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ON THE COMPLEXITY OF REINFORCEMENT IN GRAPHS

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Abstract

We show that the decision problem for p-reinforcement, p-total reinforcement, total restrained reinforcement, and k-rainbow reinforcement are NP-hard for bipartite graphs.

Keywords: domination, total domination, total restrained domination, p-domination, k-rainbow domination, reinforcement, NP-hard.

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1. Introduction

For notation and graph theory terminology, we in general follow [12]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V : uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. If the graph G is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote by $L(G)$ the set of all leaves of G. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \bigcup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called a $\gamma(G)$*-set* of G if $|S| = \gamma(G)$. A dominating set S in a graph with no isolated vertex is a *total dominating set* if the induced subgraph $G[S]$ has no isolated vertex. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set S is called a $\gamma_t(G)$-*set* of G if $|S| = \gamma_t(G)$. A total dominating set S in a graph with no isolated vertex is a *total restrained dominating set* if any vertex in $V(G)\backslash S$ is also adjacent to a vertex of $V(G)\backslash S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. A total restrained dominating set S is called a $\gamma_{tr}(G)$-*set* of G if $|S| = \gamma_{tr}(G)$. For references on domination, total domination and total restrained domination in graphs, see for example [6, 7, 12, 14].

Fink and Jacobson [10] introduced the concept of *p-domination.* Let p be a positive integer. A subset S of V is a *p-dominating set* of G if $|N(v) \cap S| \geq p$ for every vertex $v \in V(G)\backslash S$. The *p-domination number*, $\gamma_p(G)$, is the minimum cardinality among all p-dominating sets of G. A p-dominating set of G of cardinality $\gamma_p(G)$ is called a $\gamma_p(G)$-*set.* A vertex v is said to be *p-dominated* by a set S if $|N(v) \cap S| \geq p$. The p-domination number has received much research attention, see a state-of-the-art survey article by Chellali *et al.* [5]. It is clear from the definition that every p-dominating set of a graph certainly contains all vertices of degree at most $p-1$. By this simple observation, to avoid happening the trivial case, we always assume $\Delta(G) \geq p$. A total dominating set S in a graph G with no isolated vertex is a *p-total dominating set* of G if $|N(v) \cap S| \geq p$ for every vertex $v \in V(G)\backslash S$. The *p-total domination number*, $\gamma_{pt}(G)$, is the minimum cardinality among all p-total dominating sets of G. A p-total dominating set of G of cardinality $\gamma_{pt}(G)$ will be called a $\gamma_{pt}(G)$-*set.* For references in multiple domination, see for example [1, 5, 10, 20, 21].

For a graph G, let $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ be a function. If for each vertex $v \in V(G)$ with $f(v) = \emptyset$ we have $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$, then f is called a *k-rainbow dominating function* (or simply kRDF) of G. The *weight*, $w(f)$, of f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the *k-rainbow domination number* of G, and is denoted by $\gamma_{rk}(G)$. For references in rainbow domination, see for example [3, 4, 23, 24, 25, 26].

Kok and Mynhardt [18] introduced the *reinforcement number* $r(G)$ of a graph G as the minimum number of edges that have to be added to G so that the resulting graph G' satisfies $\gamma(G') < \gamma(G)$. This concept of the reinforcement number in a graph was further considered for several domination variants, including *independent domination*, *total domination*, and *total restrained domination*, see for example [8, 9, 13, 17, 22, 27]. Sridharan, Elias, and Subramanian [22] introduced the concept of *total reinforcement* in graphs. The *total reinforcement number*, $r_t(G)$, of a graph with no isolated vertex is the minimum number of edges that need to be added to the graph in order to decrease the total domination number. Total reinforcement in trees was recently studied by Blair *et al.* in [2]. Jafari Rad and Volkmann [17] introduced the concept of *total restrained reinforcement* in graphs. The *total restrained reinforcement number*, $r_{tr}(G)$, of a graph with no isolated vertex is the minimum number of edges that need to be added to the graph in order to decrease the total restrained domination number. Lu, Hu, and Xu [19] studied the p-reinforcement in graphs. The *p-reinforcement number*, $r_p(G)$, of a graph is the minimum number of edges that need to be added to the graph in order to decrease the p-domination number. Analogously, the *p-total reinforcement number*, $r_{pt}(G)$, of a graph is the minimum number of edges that need to be added to the graph in order to decrease the p-total domination number.

The *k-rainbow reinforcement number* $r_{rk}(G)$ of a graph G is the minimum number of edges that have to be added to G so that the resulting graph G' satisfies $\gamma_{rk}(G') < \gamma_{rk}(G)$. Note that $r_{r1}(G)$ is the classical reinforcement number $r(G)$. If f is a kRDF of G then we denote by $V^f_{12\ldots k}$ the set of all vertices u with $|f(u)| = k$. We refer a γ_{rk}-function in a graph G as a kRDF with minimum weight. If f is a kRDF of G, then we say that a vertex v is not k-rainbow dominated by f if $f(v) = \emptyset$ and $\bigcup_{u\in N(v)} f(u) \neq \{1, 2, \ldots, k\}$.

The complexity issue of reinforcement is studied by Lu, Hu *et al.* [15, 16, 19]. It is proved that the decision problem for the reinforcement and total reinforcement in graphs is NP-hard for bipartite graph, [15]. Lu, Hu, and Xu [19] studied the complexity of p-reinforcement in graphs.

Theorem 1 (Lu, Hu and Xu [19])**.** *The p-reinforcement problem is NP-hard for general graphs.*

A *truth assignment* for a set U of Boolean variables is a mapping $t : U \to \{T, F\}$. A variable u is said to be *true* (or *false*) under t if $t(u) = T$ (or $t(u) = F$). If u is a variable in U, then u and $\overline{u}$ are *literals* over U. The literal u is true under t if and only if the variable u is true under t, and the literal $\overline{u}$ is true if and only if the variable u is false. A *clause* over U is a set of literals over U, and it is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over U is *satisfiable* if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a *satisfying truth assignment* for $\mathcal{C}$. The 3-SAT problem is specified as follows.

3-SAT problem

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in C?

Note that the 3-SAT problem was proven to be NP-complete in [11].

In this paper we first improve Theorem 1 to bipartite graphs, and then consider the complexity of p-total reinforcement, total restrained reinforcement, and k-rainbow reinforcement. We show that the decision problems for all of these problems are NP-hard even when restricted to bipartite graphs. Our proofs are by a transformation from 3-SAT.

2. p-Reinforcement

Let $p \geq 2$, and consider the following decision problem.

***p*-reinforcement problem (*p*R)**

Instance: A nonempty graph G and a positive integer k.
Question: Is $r_p(G) \leq k$?

We show that the problem above is NP-hard, even when restricted to the case $k = 1$ and to bipartite graphs.

Theorem 2. *The p-reinforcement problem is NP-hard for bipartite graphs.*

Proof. We show the NP-hardness of the p-reinforcement problem by transforming the 3-SAT to it in polynomial time. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We construct a graph G and an integer k such that $\mathcal{C}$ is satisfiable if and only if $r_p(G) \leq k$. The graph G is constructed as follows. For $i = 1, 2, \ldots, n$, let H_i' be a 6-cycle $u_i v_i \overline{u_i} d_i b_i a_i u_i$ being consecutive vertices, and H_i be obtained from H_i' by adding $p-1$ leaves to each vertex of H_i'. For $i = 1, 2, \ldots, n$, corresponding to each variable $u_i \in U$, associate the graph H_i. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ for $j = 1, 2, \ldots, m$. Next add a star $T = K_{1,p-1}$ with center s, and join s to each vertex c_j with $1 \leq j \leq m$. Finally attach $p-1$ leaves to every vertex in $\{c_1, c_2, \ldots, c_m\}$. Let G be the resulting graph. Note that G has $p(6n + m + 1)$ vertices, and $|L(G)| = (p-1)(6n + m + 1)$. Set $k = 1$. Let S be a $\gamma_p(G)$-set. Clearly $L(G) \subseteq S$. Since any vertex of H_i' is p-dominated by S, we obtain $|S \cap V(H_i')| \geq 2$ for $i = 1, 2, \ldots, n$. Moreover, $|N[s] \cap S| \geq p$. Thus $|S| = \gamma_p(G) \geq (6n + m + 1)(p-1) + 2n + 1$. On the other hand $L(G) \cup \bigcup_{i=1}^{n} \{u_i, d_i\} \cup \{s\}$ is a p-dominating set for G of cardinality $(6n+m+1)(p-1)+2n+1$, and so $\gamma_p(G) \leq (6n+m+1)(p-1)+2n+1$. Thus $\gamma_p(G) = (6n+m+1)(p-1)+2n+1$.

We show that $\mathcal{C}$ is satisfiable if and only if $r_p(G) = 1$. Assume that $\mathcal{C}$ is satisfiable. Let $t\colon U \to \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i and d_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and a_i in D. Clearly, $|D| = 2n$. Then $D \cup L(G) \cup \{s\}$ is a p-dominating set for G, while $D \cup L(G)$ is a p-dominating set for $G + xs$, where $x \in V(H_1') \cap D$. Thus $r_p(G) = 1$. Conversely, assume that $r_p(G) = 1$. Thus there is an edge $e \in E(\overline{G})$ such that $\gamma_p(G + e) <$

$(6n+m+1)(p-1)+2n+1$. Let S_1 be a $\gamma_p(G+e)$-set. Clearly $S_1 \cap V(H'_i) \neq \emptyset$ for $i = 1, 2, \ldots, n$. Suppose that $|S_1 \cap V(H'_j)| = 1$ for some $j \in \{1, 2, \ldots, n\}$. Since a_j, b_j, d_j are p-dominated by S_1, we obtain $b_j \in S_1$. But v_j is p-dominated by S_1. Thus $v_j \in e$. Since u_j and $\overline{u_j}$ are p-dominated by S_1, there are two different integers $j_1, j_2 \in \{1, 2, \ldots, m\}$ such that $c_{j_1}, c_{j_2} \in S_1$. Moreover, we may assume that $|S_1 \cap V(H'_i)| \geq 2$ for $i \neq j$, since a_i, b_i, d_i and v_i are p-dominated by S_1. These imply that $|S_1| \geq (6n+m+1)(p-1)+2n+1$, a contradiction. Thus $|S_1 \cap V(H'_i)| \geq 2$ for each $i \in \{1, 2, \ldots, n\}$. Since $|S_1| < (6n+m+1)(p-1)+2n+1$, we deduce that $|S_1 \cap V(H'_i)| = 2$ for each $i \in \{1, 2, \ldots, n\}$. If $u_j, \overline{u_j} \in S_1$ for some j then b_j is not p-dominated by S_1, a contradiction. Thus $|S_1 \cap \{u_i, \overline{u_i}\}| \leq 1$, and we may assume that $|S_1 \cap \{u_i, \overline{u_i}\}| = 1$ for $i = 1, 2, \ldots, n$. If $s \in S_1$ then $|S_1| \geq (6n+m+1)(p-1)+2n+1$, a contradiction. Thus $s \notin S_1$. Similarly, $c_i \notin S_1$ for $i = 1, 2, \ldots, m$. Let $t\colon U \to \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S_1$, and $t(u_i) = F$ if $\overline{u_i} \in S_1$. For each $j \in \{1, 2, \ldots, m\}$, there is an integer $i \in \{1, 2, \ldots, n\}$ such that c_j is dominated by $S_1 \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S_1$ and c_j is dominated by u_i. By the construction of G, the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in S_1$ and c_j is dominated by $\overline{u_i}$. By the construction of G, the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable.

Since the construction of the p-reinforcement instance is straightforward from a 3-SAT instance, the size of the p-reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. ∎

3. p-Total Reinforcement

Let $p \geq 2$ and consider the following decision problems.

***p*-total reinforcement problem (*p*TR)**

Instance: A graph G with no isolated vertex, and a positive integer k.
Question: Is $r_{pt}(G) \leq k$?

Theorem 3. *The p-total reinforcement problem is NP-hard for bipartite graphs.*

Proof. The proof is similar to the proof of Theorem 2. By *attaching* a path P_2 to a vertex v in a graph we mean adding a path P_2 and join v to a leaf of P_2. We show the NP-hardness of the p-total reinforcement problem by transforming the 3-SAT to it in polynomial time. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We construct a graph G and an integer k such that $\mathcal{C}$ is satisfiable if and only if $r_{pt}(G) \leq k$. The graph G is constructed as follows. For $i = 1, 2, \ldots, n$, let H'_i be the 6-cycle presented in the proof of Theorem 2, and H_i be obtained from H'_i by attaching a path P_2 to every vertex of H'_i. For $i = 1, 2, \dots, n$, corresponding to each variable $u_i \in U$, associate the graph H_i. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ for $j = 1, 2, \dots, m$. Next add a graph J which is obtained from a star $K_{1,p-1}$ (with center s) by subdivision of any edge, and join s to each vertex c_j with $1 \le j \le m$. Finally attach $p - 1$ paths P_2 to every vertex in $\{c_1, c_2, \dots, c_m\}$. Let G be the resulting graph. Set $k = 1$. Now by the same argument as in the proof of Theorem 2, we obtain the result. ∎

4. Total Restrained Reinforcement

Consider the following decision problem.

Total restrained reinforcement problem (TRR)

Instance: A graph G with no isolated vertex and a positive integer k.
Question: Is $r_{tr}(G) \le k$?

Theorem 4. *The total restrained reinforcement problem is NP-hard for bipartite graphs.*

Proof. Let $U = \{u_1, u_2, \dots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We construct a bipartite graph G and an integer k such that $\mathcal{C}$ is satisfiable if and only if $r_{tr}(G) \le k$. The bipartite graph G is constructed as follows. Corresponding to each variable $u_i \in U$, we associate a graph H_i isomorphic to the complete bipartite graph $K_{3,3}$, where its partite sets are $\{u_i, \overline{u_i}, d_i\}$ and $\{a_i, b_i, e_i\}$. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$. Add a path P_2=$s_1 s_2$, join s_1 to each vertex c_j with $1 \le j \le m$. Let G be the resulting graph. Set $k = 1$. Let S be a $\gamma_{tr}(G)$-set. Clearly $|S \cap V(H_i)| \ge 2$ for $i = 1, 2, \dots, n$. Furthermore, $s_1, s_2 \in S$, and thus $\gamma_{tr}(G) = |S| \ge 2n + 2$. On the other hand, $\{d_i, a_i : i = 1, 2, \dots, n\} \cup \{s_1, s_2\}$ is a total restrained dominating set for G of cardinality $2n + 2$, and thus $\gamma_{tr}(G) \le 2n + 2$. Hence $\gamma_{tr}(G) = 2n + 2$.

We show that $\mathcal{C}$ is satisfiable if and only if $r_{tr}(G) = 1$. Assume that $\mathcal{C}$ is satisfiable. Let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i and a_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and a_i in D. Clearly, $|D| = 2n$. Now $D \cup \{s_2\}$ is a total restrained dominating set for $G + s_2 x$, where $x \in D \cap V(H_1)$. Thus $r_{tr}(G) = 1$.

Conversely, assume that $r_{tr}(G) = 1$. There is an edge $e \in E(\overline{G})$ such that $\gamma_{tr}(G + e) < 2n + 2$. Let S_1 be a $\gamma_{tr}(G + e)$-set. It is obvious that $|S_1 \cap V(G_i)| = 2$ for $i = 1, 2, \dots, n$. Since s_1 and s_2 are dominated by S, we obtain

$|S_1| = 2n+1$, and since S_1 contains any leaf and support vertex of $G+e$, we obtain $e = s_2x$, where $x \in S_1 \cap V(H_i)$, for some integer $i \in \{1, 2, \ldots, n\}$. Thus $S_1 \cap \{c_1, c_2, \ldots, c_m\} = \emptyset$, and any vertex of $\{c_1, c_2, \ldots, c_m\}$ is dominated by some vertex of $S_1 \cap \bigcup_{i=1}^{n}\{u_i, \overline{u_i}\}$. Let $t : U \longrightarrow \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S_1$ and $t(u_i) = F$ if $\overline{u_i} \in S_1$. For each $j \in \{1, 2, \ldots, m\}$, there is an integer $i \in \{1, 2, \ldots, n\}$ such that c_j is dominated by $S_1 \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S_1$, and c_j is dominated by u_i. By the construction of G the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in S_1$, and c_j is dominated by $\overline{u_i}$. By the construction of G the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable. Since the construction of the total restrained reinforcement instance is straightforward from a 3-SAT instance, the size of the total restrained reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. ∎

5. k-Rainbow Reinforcement

Consider the following decision problem.

***k*-rainbow reinforcement problem (*k*RR)**

Instance: A nonempty graph G, and two positive integers $k \geq 2$ and $t \geq 1$.
Question: Is $r_{rk}(G) \leq t$?

Theorem 5. *For $k \geq 2$, the k-rainbow reinforcement problem is NP-complete for bipartite graphs.*

Proof. We show the NP-hardness of the k-rainbow reinforcement by transforming the 3-SAT to it in polynomial time. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We construct a bipartite graph G and an integer t such that $\mathcal{C}$ is satisfiable if and only if $r_{kr}(G) \leq t$. The bipartite graph G is constructed as follows. For $i = 1, 2, \ldots, n$, let H_i be a graph with $V(H_i) = \{u_i, \overline{u_i}, b_i, d_i\} \cup \{c_{ij}, e_{ij} : j = 1, 2, \ldots, k+1\}$ and $E(H_i) = \{u_id_i, \overline{u_i}b_i\} \cup \{c_{ij}e_{ij}, c_{ij}d_i, c_{ij}b_i, e_{ij}u_i, e_{ij}\overline{u_i} : j = 1, 2, \ldots, k+1\}$. Figure 1 shows the graph H_i for $k = 2$. Corresponding to each variable $u_i \in U$, we associate a graph H_i.

Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Finally, add a star $K_{1,k-1}$ with central vertex s and leaves $s_1, \ldots, s_{k-1}$, and join s to each vertex c_j with $1 \leq j \leq m$. Let G be the resulting graph. Set $t = 1$. Let f be a $\gamma_{rk}(G)$-function. We show that $\sum_{v \in V(H_i)} |f(v)| \geq 2k$ for $i = 1, 2, \ldots, n$. Let $i \in \{1, 2, \ldots, n\}$. If $|f(c_{ij})| = |f(e_{ij})| = 0$ for all $j = 1, 2, \ldots, k+1$, then clearly $\sum_{v \in V(H_i)} |f(v)| \geq$

$2k+2 > 2k$. Thus without loss of generality assume that $|f(c_{i1})| \neq 0$. Then $|f(d_i)| + |f(b_i)| + |f(e_{i1})| \geq k$. If $|f(e_{il})| = 0$ for some $l \in \{1, 2, \ldots, k+1\}$, then $|f(c_{il})| + |f(u_i)| + |f(\overline{u_i})| \geq k$, and so $\sum_{v \in V(H_i)} |f(v)| \geq 2k$. We thus assume that $|f(e_{il})| \neq 0$ for $l = 1, 2, \ldots, k+1$. Then $\sum_{v \in V(H_i)} |f(v)| \geq (|f(d_i)| + |f(b_i)| + \sum_{j=1}^{k+1} |f(e_{ij})| \geq 2k$, as desired. Since $|f(s)| + \sum_{j=1}^{k-1} |f(s_i)| + \sum_{j=1}^{m} |f(c_j)| \geq k$, we obtain $\gamma_{rk}(G) = w(f) \geq 2kn + k$. On the other hand f_1 defined on $V(G)$, by $f_1(s) = f_1(u_i) = f_1(b_i) = \{1, 2, \ldots, k\}$ for $i = 1, 2, \ldots, n$, and $f_1(u) = \emptyset$ otherwise, is a k-rainbow dominating function of weight $2kn+k$. Hence $\gamma_{rk}(G) = 2kn + k$.

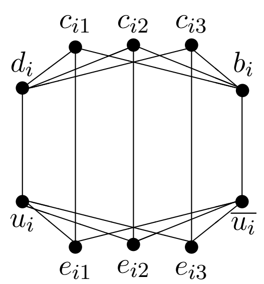

Figure 1. The graph H_i for $k = 2$.

We show that $\mathcal{C}$ is satisfiable if and only if $r_{rk}(G) = 1$. Assume that $\mathcal{C}$ is satisfiable. Let $t' : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset D of vertices of G as follows. If $t'(u_i) = T$, then we put the vertices u_i and b_i in D; if $t'(u_i) = F$, then put the vertices $\overline{u_i}$ and d_i in D. Clearly, $|D| = 2n$. Now f_2 defined on $V(G)$ by $f_2(u) = \{1, 2, \ldots, k\}$ if $u \in D$, $f_2(s) = f_2(s_i) = \{1\}$ for $i = 1, 2, \ldots, k-1$ and $f_2(u) = \emptyset$ otherwise is a $\gamma_{rk}(G)$-function, while f_3 defined on $V(G)$ by $f_3(u) = \{1, 2, \ldots k\}$ if $u \in D$, $f_3(s) = f_3(s_i) = \{1\}$ for $i = 1, 2, \ldots, k-2$, and $f_3(u) = \emptyset$ otherwise is a kRDF for $G + xs_{k-1}$, where $x \in D \cap V(H_1)$. Thus $r_{kr}(G) = 1$. Conversely, assume that $r_{rk}(G) = 1$. Thus there is an edge $e \in E(\overline{G})$ such that $\gamma_{rk}(G + e) < 2kn + k$. Let g be a $\gamma_{rk}(G + e)$-function. Suppose that $\sum_{v \in V(H_i)} |g(v)| \leq 2k - 1$, for some $i \in \{1, 2, \ldots, n\}$. Then there is an integer l such that c_{il} or e_{il} is not k-rainbow dominated by g, a contradiction. Thus $\sum_{v \in V(H_i)} |g(v)| \geq 2kn$, for each $i \in \{1, 2, \ldots, n\}$. Since $|g(s)| + \sum_{i=1}^{k-1} |g(s_i)| \geq k-1$, we obtain $\sum_{v \in V(H_i)} |g(v)| = 2kn$, for each $i \in \{1, 2, \ldots, n\}$. If $g(u_i) = g(\overline{u_i}) = \{1, 2, \ldots, k\}$ for some i, then c_{ij} is not k-rainbow dominated by g, for $j = 1, 2, \ldots, k+1$, a contradiction. Thus $|\{u_i, \overline{u_i}\} \cap V^g_{12\ldots k}| \leq 1$. Since $\sum_{v \in V(H_i)} |g(v)| = 2k$ for each $i \in \{1, 2, \ldots, n\}$, and $w(g) \leq 2kn+k-1$, we obtain $w(g) = 2kn+k-1$, $\sum_{j=1}^{m} |g(c_j)| = 0$, and $|g(s)| \neq k$. Thus any vertex of $\{c_1, c_2, \ldots, c_m\}$ is dominated by a vertex in $\{u_i, \overline{u_i}\}$, for some $i \in \{1, 2, \ldots, n\}$.

Let $t' : U \longrightarrow \{T, F\}$ be a mapping defined by $t'(u_i) = T$ if $u_i \in V^g_{12\ldots k}$

and $t'(u_i) = F$ if $\overline{u_i} \in V^g_{12\ldots k}$. For each $j \in \{1, 2, \ldots, m\}$, there is an integer $i \in \{1, 2, \ldots, n\}$ such that c_j is dominated by $V^g_{12\ldots k} \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in V^g_{12\ldots k}$, and c_j is dominated by u_i. By the construction of G the literal u_i is in the clause C_j. Then $t'(u_i) = T$, which implies that the clause C_j is satisfied by t'. Next assume that $\overline{u_i} \in V^g_{12\ldots k}$, and c_j is dominated by $\overline{u_i}$. By the construction of G the literal $\overline{u_i}$ is in the clause C_j. Then $t'(u_i) = F$. Thus, t' assigns $\overline{u_i}$ the truth value T, that is, t' satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable.

Since the construction of the k-rainbow reinforcement instance is straightforward from a 3-SAT instance, the size of the k-rainbow reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. ■

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