# ON THE COMPLEXITY OF REINFORCEMENT IN GRAPHS 

Nader Jafari Rad<br>Department of Mathematics<br>Shahrood University of Technology<br>Shahrood, Iran<br>e-mail: n.jafarirad@gmail.com


#### Abstract

We show that the decision problem for $p$-reinforcement, $p$-total reinforcement, total restrained reinforcement, and $k$-rainbow reinforcement are NP-hard for bipartite graphs. Keywords: domination, total domination, total restrained domination, $p$ domination, $k$-rainbow domination, reinforcement, NP-hard.


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## 1. Introduction

For notation and graph theory terminology, we in general follow [12]. Specifically, let $G$ be a graph with vertex set $V(G)=V$ of order $|V|=n$ and size $|E(G)|=m$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_{G}(v)=\{u \in V: u v \in$ $E(G)\}$ and the closed neighborhood of $v$ is $N_{G}[v]=\{v\} \cup N(v)$. If the graph $G$ is clear from the context, we simply write $N(v)$ rather than $N_{G}(v)$. The degree of a vertex $v$, is $\operatorname{deg}(v)=|N(v)|$. A vertex of degree one is called a leaf and its neighbor a support vertex. We denote by $L(G)$ the set of all leaves of $G$. For a set $S \subseteq V$, its open neighborhood is the set $N(S)=\bigcup_{v \in S} N(v)$, and its closed neighborhood is the set $N[S]=N(S) \cup S$. A subset $S \subseteq V$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $S$ is called a $\gamma(G)$-set of $G$ if $|S|=\gamma(G)$. A dominating set $S$ in a graph with no isolated vertex is a total dominating set if the induced subgraph $G[S]$ has no isolated vertex. The total domination number of $G$, denoted by $\gamma_{t}(G)$, is the minimum cardinality of a total dominating set of $G$. A total dominating
set $S$ is called a $\gamma_{t}(G)$-set of $G$ if $|S|=\gamma_{t}(G)$. A total dominating set $S$ in a graph with no isolated vertex is a total restrained dominating set if any vertex in $V(G) \backslash S$ is also adjacent to a vertex of $V(G) \backslash S$. The total restrained domination number of $G$, denoted by $\gamma_{t r}(G)$, is the minimum cardinality of a total restrained dominating set of $G$. A total restrained dominating set $S$ is called a $\gamma_{t r}(G)$-set of $G$ if $|S|=\gamma_{t r}(G)$. For references on domination, total domination and total restrained domination in graphs, see for example $[6,7,12,14]$.

Fink and Jacobson [10] introduced the concept of $p$-domination. Let $p$ be a positive integer. A subset $S$ of $V$ is a $p$-dominating set of $G$ if $|N(v) \cap S| \geq p$ for every vertex $v \in V(G) \backslash S$. The $p$-domination number, $\gamma_{p}(G)$, is the minimum cardinality among all $p$-dominating sets of $G$. A $p$-dominating set of $G$ of cardinality $\gamma_{p}(G)$ is called a $\gamma_{p}(G)$-set. A vertex $v$ is said to be $p$-dominated by a set $S$ if $|N(v) \cap S| \geq p$. The $p$-domination number has received much research attention, see a state-of-the-art survey article by Chellali et al. [5]. It is clear from the definition that every $p$-dominating set of a graph certainly contains all vertices of degree at most $p-1$. By this simple observation, to avoid happening the trivial case, we always assume $\Delta(G) \geq p$. A total dominating set $S$ in a graph $G$ with no isolated vertex is a $p$-total dominating set of $G$ if $|N(v) \cap S| \geq p$ for every vertex $v \in V(G) \backslash S$. The $p$-total domination number, $\gamma_{p t}(G)$, is the minimum cardinality among all $p$-total dominating sets of $G$. A $p$-total dominating set of $G$ of cardinality $\gamma_{p t}(G)$ will be called a $\gamma_{p t}(G)$-set. For references in multiple domination, see for example $[1,5,10,20,21]$.

For a graph $G$, let $f: V(G) \rightarrow \mathcal{P}(\{1,2, \ldots, k\})$ be a function. If for each vertex $v \in V(G)$ with $f(v)=\emptyset$ we have $\bigcup_{u \in N(v)} f(u)=\{1,2, \ldots, k\}$, then $f$ is called a $k$-rainbow dominating function (or simply $k \mathrm{RDF}$ ) of $G$. The weight, $w(f)$, of $f$ is defined as $w(f)=\sum_{v \in V(G)}|f(v)|$. The minimum weight of a $k$ RDF of $G$ is called the $k$-rainbow domination number of $G$, and is denoted by $\gamma_{r k}(G)$. For references in rainbow domination, see for example $[3,4,23,24,25,26]$.

Kok and Mynhardt [18] introduced the reinforcement number $r(G)$ of a graph $G$ as the minimum number of edges that have to be added to $G$ so that the resulting graph $G^{\prime}$ satisfies $\gamma\left(G^{\prime}\right)<\gamma(G)$. This concept of the reinforcement number in a graph was further considered for several domination variants, including independent domination, total domination, and total restrained domination, see for example $[8,9,13,17,22,27]$. Sridharan, Elias, and Subramanian [22] introduced the concept of total reinforcement in graphs. The total reinforcement number, $r_{t}(G)$, of a graph with no isolated vertex is the minimum number of edges that need to be added to the graph in order to decrease the total domination number. Total reinforcement in trees was recently studied by Blair et al. in [2]. Jafari Rad and Volkmann [17] introduced the concept of total restrained reinforcement in graphs. The total restrained reinforcement number, $r_{t r}(G)$, of a graph with no isolated vertex is the minimum number of edges that need to be added to
the graph in order to decrease the total restrained domination number. $\mathrm{Lu}, \mathrm{Hu}$, and $\mathrm{Xu}[19]$ studied the $p$-reinforcement in graphs. The $p$-reinforcement number, $r_{p}(G)$, of a graph is the minimum number of edges that need to be added to the graph in order to decrease the $p$-domination number. Analogously, the $p$ total reinforcement number, $r_{p t}(G)$, of a graph is the minimum number of edges that need to be added to the graph in order to decrease the $p$-total domination number.

The $k$-rainbow reinforcement number $r_{r k}(G)$ of a graph $G$ is the minimum number of edges that have to be added to $G$ so that the resulting graph $G^{\prime}$ satisfies $\gamma_{r k}\left(G^{\prime}\right)<\gamma_{r k}(G)$. Note that $r_{r 1}(G)$ is the classical reinforcement number $r(G)$. If $f$ is a $k \operatorname{RDF}$ of $G$ then we denote by $V_{12 \ldots k}^{f}$ the set of all vertices $u$ with $|f(u)|=k$. We refer a $\gamma_{r k}$-function in a graph $G$ as a $k \mathrm{RDF}$ with minimum weight. If $f$ is a $k \operatorname{RDF}$ of $G$, then we say that a vertex $v$ is not $k$-rainbow dominated by $f$ if $f(v)=\emptyset$ and $\bigcup_{u \in N(v)} f(u) \neq\{1,2, \ldots, k\}$.

The complexity issue of reinforcement is studied by Lu, Hu et al. [15, 16, 19]. It is proved that the decision problem for the reinforcement and total reinforcement in graphs is NP-hard for bipartite graph, [15]. Lu, Hu , and Xu [19] studied the complexity of $p$-reinforcement in graphs.

Theorem 1 (Lu, Hu and Xu [19]). The p-reinforcement problem is NP-hard for general graphs.

A truth assignment for a set $U$ of Boolean variables is a mapping $t: U \rightarrow$ $\{T, F\}$. A variable $u$ is said to be true (or false) under $t$ if $t(u)=T$ (or $t(u)=F$ ). If $u$ is a variable in $U$, then $u$ and $\bar{u}$ are literals over $U$. The literal $u$ is true under $t$ if and only if the variable $u$ is true under $t$, and the literal $\bar{u}$ is true if and only if the variable $u$ is false. A clause over $U$ is a set of literals over $U$, and it is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over $U$ is satisfiable if and only if there exists some truth assignment for $U$ that simultaneously satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a satisfying truth assignment for $\mathcal{C}$. The 3-SAT problem is specified as follows.

## 3-SAT problem

Instance: A collection $\mathcal{C}=\left\{C_{1}, C_{2}, \ldots, C_{m}\right\}$ of clauses over a finite set $U$ of variables such that $\left|C_{j}\right|=3$ for $j=1,2, \ldots, m$.

Question: Is there a truth assignment for $U$ that satisfies all the clauses in $C$ ?
Note that the 3-SAT problem was proven to be NP-complete in [11].
In this paper we first improve Theorem 1 to bipartite graphs, and then consider the complexity of $p$-total reinforcement, total restrained reinforcement, and $k$-rainbow reinforcement. We show that the decision problems for all of these
problems are NP-hard even when restricted to bipartite graphs. Our proofs are by a transformation from 3-SAT.

## 2. $p$-REINFORCEMENT

Let $p \geq 2$, and consider the following decision problem.

## $p$-reinforcement problem ( $p \mathrm{R}$ )

Instance: A nonempty graph $G$ and a positive integer $k$.
Question: Is $r_{p}(G) \leq k$ ?
We show that the problem above is NP-hard, even when restricted to the case $k=1$ and to bipartite graphs.

Theorem 2. The p-reinforcement problem is NP-hard for bipartite graphs.
Proof. We show the NP-hardness of the p-reinforcement problem by transforming the 3 -SAT to it in polynomial time. Let $U=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $\mathcal{C}=\left\{C_{1}, C_{2}, \ldots, C_{m}\right\}$ be an arbitrary instance of the 3 -SAT problem. We construct a graph $G$ and an integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $r_{p}(G) \leq k$. The graph $G$ is constructed as follows. For $i=1,2, \ldots, n$, let $H_{i}^{\prime}$ be a 6 -cycle $u_{i} v_{i} \overline{u_{i}} d_{i} b_{i} a_{i} u_{i}$ being consecutive vertices, and $H_{i}$ be obtained from $H_{i}^{\prime}$ by adding $p-1$ leaves to each vertex of $H_{i}^{\prime}$. For $i=1,2, \ldots, n$, corresponding to each variable $u_{i} \in U$, associate the graph $H_{i}$. Corresponding to each clause $C_{j}=\left\{x_{j}, y_{j}, z_{j}\right\} \in \mathcal{C}$, associate a single vertex $c_{j}$ and add the edge-set $E_{j}=\left\{c_{j} x_{j}, c_{j} y_{j}, c_{j} z_{j}\right\}$ for $j=1,2, \ldots, m$. Next add a star $T=K_{1, p-1}$ with center $s$, and join $s$ to each vertex $c_{j}$ with $1 \leq j \leq m$. Finally attach $p-1$ leaves to every vertex in $\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$. Let $G$ be the resulting graph. Note that $G$ has $p(6 n+m+1)$ vertices, and $|L(G)|=(p-1)(6 n+m+1)$. Set $k=1$. Let $S$ be a $\gamma_{p}(G)$-set. Clearly $L(G) \subseteq S$. Since any vertex of $H_{i}^{\prime}$ is $p$-dominated by $S$, we obtain $\left|S \cap V\left(H_{i}^{\prime}\right)\right| \geq 2$ for $i=1,2, \ldots, n$. Moreover, $|N[s] \cap S| \geq p$. Thus $|S|=\gamma_{p}(G) \geq(6 n+m+1)(p-1)+2 n+1$. On the other hand $L(G) \cup \bigcup_{i=1}^{n}\left\{u_{i}, d_{i}\right\} \cup\{s\}$ is a $p$-dominating set for $G$ of cardinality $(6 n+m+1)(p-1)+2 n+1$, and so $\gamma_{p}(G) \leq(6 n+m+1)(p-1)+2 n+1$. Thus $\gamma_{p}(G)=(6 n+m+1)(p-1)+2 n+1$.

We show that $\mathcal{C}$ is satisfiable if and only if $r_{p}(G)=1$. Assume that $\mathcal{C}$ is satisfiable. Let $t: U \rightarrow\{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset $D$ of vertices of $G$ as follows. If $t\left(u_{i}\right)=T$, then we put the vertices $u_{i}$ and $d_{i}$ in $D$; if $t\left(u_{i}\right)=F$, then put the vertices $\overline{u_{i}}$ and $a_{i}$ in $D$. Clearly, $|D|=2 n$. Then $D \cup L(G) \cup\{s\}$ is a $p$-dominating set for $G$, while $D \cup L(G)$ is a $p$-dominating set for $G+x s$, where $x \in V\left(H_{1}^{\prime}\right) \cap D$. Thus $r_{p}(G)=1$. Conversely, assume that $r_{p}(G)=1$. Thus there is an edge $e \in E(\bar{G})$ such that $\gamma_{p}(G+e)<$
$(6 n+m+1)(p-1)+2 n+1$. Let $S_{1}$ be a $\gamma_{p}(G+e)$-set. Clearly $S_{1} \cap V\left(H_{i}^{\prime}\right) \neq \emptyset$ for $i=1,2, \ldots, n$. Suppose that $\left|S_{1} \cap V\left(H_{j}^{\prime}\right)\right|=1$ for some $j \in\{1,2, \ldots, n\}$. Since $a_{j}, b_{j}, d_{j}$ are $p$-dominated by $S_{1}$, we obtain $b_{j} \in S_{1}$. But $v_{j}$ is $p$-dominated by $S_{1}$. Thus $v_{j} \in e$. Since $u_{j}$ and $\overline{u_{j}}$ are $p$-dominated by $S_{1}$, there are two different integers $j_{1}, j_{2} \in\{1,2, \ldots, m\}$ such that $c_{j_{1}}, c_{j_{2}} \in S_{1}$. Moreover, we may assume that $\left|S_{1} \cap V\left(H_{i}^{\prime}\right)\right| \geq 2$ for $i \neq j$, since $a_{i}, b_{i}, d_{i}$ and $v_{i}$ are $p$-dominated by $S_{1}$. These imply that $\left|S_{1}\right| \geq(6 n+m+1)(p-1)+2 n+1$, a contradiction. Thus $\left|S_{1} \cap V\left(H_{i}^{\prime}\right)\right| \geq 2$ for each $i \in\{1,2, \ldots, n\}$. Since $\left|S_{1}\right|<(6 n+m+1)(p-1)+2 n+1$, we deduce that $\left|S_{1} \cap V\left(H_{i}^{\prime}\right)\right|=2$ for each $i \in\{1,2, \ldots, n\}$. If $u_{j}, \overline{u_{j}} \in S_{1}$ for some $j$ then $b_{j}$ is not $p$-dominated by $S_{1}$, a contradiction. Thus $\left|S_{1} \cap\left\{u_{i}, \overline{u_{i}}\right\}\right| \leq 1$, and we may assume that $\left|S_{1} \cap\left\{u_{i}, \overline{u_{i}}\right\}\right|=1$ for $i=1,2, \ldots, n$. If $s \in S_{1}$ then $\left|S_{1}\right| \geq(6 n+m+1)(p-1)+2 n+1$, a contradiction. Thus $s \notin S_{1}$. Similarly, $c_{i} \notin S_{1}$ for $i=1,2, \ldots, m$. Let $t: U \rightarrow\{T, F\}$ be a mapping defined by $t\left(u_{i}\right)=T$ if $u_{i} \in S_{1}$, and $t\left(u_{i}\right)=F$ if $\overline{u_{i}} \in S_{1}$. For each $j \in\{1,2, \ldots, m\}$, there is an integer $i \in\{1,2, \ldots, n\}$ such that $c_{j}$ is dominated by $S_{1} \cap\left\{u_{i}, \overline{u_{i}}\right\}$. Assume that $u_{i} \in S_{1}$ and $c_{j}$ is dominated by $u_{i}$. By the construction of $G$, the literal $u_{i}$ is in the clause $C_{j}$. Then $t\left(u_{i}\right)=T$, which implies that the clause $C_{j}$ is satisfied by $t$. Next assume that $\overline{u_{i}} \in S_{1}$ and $c_{j}$ is dominated by $\overline{u_{i}}$. By the construction of $G$, the literal $\overline{u_{i}}$ is in the clause $C_{j}$. Then $t\left(u_{i}\right)=F$. Thus $t$ assigns $\overline{u_{i}}$ the truth value $T$, that is, $t$ satisfies the clause $C_{j}$. Hence $\mathcal{C}$ is satisfiable.

Since the construction of the $p$-reinforcement instance is straightforward from a 3 -SAT instance, the size of the $p$-reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired.

## 3. $p$-Total Reinforcement

Let $p \geq 2$ and consider the following decision problems.

## $p$-total reinforcement problem ( $p \mathbf{T R}$ )

Instance: A graph $G$ with no isolated vertex, and a positive integer $k$.
Question: Is $r_{p t}(G) \leq k$ ?
Theorem 3. The p-total reinforcement problem is NP-hard for bipartite graphs.
Proof. The proof is similar to the proof of Theorem 2. By attaching a path $P_{2}$ to a vertex $v$ in a graph we mean adding a path $P_{2}$ and join $v$ to a leaf of $P_{2}$. We show the NP-hardness of the $p$-total reinforcement problem by transforming the 3-SAT to it in polynomial time. Let $U=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $\mathcal{C}=\left\{C_{1}, C_{2}, \ldots, C_{m}\right\}$ be an arbitrary instance of the 3-SAT problem. We construct a graph $G$ and an integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $r_{p t}(G) \leq k$. The graph $G$ is constructed as follows. For $i=1,2, \ldots, n$, let $H_{i}^{\prime}$ be the 6 -cycle presented in the
proof of Theorem 2, and $H_{i}$ be obtained from $H_{i}^{\prime}$ by attaching a path $P_{2}$ to every vertex of $H_{i}^{\prime}$. For $i=1,2, \ldots, n$, corresponding to each variable $u_{i} \in U$, associate the graph $H_{i}$. Corresponding to each clause $C_{j}=\left\{x_{j}, y_{j}, z_{j}\right\} \in \mathcal{C}$, associate a single vertex $c_{j}$ and add the edge-set $E_{j}=\left\{c_{j} x_{j}, c_{j} y_{j}, c_{j} z_{j}\right\}$ for $j=1,2, \ldots, m$. Next add a graph $J$ which is obtained from a star $K_{1, p-1}$ (with center $s$ ) by subdivision of any edge, and join $s$ to each vertex $c_{j}$ with $1 \leq j \leq m$. Finally attach $p-1$ paths $P_{2}$ to every vertex in $\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$. Let $G$ be the resulting graph. Set $k=1$. Now by the same argument as in the proof of Theorem 2, we obtain the result.

## 4. Total Restrained Reinforcement

Consider the following decision problem.

## Total restrained reinforcement problem (TRR)

Instance: A graph $G$ with no isolated vertex and a positive integer $k$.
Question: Is $r_{t r}(G) \leq k$ ?
Theorem 4. The total restrained reinforcement problem is NP-hard for bipartite graphs.

Proof. Let $U=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $\mathcal{C}=\left\{C_{1}, C_{2}, \ldots, C_{m}\right\}$ be an arbitrary instance of the 3 -SAT problem. We construct a bipartite graph $G$ and an integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $r_{t r}(G) \leq k$. The bipartite graph $G$ is constructed as follows. Corresponding to each variable $u_{i} \in U$, we associate a graph $H_{i}$ isomorphic to the complete bipartite graph $K_{3,3}$, where its partite sets are $\left\{u_{i}, \overline{u_{i}}, d_{i}\right\}$ and $\left\{a_{i}, b_{i}, e_{i}\right\}$. Corresponding to each clause $C_{j}=\left\{x_{j}, y_{j}, z_{j}\right\} \in \mathcal{C}$, associate a single vertex $c_{j}$ and add the edge-set $E_{j}=\left\{c_{j} x_{j}, c_{j} y_{j}, c_{j} z_{j}\right\}$. Add a path $P_{2}=s_{1} s_{2}$, join $s_{1}$ to each vertex $c_{j}$ with $1 \leq j \leq m$. Let $G$ be the resulting graph. Set $k=1$. Let $S$ be a $\gamma_{t r}(G)$-set. Clearly $\left|S \cap V\left(H_{i}\right)\right| \geq 2$ for $i=1,2, \ldots, n$. Furthermore, $s_{1}, s_{2} \in S$, and thus $\gamma_{t r}(G)=|S| \geq 2 n+2$. On the other hand, $\left\{d_{i}, a_{i}: i=1,2, \ldots, n\right\} \cup\left\{s_{1}, s_{2}\right\}$ is a total restrained dominating set for $G$ of cardinality $2 n+2$, and thus $\gamma_{t r}(G) \leq 2 n+2$. Hence $\gamma_{t r}(G)=2 n+2$.

We show that $\mathcal{C}$ is satisfiable if and only if $r_{t r}(G)=1$. Assume that $\mathcal{C}$ is satisfiable. Let $t: U \longrightarrow\{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset $D$ of vertices of $G$ as follows. If $t\left(u_{i}\right)=T$, then we put the vertices $u_{i}$ and $a_{i}$ in $D$; if $t\left(u_{i}\right)=F$, then put the vertices $\overline{u_{i}}$ and $a_{i}$ in $D$. Clearly, $|D|=2 n$. Now $D \cup\left\{s_{2}\right\}$ is a total restrained dominating set for $G+s_{2} x$, where $x \in D \cap V\left(H_{1}\right)$. Thus $r_{t r}(G)=1$.

Conversely, assume that $r_{t r}(G)=1$. There is an edge $e \in E(\bar{G})$ such that $\gamma_{t r}(G+e)<2 n+2$. Let $S_{1}$ be a $\gamma_{t r}(G+e)$-set. It is obvious that $\mid S_{1} \cap$ $V\left(G_{i}\right) \mid=2$ for $i=1,2, \ldots, n$. Since $s_{1}$ and $s_{2}$ are dominated by $S$, we obtain
$\left|S_{1}\right|=2 n+1$, and since $S_{1}$ contains any leaf and support vertex of $G+e$, we obtain $e=s_{2} x$, where $x \in S_{1} \cap V\left(H_{i}\right)$, for some integer $i \in\{1,2, \ldots, n\}$. Thus $S_{1} \cap\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}=\emptyset$, and any vertex of $\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$ is dominated by some vertex of $S_{1} \cap \bigcup_{i=1}^{n}\left\{u_{i}, \overline{u_{i}}\right\}$. Let $t: U \longrightarrow\{T, F\}$ be a mapping defined by $t\left(u_{i}\right)=T$ if $u_{i} \in S_{1}$ and $t\left(u_{i}\right)=F$ if $\overline{u_{i}} \in S_{1}$. For each $j \in\{1,2, \ldots, m\}$, there is an integer $i \in\{1,2, \ldots, n\}$ such that $c_{j}$ is dominated by $S_{1} \cap\left\{u_{i}, \overline{u_{i}}\right\}$. Assume that $u_{i} \in S_{1}$, and $c_{j}$ is dominated by $u_{i}$. By the construction of $G$ the literal $u_{i}$ is in the clause $C_{j}$. Then $t\left(u_{i}\right)=T$, which implies that the clause $C_{j}$ is satisfied by $t$. Next assume that $\overline{u_{i}} \in S_{1}$, and $c_{j}$ is dominated by $\overline{u_{i}}$. By the construction of $G$ the literal $\overline{u_{i}}$ is in the clause $C_{j}$. Then $t\left(u_{i}\right)=F$. Thus, $t$ assigns $\overline{u_{i}}$ the truth value $T$, that is, $t$ satisfies the clause $C_{j}$. Hence $\mathcal{C}$ is satisfiable. Since the construction of the total restrained reinforcement instance is straightforward from a 3-SAT instance, the size of the total restrained reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired.

## 5. $k$-Rainbow Reinforcement

Consider the following decision problem.

## $k$-rainbow reinforcement problem ( $\boldsymbol{k R R}$ )

Instance: A nonempty graph $G$, and two positive integers $k \geq 2$ and $t \geq 1$.
Question: Is $r_{r k}(G) \leq t$ ?
Theorem 5. For $k \geq 2$, the $k$-rainbow reinforcement problem is NP-complete for bipartite graphs.

Proof. We show the NP-hardness of the $k$-rainbow reinforcement by transforming the 3 -SAT to it in polynomial time. Let $U=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $\mathcal{C}=\left\{C_{1}, C_{2}, \ldots, C_{m}\right\}$ be an arbitrary instance of the 3 -SAT problem. We construct a bipartite graph $G$ and an integer $t$ such that $\mathcal{C}$ is satisfiable if and only if $r_{k r}(G) \leq t$. The bipartite graph $G$ is constructed as follows. For $i=1,2, \ldots, n$, let $H_{i}$ be a graph with $V\left(H_{i}\right)=\left\{u_{i}, \overline{u_{i}}, b_{i}, d_{i}\right\} \cup\left\{c_{i j}, e_{i j}: j=1,2, \ldots, k+1\right\}$ and $E\left(H_{i}\right)=\left\{u_{i} d_{i}, \overline{u_{i}} b_{i}\right\} \cup\left\{c_{i j} e_{i j}, c_{i j} d_{i}, c_{i j} b_{i}, e_{i j} u_{i}, e_{i j} \overline{u_{i}}: j=1,2, \ldots, k+1\right\}$. Figure 1 shows the graph $H_{i}$ for $k=2$. Corresponding to each variable $u_{i} \in U$, we associate a graph $H_{i}$.

Corresponding to each clause $C_{j}=\left\{x_{j}, y_{j}, z_{j}\right\} \in \mathcal{C}$, associate a single vertex $c_{j}$ and add the edge-set $E_{j}=\left\{c_{j} x_{j}, c_{j} y_{j}, c_{j} z_{j}\right\}$. Finally, add a star $K_{1, k-1}$ with central vertex $s$ and leaves $s_{1}, \ldots, s_{k-1}$, and join $s$ to each vertex $c_{j}$ with $1 \leq$ $j \leq m$. Let $G$ be the resulting graph. Set $t=1$. Let $f$ be a $\gamma_{r k}(G)$-function. We show that $\sum_{v \in V\left(H_{i}\right)}|f(v)| \geq 2 k$ for $i=1,2, \ldots, n$. Let $i \in\{1,2, \ldots, n\}$. If $\left|f\left(c_{i j}\right)\right|=\left|f\left(e_{i j}\right)\right|=0$ for all $j=1,2, \ldots, k+1$, then clearly $\sum_{v \in V\left(H_{i}\right)}|f(v)| \geq$
$2 k+2>2 k$. Thus without loss of generality assume that $\left|f\left(c_{i 1}\right)\right| \neq 0$. Then $\left|f\left(d_{i}\right)\right|+\left|f\left(b_{i}\right)\right|+\left|f\left(e_{i 1}\right)\right| \geq k$. If $\left|f\left(e_{i l}\right)\right|=0$ for some $l \in\{1,2, \ldots, k+1\}$, then $\left|f\left(c_{i l}\right)\right|+\left|f\left(u_{i}\right)\right|+\left|f\left(\overline{u_{i}}\right)\right| \geq k$, and so $\sum_{v \in V\left(H_{i}\right)}|f(v)| \geq 2 k$. We thus assume that $\left|f\left(e_{i l}\right)\right| \neq 0$ for $l=1,2, \ldots, k+1$. Then $\sum_{v \in V\left(H_{i}\right)}|f(v)| \geq\left(\left|f\left(d_{i}\right)\right|+\left|f\left(b_{i}\right)\right|+\right.$ $\sum_{j=1}^{k+1}\left|f\left(e_{i j}\right)\right| \geq 2 k$, as desired. Since $|f(s)|+\sum_{j=1}^{k-1}\left|f\left(s_{i}\right)\right|+\sum_{j=1}^{m}\left|f\left(c_{j}\right)\right| \geq k$, we obtain $\gamma_{r k}(G)=w(f) \geq 2 k n+k$. On the other hand $f_{1}$ defined on $V(G)$, by $f_{1}(s)=f_{1}\left(u_{i}\right)=f_{1}\left(b_{i}\right)=\{1,2, \ldots, k\}$ for $i=1,2, \ldots, n$, and $f_{1}(u)=\emptyset$ otherwise, is a $k$-rainbow dominating function of weight $2 k n+k$. Hence $\gamma_{r k}(G)=$ $2 k n+k$.


Figure 1. The graph $H_{i}$ for $k=2$.
We show that $\mathcal{C}$ is satisfiable if and only if $r_{r k}(G)=1$. Assume that $\mathcal{C}$ is satisfiable. Let $t^{\prime}: U \longrightarrow\{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset $D$ of vertices of $G$ as follows. If $t^{\prime}\left(u_{i}\right)=T$, then we put the vertices $u_{i}$ and $b_{i}$ in $D$; if $t^{\prime}\left(u_{i}\right)=F$, then put the vertices $\overline{u_{i}}$ and $d_{i}$ in $D$. Clearly, $|D|=2 n$. Now $f_{2}$ defined on $V(G)$ by $f_{2}(u)=\{1,2, \ldots, k\}$ if $u \in D, f_{2}(s)=f_{2}\left(s_{i}\right)=\{1\}$ for $i=1,2, \ldots, k-1$ and $f_{2}(u)=\emptyset$ otherwise is a $\gamma_{r k}(G)$-function, while $f_{3}$ defined on $V(G)$ by $f_{3}(u)=\{1,2, \ldots k\}$ if $u \in D$, $f_{3}(s)=f_{3}\left(s_{i}\right)=\{1\}$ for $i=1,2, \ldots, k-2$, and $f_{3}(u)=\emptyset$ otherwise is a $k \operatorname{RDF}$ for $G+x s_{k-1}$, where $x \in D \cap V\left(H_{1}\right)$. Thus $r_{k r}(G)=1$. Conversely, assume that $r_{r k}(G)=1$. Thus there is an edge $e \in E(\bar{G})$ such that $\gamma_{r k}(G+e)<2 k n+k$. Let $g$ be a $\gamma_{r k}(G+e)$-function. Suppose that $\sum_{v \in V\left(H_{i}\right)}|g(v)| \leq 2 k-1$, for some $i \in\{1,2, \ldots, n\}$. Then there is an integer $l$ such that $c_{i l}$ or $e_{i l}$ is not $k$ rainbow dominated by $g$, a contradiction. Thus $\sum_{v \in V\left(H_{i}\right)}|g(v)| \geq 2 k n$, for each $i \in\{1,2, \ldots, n\}$. Since $|g(s)|+\sum_{i=1}^{k-1}\left|g\left(s_{i}\right)\right| \geq k-1$, we obtain $\sum_{v \in V\left(H_{i}\right)}|g(v)|=$ $2 k n$, for each $i \in\{1,2, \ldots, n\}$. If $g\left(u_{i}\right)=g\left(\overline{u_{i}}\right)=\{1,2, \ldots, k\}$ for some $i$, then $c_{i j}$ is not $k$-rainbow dominated by $g$, for $j=1,2, \ldots, k+1$, a contradiction. Thus $\left|\left\{u_{i}, \overline{u_{i}}\right\} \cap V_{12 \ldots k}^{g}\right| \leq 1$. Since $\sum_{v \in V\left(H_{i}\right)}|g(v)|=2 k$ for each $i \in\{1,2, \ldots, n\}$, and $w(g) \leq 2 k n+k-1$, we obtain $w(g)=2 k n+k-1, \sum_{j=1}^{m}\left|g\left(c_{j}\right)\right|=0$, and $|g(s)| \neq k$. Thus any vertex of $\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$ is dominated by a vertex in $\left\{u_{i}, \overline{u_{i}}\right\}$, for some $i \in\{1,2, \ldots, n\}$.

Let $t^{\prime}: U \longrightarrow\{T, F\}$ be a mapping defined by $t^{\prime}\left(u_{i}\right)=T$ if $u_{i} \in V_{12 \ldots k}^{g}$
and $t^{\prime}\left(u_{i}\right)=F$ if $\overline{u_{i}} \in V_{12 \ldots k}^{g}$. For each $j \in\{1,2, \ldots, m\}$, there is an integer $i \in\{1,2, \ldots, n\}$ such that $c_{j}$ is dominated by $V_{12 \ldots k}^{g} \cap\left\{u_{i}, \overline{u_{i}}\right\}$. Assume that $u_{i} \in V_{12 \ldots k}^{g}$, and $c_{j}$ is dominated by $u_{i}$. By the construction of $G$ the literal $u_{i}$ is in the clause $C_{j}$. Then $t^{\prime}\left(u_{i}\right)=T$, which implies that the clause $C_{j}$ is satisfied by $t^{\prime}$. Next assume that $\overline{u_{i}} \in V_{12 \ldots k}^{g}$, and $c_{j}$ is dominated by $\overline{u_{i}}$. By the construction of $G$ the literal $\overline{u_{i}}$ is in the clause $C_{j}$. Then $t^{\prime}\left(u_{i}\right)=F$. Thus, $t^{\prime}$ assigns $\overline{u_{i}}$ the truth value $T$, that is, $t^{\prime}$ satisfies the clause $C_{j}$. Hence $\mathcal{C}$ is satisfiable.

Since the construction of the $k$-rainbow reinforcement instance is straightforward from a 3 -SAT instance, the size of the $k$-rainbow reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired.

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