesian product $C_4 \,\square\, C_4$. It is vertex-transitive and realizes $(5, 6)$, so it is PLUS and double bluff. The other example is the Cartesian product $C_6 \,\square\, C_6$. This one realizes $(13, 13)$, so it is not PLUS, however we have checked by computer that it is a double bluff graph.

We conclude this section with a quick observation about the so-called "total bluff graphs," that is the graphs in which all sequences of legal moves bring to the same score. These graphs were studied in [2] under the name *k-uniform dominating sequence graphs*, where k is the length of such a sequence. In particular, the following observation was noticed.

Observation 4.4. *If G is a uniform dominating sequence graph, then $\gamma_g(G) = \gamma'_g(G) = \gamma(G)$.*

The converse of the observation is not true. For instance, if G_n is the *corona* of the complete graph K_n (which is obtained by adding a pendant vertex to each vertex of K_n), then $\gamma_g(G_n) = \gamma'_g(G_n) = \gamma(G_n) = n$, but G_n is not even a bluff graph.

By Observation 4.4, "total bluff graphs" are all EQUAL. It was also shown that twin-free 2-uniform dominating sequence graphs are precisely the cocktail-party graphs, and that every 3-uniform dominating sequence graph is a disjoint union of a 2-uniform graph and a 1-uniform graph [2].

5. Generalized Petersen Graphs and Hamming Graphs

In this section we consider the domination game on generalized Petersen graphs and on Hamming graphs. Recall that the *generalized Petersen graph $P(n, k)$*, $n \geq 3$, $1 \leq k < n/2$, is the graph with vertices $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ and edges u_iu_{i+1}, u_iv_i, and v_iv_{i+k}, where addition is modulo n.

As it is well-known that the Petersen graph is a MINUS graph we have computed the game domination numbers for the generalized Petersen graphs $P(n, k)$, $3 \leq n \leq 18$, $1 \leq k < n/2$. The computations were made using the algorithm from [3] and are presented in Table 1.

Contrary to possible expectations, the obtained results show an almost random behaviour of the MINUS, EQUAL, PLUS type of these graphs. Nevertheless in this way we have found numerous additional MINUS graphs and hence bluff graphs. We have also checked the EQUAL and PLUS instances for being bluff or not. In the table we have marked with squares those entries that are not bluff (so that the not squared entries of the table are bluff graphs). As vertex-transitive graphs are bluff graphs, the squared generalized Petersen graphs cannot be vertex-transitive. Recall from [13] that $P(n, k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$, or $n = 10$ and $k = 2$. Hence there are many more bluff graphs than vertex-transitive graphs among the generalized Petersen graphs.

We next consider Hamming graphs $K_m \,\square\, K_n$, $n \geq m$. These graphs are vertex-transitive, so they are necessarily bluff graphs. In the rest of the section

$k \backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	$-$	$-$	$=$	$-$	$-$	$-$	$-$	$=$	$=$	$=$	$-$	$-$	$-$	$=$	$=$	$=$
2			$-$	$\boxed{=}$	$=$	$=$	$=$	$=$	$-$	$\boxed{=}$	$=$	$\boxed{=}$	$=$	$\boxed{=}$	$=$	$\boxed{=}$
3					$=$	$-$	$\boxed{=}$	$=$	$-$	$\boxed{=}$	$-$	$=$	$\boxed{=}$	$-$	$=$	$\boxed{+}$
4							$=$	$\boxed{=}$	$-$	$\boxed{+}$	$-$	$\boxed{=}$	$=$	$\boxed{+}$	$=$	$\boxed{+}$
5									$-$	$-$	$=$	$=$	$\boxed{=}$	$-$	$=$	$=$
6											$=$	$\boxed{=}$	$\boxed{=}$	$\boxed{=}$	$=$	$\boxed{+}$
7													$=$	$-$	$=$	$=$
8															$=$	$\boxed{+}$

Table 1. Generalized Petersen graphs $P(n, k)$ as MINUS, EQUAL or PLUS graphs. The squared entries are **not** bluff graphs.

we determine the values of their game domination numbers and prove that they are not double bluff graphs.

First, we consider partially dominated products for which we use the following notation. Let $G_{m,n}$ be a partially dominated subgraph of $K_r \,\Box\, K_s$, where $r > m$ and $s > n$, in which exactly $r - m$ rows and $s - n$ columns are dominated. This implies that the undominated vertices induce a copy of $K_m \,\Box\, K_n$. Note that the (Staller-start) game domination number of $G_{m,n}$ is not dependent on how big are r and s, it is only dependent on m and n.

In the course of the game, whenever a player chooses a vertex that dominates a previously undominated vertex, this vertex either dominates all the vertices of a column or of a row, or dominates all the vertices of both its row and its column. Then from $G_{m,n}$, the legal moves in the domination game bring to the games on one of $G_{m-1,n}$, $G_{m,n-1}$, and $G_{m-1,n-1}$.

Using the Continuation Principle, we easily infer that choosing an undominated vertex and dominating a full row and column is always an optimal move for Dominator, while optimal moves for Staller consist in dominating only a row or only a column. Thus we get:

$$\gamma_g(G_{m,n}) = 1 + \gamma_g'(G_{m-1,n-1})\,, \tag{1}$$

$$\gamma_g'(G_{m,n}) = 1 + \max\{\gamma_g(G_{m-1,n}), \gamma_g(G_{m,n-1})\}. \tag{2}$$

We first assume that $m \le n \le 2m$. Using (1), (2), and induction, we can prove the following formulas for $n + m \le 3m \le 3n$. Letting $k = \lfloor \frac{n+m}{3} \rfloor$ we get

$$(\gamma_g(G_{m,n}), \gamma_g'(G_{m,n})) = \begin{cases} (2k-1, 2k), & \text{if } n+m \equiv 0 \pmod 3, \\ (2k, 2k), & \text{if } n+m \equiv 1 \pmod 3, \\ (2k+1, 2k+1), & \text{if } n+m \equiv 2 \pmod 3. \end{cases} \tag{3}$$

Using induction again, we get that if $n \geq 2m$, then the partially dominated graph $G_{m,n}$ realizes $(2m-1, 2m)$. This is easy to figure out because in the course of the game, Dominator is always picking a previously undominated row (and since there are m rows he cannot finish before he made m moves), while Staller is picking previously undominated columns by using vertices in rows that are already dominated.

Now, consider the game is played on $K_{m+1} \,\square\, K_{n+1}$ with $1 \leq m \leq n$. Then after the first move of Dominator (respectively Staller), the players arrive at Game 2 (respectively Game 1) played in $G_{m,n}$. Hence by the above we infer the following result.

Proposition 5.1. *For $m, n \geq 0$ the following holds.*

(i) *If $n \geq 2m$, then*

$$\big(\gamma_g(K_{m+1} \,\square\, K_{n+1}), \gamma_g'(K_{m+1} \,\square\, K_{n+1})\big) = (2m+1, 2m).$$

(ii) *If $n + m \leq 3m \leq 3n$, $k = \lfloor \frac{n+m}{3} \rfloor$ and $G = K_{m+1} \,\square\, K_{n+1}$, then*

$$\big(\gamma_g(G), \gamma_g'(G)\big) = \begin{cases} (2k+1, 2k), & \textit{if } n+m \equiv 0 \pmod 3, \\ (2k+1, 2k+1), & \textit{if } n+m \equiv 1 \pmod 3, \\ (2k+2, 2k+2), & \textit{if } n+m \equiv 2 \pmod 3. \end{cases}$$

Note that there is a big family of Hamming graphs that are minus, in particular all graphs $K_m \,\square\, K_n$, where $n \geq 2m$ or $m \geq 2n$, are such. Although $K_m \,\square\, K_n$ is never a plus, Hamming graphs seemed to be natural candidates for being double bluff. However the next result shows that this is not the case.

Proposition 5.2. *Graphs $K_m \,\square\, K_n$, $m, n \geq 3$, are not double bluff.*

Proof. Assume by way of contradiction that some Hamming graph $K_{m+1} \,\square\, K_{n+1}$ is double bluff. All first moves are equivalent in Hamming graphs, and lead to the graph $G_{m,n}$ as defined above, which must thus be bluff. Therefore, all the moves from $G_{m,n}$ should bring to games with the same game domination number, and we deduce $\gamma_g(G_{m-1,n-1}) = \gamma_g(G_{m,n-1}) = \gamma_g(G_{m-1,n})$. By equality (3), $\gamma_g(G_{m-1,n-1}) = \gamma_g(G_{m-1,n})$ may not occur if $m-1 \leq n-1 \leq 2(m-1)$. Now if $n-1 \geq 2(m-1)$, then $\gamma_g(G_{m-1,n-1}) < \gamma_g(G_{m,n-1})$, a contradiction. This concludes the proof. ∎

We conclude the paper with Figure 3 that shows relations between bluff graphs, vertex-transitive graphs, and double bluff graphs.

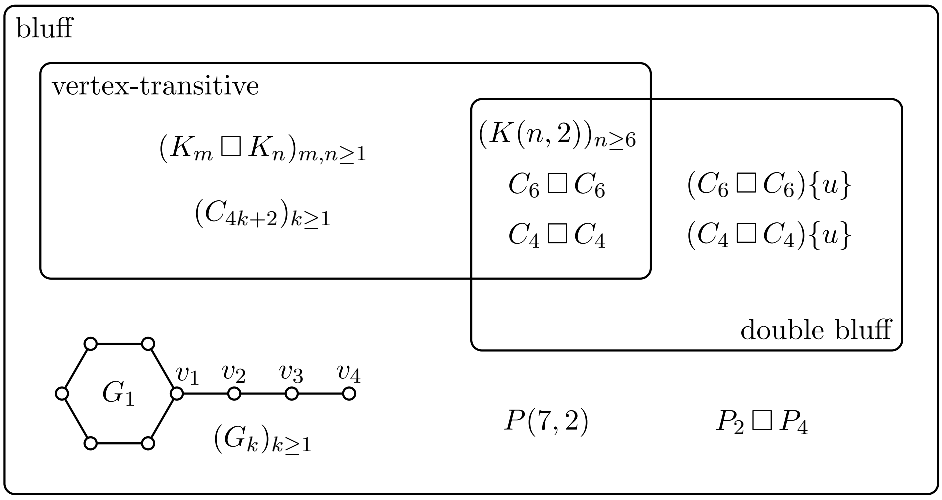

Figure 3. Bluff graphs and their subclasses.

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Appendix: γ_g and γ_g' of Generalized Petersen Graphs

$n\backslash k$	1	2	3	4	5	6	7	8
3	(3,2)							
4	(3,2)							
5	(4,4)	(5,4)						
6	(5,4)	(6,6)						
7	(6,5)	(6,6)	(6,6)					
8	(7,6)	(8,8)	(7,6)					
9	(8,7)	(8,8)	(9,9)	(8,8)				
10	(8,8)	(10,10)	(9,9)	(10,10)				
11	(9,9)	(10,9)	(10,9)	(10,9)	(10,9)			
12	(10,10)	(12,12)	(12,12)	(12,13)	(11,10)			
13	(11,10)	(11,11)	(12,11)	(12,11)	(11,11)	(11,11)		
14	(12,11)	(14,14)	(12,12)	(14,14)	(12,12)	(14,14)		
15	(13,12)	(13,13)	(15,15)	(13,13)	(16,16)	(15,15)	(13,13)	
16	(13,13)	(16,16)	(14,13)	(16,17)	(14,13)	(16,16)	(14,13)	
17	(14,14)	(15,15)	(15,15)	(15,15)	(15,15)	(15,15)	(15,15)	(15,15)
18	(15,15)	(18,18)	(18,19)	(17,18)	(16,16)	(19,20)	(16,16)	(17,18)

Table 2. $\big(\gamma_g(P(n,k)), \gamma_g'(P(n,k))\big)$ for $3 \le n \le 18$, $1 \le k < \frac{n}{2}$.