

## ON THE COMPLEXITY OF REINFORCEMENT IN GRAPHS

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### Abstract

We show that the decision problem for  $p$ -reinforcement,  $p$ -total reinforcement, total restrained reinforcement, and  $k$ -rainbow reinforcement are NP-hard for bipartite graphs.

**Keywords:** domination, total domination, total restrained domination,  $p$ -domination,  $k$ -rainbow domination, reinforcement, NP-hard.

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### 1. INTRODUCTION

For notation and graph theory terminology, we in general follow [12]. Specifically, let  $G$  be a graph with vertex set  $V(G) = V$  of order  $|V| = n$  and size  $|E(G)| = m$ , and let  $v$  be a vertex in  $V$ . The *open neighborhood* of  $v$  is  $N_G(v) = \{u \in V : uv \in E(G)\}$  and the *closed neighborhood* of  $v$  is  $N_G[v] = \{v\} \cup N(v)$ . If the graph  $G$  is clear from the context, we simply write  $N(v)$  rather than  $N_G(v)$ . The *degree* of a vertex  $v$ , is  $\deg(v) = |N(v)|$ . A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote by  $L(G)$  the set of all leaves of  $G$ . For a set  $S \subseteq V$ , its *open neighborhood* is the set  $N(S) = \bigcup_{v \in S} N(v)$ , and its *closed neighborhood* is the set  $N[S] = N(S) \cup S$ . A subset  $S \subseteq V$  is a *dominating set* of  $G$  if every vertex not in  $S$  is adjacent to a vertex in  $S$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set of  $G$ . A dominating set  $S$  is called a  $\gamma(G)$ -*set* of  $G$  if  $|S| = \gamma(G)$ . A dominating set  $S$  in a graph with no isolated vertex is a *total dominating set* if the induced subgraph  $G[S]$  has no isolated vertex. The *total domination number* of  $G$ , denoted by  $\gamma_t(G)$ , is the minimum cardinality of a total dominating set of  $G$ . A total dominating

set  $S$  is called a  $\gamma_t(G)$ -set of  $G$  if  $|S| = \gamma_t(G)$ . A total dominating set  $S$  in a graph with no isolated vertex is a *total restrained dominating set* if any vertex in  $V(G) \setminus S$  is also adjacent to a vertex of  $V(G) \setminus S$ . The *total restrained domination number* of  $G$ , denoted by  $\gamma_{tr}(G)$ , is the minimum cardinality of a total restrained dominating set of  $G$ . A total restrained dominating set  $S$  is called a  $\gamma_{tr}(G)$ -set of  $G$  if  $|S| = \gamma_{tr}(G)$ . For references on domination, total domination and total restrained domination in graphs, see for example [6, 7, 12, 14].

Fink and Jacobson [10] introduced the concept of *p-domination*. Let  $p$  be a positive integer. A subset  $S$  of  $V$  is a *p-dominating set* of  $G$  if  $|N(v) \cap S| \geq p$  for every vertex  $v \in V(G) \setminus S$ . The *p-domination number*,  $\gamma_p(G)$ , is the minimum cardinality among all *p-dominating sets* of  $G$ . A *p-dominating set* of  $G$  of cardinality  $\gamma_p(G)$  is called a  $\gamma_p(G)$ -set. A vertex  $v$  is said to be *p-dominated* by a set  $S$  if  $|N(v) \cap S| \geq p$ . The *p-domination number* has received much research attention, see a state-of-the-art survey article by Chellali *et al.* [5]. It is clear from the definition that every *p-dominating set* of a graph certainly contains all vertices of degree at most  $p-1$ . By this simple observation, to avoid happening the trivial case, we always assume  $\Delta(G) \geq p$ . A total dominating set  $S$  in a graph  $G$  with no isolated vertex is a *p-total dominating set* of  $G$  if  $|N(v) \cap S| \geq p$  for every vertex  $v \in V(G) \setminus S$ . The *p-total domination number*,  $\gamma_{pt}(G)$ , is the minimum cardinality among all *p-total dominating sets* of  $G$ . A *p-total dominating set* of  $G$  of cardinality  $\gamma_{pt}(G)$  will be called a  $\gamma_{pt}(G)$ -set. For references in multiple domination, see for example [1, 5, 10, 20, 21].

For a graph  $G$ , let  $f : V(G) \rightarrow \mathcal{P}(\{1, 2, \dots, k\})$  be a function. If for each vertex  $v \in V(G)$  with  $f(v) = \emptyset$  we have  $\bigcup_{u \in N(v)} f(u) = \{1, 2, \dots, k\}$ , then  $f$  is called a *k-rainbow dominating function* (or simply *kRDF*) of  $G$ . The *weight*,  $w(f)$ , of  $f$  is defined as  $w(f) = \sum_{v \in V(G)} |f(v)|$ . The minimum weight of a *kRDF* of  $G$  is called the *k-rainbow domination number* of  $G$ , and is denoted by  $\gamma_{rk}(G)$ . For references in rainbow domination, see for example [3, 4, 23, 24, 25, 26].

Kok and Mynhardt [18] introduced the *reinforcement number*  $r(G)$  of a graph  $G$  as the minimum number of edges that have to be added to  $G$  so that the resulting graph  $G'$  satisfies  $\gamma(G') < \gamma(G)$ . This concept of the reinforcement number in a graph was further considered for several domination variants, including *independent domination*, *total domination*, and *total restrained domination*, see for example [8, 9, 13, 17, 22, 27]. Sridharan, Elias, and Subramanian [22] introduced the concept of *total reinforcement* in graphs. The *total reinforcement number*,  $r_t(G)$ , of a graph with no isolated vertex is the minimum number of edges that need to be added to the graph in order to decrease the total domination number. Total reinforcement in trees was recently studied by Blair *et al.* in [2]. Jafari Rad and Volkmann [17] introduced the concept of *total restrained reinforcement* in graphs. The *total restrained reinforcement number*,  $r_{tr}(G)$ , of a graph with no isolated vertex is the minimum number of edges that need to be added to

the graph in order to decrease the total restrained domination number. Lu, Hu, and Xu [19] studied the  $p$ -reinforcement in graphs. The  $p$ -reinforcement number,  $r_p(G)$ , of a graph is the minimum number of edges that need to be added to the graph in order to decrease the  $p$ -domination number. Analogously, the  $p$ -total reinforcement number,  $r_{pt}(G)$ , of a graph is the minimum number of edges that need to be added to the graph in order to decrease the  $p$ -total domination number.

The  $k$ -rainbow reinforcement number  $r_{rk}(G)$  of a graph  $G$  is the minimum number of edges that have to be added to  $G$  so that the resulting graph  $G'$  satisfies  $\gamma_{rk}(G') < \gamma_{rk}(G)$ . Note that  $r_{r1}(G)$  is the classical reinforcement number  $r(G)$ . If  $f$  is a  $k$ RDF of  $G$  then we denote by  $V_{12\dots k}^f$  the set of all vertices  $u$  with  $|f(u)| = k$ . We refer a  $\gamma_{rk}$ -function in a graph  $G$  as a  $k$ RDF with minimum weight. If  $f$  is a  $k$ RDF of  $G$ , then we say that a vertex  $v$  is not  $k$ -rainbow dominated by  $f$  if  $f(v) = \emptyset$  and  $\bigcup_{u \in N(v)} f(u) \neq \{1, 2, \dots, k\}$ .

The complexity issue of reinforcement is studied by Lu, Hu *et al.* [15, 16, 19]. It is proved that the decision problem for the reinforcement and total reinforcement in graphs is NP-hard for bipartite graph, [15]. Lu, Hu, and Xu [19] studied the complexity of  $p$ -reinforcement in graphs.

**Theorem 1** (Lu, Hu and Xu [19]). *The  $p$ -reinforcement problem is NP-hard for general graphs.*

A *truth assignment* for a set  $U$  of Boolean variables is a mapping  $t : U \rightarrow \{T, F\}$ . A variable  $u$  is said to be *true* (or *false*) under  $t$  if  $t(u) = T$  (or  $t(u) = F$ ). If  $u$  is a variable in  $U$ , then  $u$  and  $\bar{u}$  are *literals* over  $U$ . The literal  $u$  is true under  $t$  if and only if the variable  $u$  is true under  $t$ , and the literal  $\bar{u}$  is true if and only if the variable  $u$  is false. A *clause* over  $U$  is a set of literals over  $U$ , and it is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection  $\mathcal{C}$  of clauses over  $U$  is *satisfiable* if and only if there exists some truth assignment for  $U$  that simultaneously satisfies all the clauses in  $\mathcal{C}$ . Such a truth assignment is called a *satisfying truth assignment* for  $\mathcal{C}$ . The 3-SAT problem is specified as follows.

### 3-SAT problem

**Instance:** A collection  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  of clauses over a finite set  $U$  of variables such that  $|C_j| = 3$  for  $j = 1, 2, \dots, m$ .

**Question:** Is there a truth assignment for  $U$  that satisfies all the clauses in  $\mathcal{C}$ ?

Note that the 3-SAT problem was proven to be NP-complete in [11].

In this paper we first improve Theorem 1 to bipartite graphs, and then consider the complexity of  $p$ -total reinforcement, total restrained reinforcement, and  $k$ -rainbow reinforcement. We show that the decision problems for all of these

problems are NP-hard even when restricted to bipartite graphs. Our proofs are by a transformation from 3-SAT.

## 2. $p$ -REINFORCEMENT

Let  $p \geq 2$ , and consider the following decision problem.

### $p$ -reinforcement problem ( $pR$ )

**Instance:** A nonempty graph  $G$  and a positive integer  $k$ .

**Question:** Is  $r_p(G) \leq k$ ?

We show that the problem above is NP-hard, even when restricted to the case  $k = 1$  and to bipartite graphs.

**Theorem 2.** *The  $p$ -reinforcement problem is NP-hard for bipartite graphs.*

**Proof.** We show the NP-hardness of the  $p$ -reinforcement problem by transforming the 3-SAT to it in polynomial time. Let  $U = \{u_1, u_2, \dots, u_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  be an arbitrary instance of the 3-SAT problem. We construct a graph  $G$  and an integer  $k$  such that  $\mathcal{C}$  is satisfiable if and only if  $r_p(G) \leq k$ . The graph  $G$  is constructed as follows. For  $i = 1, 2, \dots, n$ , let  $H'_i$  be a 6-cycle  $u_i v_i \bar{u}_i d_i b_i a_i u_i$  being consecutive vertices, and  $H_i$  be obtained from  $H'_i$  by adding  $p - 1$  leaves to each vertex of  $H'_i$ . For  $i = 1, 2, \dots, n$ , corresponding to each variable  $u_i \in U$ , associate the graph  $H_i$ . Corresponding to each clause  $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$ , associate a single vertex  $c_j$  and add the edge-set  $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$  for  $j = 1, 2, \dots, m$ . Next add a star  $T = K_{1,p-1}$  with center  $s$ , and join  $s$  to each vertex  $c_j$  with  $1 \leq j \leq m$ . Finally attach  $p - 1$  leaves to every vertex in  $\{c_1, c_2, \dots, c_m\}$ . Let  $G$  be the resulting graph. Note that  $G$  has  $p(6n + m + 1)$  vertices, and  $|L(G)| = (p - 1)(6n + m + 1)$ . Set  $k = 1$ . Let  $S$  be a  $\gamma_p(G)$ -set. Clearly  $L(G) \subseteq S$ . Since any vertex of  $H'_i$  is  $p$ -dominated by  $S$ , we obtain  $|S \cap V(H'_i)| \geq 2$  for  $i = 1, 2, \dots, n$ . Moreover,  $|N[s] \cap S| \geq p$ . Thus  $|S| = \gamma_p(G) \geq (6n + m + 1)(p - 1) + 2n + 1$ . On the other hand  $L(G) \cup \bigcup_{i=1}^n \{u_i, d_i\} \cup \{s\}$  is a  $p$ -dominating set for  $G$  of cardinality  $(6n + m + 1)(p - 1) + 2n + 1$ , and so  $\gamma_p(G) \leq (6n + m + 1)(p - 1) + 2n + 1$ . Thus  $\gamma_p(G) = (6n + m + 1)(p - 1) + 2n + 1$ .

We show that  $\mathcal{C}$  is satisfiable if and only if  $r_p(G) = 1$ . Assume that  $\mathcal{C}$  is satisfiable. Let  $t: U \rightarrow \{T, F\}$  be a satisfying truth assignment for  $\mathcal{C}$ . We construct a subset  $D$  of vertices of  $G$  as follows. If  $t(u_i) = T$ , then we put the vertices  $u_i$  and  $d_i$  in  $D$ ; if  $t(u_i) = F$ , then put the vertices  $\bar{u}_i$  and  $a_i$  in  $D$ . Clearly,  $|D| = 2n$ . Then  $D \cup L(G) \cup \{s\}$  is a  $p$ -dominating set for  $G$ , while  $D \cup L(G)$  is a  $p$ -dominating set for  $G + xs$ , where  $x \in V(H'_1) \cap D$ . Thus  $r_p(G) = 1$ . Conversely, assume that  $r_p(G) = 1$ . Thus there is an edge  $e \in E(\bar{G})$  such that  $\gamma_p(G + e) <$

$(6n+m+1)(p-1)+2n+1$ . Let  $S_1$  be a  $\gamma_p(G+e)$ -set. Clearly  $S_1 \cap V(H'_i) \neq \emptyset$  for  $i = 1, 2, \dots, n$ . Suppose that  $|S_1 \cap V(H'_j)| = 1$  for some  $j \in \{1, 2, \dots, n\}$ . Since  $a_j, b_j, d_j$  are  $p$ -dominated by  $S_1$ , we obtain  $b_j \in S_1$ . But  $v_j$  is  $p$ -dominated by  $S_1$ . Thus  $v_j \in e$ . Since  $u_j$  and  $\bar{u}_j$  are  $p$ -dominated by  $S_1$ , there are two different integers  $j_1, j_2 \in \{1, 2, \dots, m\}$  such that  $c_{j_1}, c_{j_2} \in S_1$ . Moreover, we may assume that  $|S_1 \cap V(H'_i)| \geq 2$  for  $i \neq j$ , since  $a_i, b_i, d_i$  and  $v_i$  are  $p$ -dominated by  $S_1$ . These imply that  $|S_1| \geq (6n+m+1)(p-1)+2n+1$ , a contradiction. Thus  $|S_1 \cap V(H'_i)| \geq 2$  for each  $i \in \{1, 2, \dots, n\}$ . Since  $|S_1| < (6n+m+1)(p-1)+2n+1$ , we deduce that  $|S_1 \cap V(H'_i)| = 2$  for each  $i \in \{1, 2, \dots, n\}$ . If  $u_j, \bar{u}_j \in S_1$  for some  $j$  then  $b_j$  is not  $p$ -dominated by  $S_1$ , a contradiction. Thus  $|S_1 \cap \{u_i, \bar{u}_i\}| \leq 1$ , and we may assume that  $|S_1 \cap \{u_i, \bar{u}_i\}| = 1$  for  $i = 1, 2, \dots, n$ . If  $s \in S_1$  then  $|S_1| \geq (6n+m+1)(p-1)+2n+1$ , a contradiction. Thus  $s \notin S_1$ . Similarly,  $c_i \notin S_1$  for  $i = 1, 2, \dots, m$ . Let  $t: U \rightarrow \{T, F\}$  be a mapping defined by  $t(u_i) = T$  if  $u_i \in S_1$ , and  $t(u_i) = F$  if  $\bar{u}_i \in S_1$ . For each  $j \in \{1, 2, \dots, m\}$ , there is an integer  $i \in \{1, 2, \dots, n\}$  such that  $c_j$  is dominated by  $S_1 \cap \{u_i, \bar{u}_i\}$ . Assume that  $u_i \in S_1$  and  $c_j$  is dominated by  $u_i$ . By the construction of  $G$ , the literal  $u_i$  is in the clause  $C_j$ . Then  $t(u_i) = T$ , which implies that the clause  $C_j$  is satisfied by  $t$ . Next assume that  $\bar{u}_i \in S_1$  and  $c_j$  is dominated by  $\bar{u}_i$ . By the construction of  $G$ , the literal  $\bar{u}_i$  is in the clause  $C_j$ . Then  $t(u_i) = F$ . Thus  $t$  assigns  $\bar{u}_i$  the truth value  $T$ , that is,  $t$  satisfies the clause  $C_j$ . Hence  $\mathcal{C}$  is satisfiable.

Since the construction of the  $p$ -reinforcement instance is straightforward from a 3-SAT instance, the size of the  $p$ -reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. ■

### 3. $p$ -TOTAL REINFORCEMENT

Let  $p \geq 2$  and consider the following decision problems.

#### **$p$ -total reinforcement problem ( $p$ TR)**

**Instance:** A graph  $G$  with no isolated vertex, and a positive integer  $k$ .

**Question:** Is  $r_{pt}(G) \leq k$ ?

**Theorem 3.** *The  $p$ -total reinforcement problem is NP-hard for bipartite graphs.*

**Proof.** The proof is similar to the proof of Theorem 2. By *attaching* a path  $P_2$  to a vertex  $v$  in a graph we mean adding a path  $P_2$  and join  $v$  to a leaf of  $P_2$ . We show the NP-hardness of the  $p$ -total reinforcement problem by transforming the 3-SAT to it in polynomial time. Let  $U = \{u_1, u_2, \dots, u_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  be an arbitrary instance of the 3-SAT problem. We construct a graph  $G$  and an integer  $k$  such that  $\mathcal{C}$  is satisfiable if and only if  $r_{pt}(G) \leq k$ . The graph  $G$  is constructed as follows. For  $i = 1, 2, \dots, n$ , let  $H'_i$  be the 6-cycle presented in the

proof of Theorem 2, and  $H_i$  be obtained from  $H'_i$  by attaching a path  $P_2$  to every vertex of  $H'_i$ . For  $i = 1, 2, \dots, n$ , corresponding to each variable  $u_i \in U$ , associate the graph  $H_i$ . Corresponding to each clause  $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$ , associate a single vertex  $c_j$  and add the edge-set  $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$  for  $j = 1, 2, \dots, m$ . Next add a graph  $J$  which is obtained from a star  $K_{1,p-1}$  (with center  $s$ ) by subdivision of any edge, and join  $s$  to each vertex  $c_j$  with  $1 \leq j \leq m$ . Finally attach  $p - 1$  paths  $P_2$  to every vertex in  $\{c_1, c_2, \dots, c_m\}$ . Let  $G$  be the resulting graph. Set  $k = 1$ . Now by the same argument as in the proof of Theorem 2, we obtain the result. ■

#### 4. TOTAL RESTRAINED REINFORCEMENT

Consider the following decision problem.

##### **Total restrained reinforcement problem (TRR)**

**Instance:** A graph  $G$  with no isolated vertex and a positive integer  $k$ .

**Question:** Is  $r_{tr}(G) \leq k$ ?

**Theorem 4.** *The total restrained reinforcement problem is NP-hard for bipartite graphs.*

**Proof.** Let  $U = \{u_1, u_2, \dots, u_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  be an arbitrary instance of the 3-SAT problem. We construct a bipartite graph  $G$  and an integer  $k$  such that  $\mathcal{C}$  is satisfiable if and only if  $r_{tr}(G) \leq k$ . The bipartite graph  $G$  is constructed as follows. Corresponding to each variable  $u_i \in U$ , we associate a graph  $H_i$  isomorphic to the complete bipartite graph  $K_{3,3}$ , where its partite sets are  $\{u_i, \bar{u}_i, d_i\}$  and  $\{a_i, b_i, e_i\}$ . Corresponding to each clause  $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$ , associate a single vertex  $c_j$  and add the edge-set  $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ . Add a path  $P_2 = s_1 s_2$ , join  $s_1$  to each vertex  $c_j$  with  $1 \leq j \leq m$ . Let  $G$  be the resulting graph. Set  $k = 1$ . Let  $S$  be a  $\gamma_{tr}(G)$ -set. Clearly  $|S \cap V(H_i)| \geq 2$  for  $i = 1, 2, \dots, n$ . Furthermore,  $s_1, s_2 \in S$ , and thus  $\gamma_{tr}(G) = |S| \geq 2n + 2$ . On the other hand,  $\{d_i, a_i : i = 1, 2, \dots, n\} \cup \{s_1, s_2\}$  is a total restrained dominating set for  $G$  of cardinality  $2n + 2$ , and thus  $\gamma_{tr}(G) \leq 2n + 2$ . Hence  $\gamma_{tr}(G) = 2n + 2$ .

We show that  $\mathcal{C}$  is satisfiable if and only if  $r_{tr}(G) = 1$ . Assume that  $\mathcal{C}$  is satisfiable. Let  $t : U \rightarrow \{T, F\}$  be a satisfying truth assignment for  $\mathcal{C}$ . We construct a subset  $D$  of vertices of  $G$  as follows. If  $t(u_i) = T$ , then we put the vertices  $u_i$  and  $a_i$  in  $D$ ; if  $t(u_i) = F$ , then put the vertices  $\bar{u}_i$  and  $a_i$  in  $D$ . Clearly,  $|D| = 2n$ . Now  $D \cup \{s_2\}$  is a total restrained dominating set for  $G + s_2 x$ , where  $x \in D \cap V(H_1)$ . Thus  $r_{tr}(G) = 1$ .

Conversely, assume that  $r_{tr}(G) = 1$ . There is an edge  $e \in E(\bar{G})$  such that  $\gamma_{tr}(G + e) < 2n + 2$ . Let  $S_1$  be a  $\gamma_{tr}(G + e)$ -set. It is obvious that  $|S_1 \cap V(G_i)| = 2$  for  $i = 1, 2, \dots, n$ . Since  $s_1$  and  $s_2$  are dominated by  $S$ , we obtain

$|S_1| = 2n + 1$ , and since  $S_1$  contains any leaf and support vertex of  $G + e$ , we obtain  $e = s_2x$ , where  $x \in S_1 \cap V(H_i)$ , for some integer  $i \in \{1, 2, \dots, n\}$ . Thus  $S_1 \cap \{c_1, c_2, \dots, c_m\} = \emptyset$ , and any vertex of  $\{c_1, c_2, \dots, c_m\}$  is dominated by some vertex of  $S_1 \cap \bigcup_{i=1}^n \{u_i, \bar{u}_i\}$ . Let  $t : U \rightarrow \{T, F\}$  be a mapping defined by  $t(u_i) = T$  if  $u_i \in S_1$  and  $t(u_i) = F$  if  $\bar{u}_i \in S_1$ . For each  $j \in \{1, 2, \dots, m\}$ , there is an integer  $i \in \{1, 2, \dots, n\}$  such that  $c_j$  is dominated by  $S_1 \cap \{u_i, \bar{u}_i\}$ . Assume that  $u_i \in S_1$ , and  $c_j$  is dominated by  $u_i$ . By the construction of  $G$  the literal  $u_i$  is in the clause  $C_j$ . Then  $t(u_i) = T$ , which implies that the clause  $C_j$  is satisfied by  $t$ . Next assume that  $\bar{u}_i \in S_1$ , and  $c_j$  is dominated by  $\bar{u}_i$ . By the construction of  $G$  the literal  $\bar{u}_i$  is in the clause  $C_j$ . Then  $t(u_i) = F$ . Thus,  $t$  assigns  $\bar{u}_i$  the truth value  $T$ , that is,  $t$  satisfies the clause  $C_j$ . Hence  $\mathcal{C}$  is satisfiable. Since the construction of the total restrained reinforcement instance is straightforward from a 3-SAT instance, the size of the total restrained reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. ■

## 5. $k$ -RAINBOW REINFORCEMENT

Consider the following decision problem.

### **$k$ -rainbow reinforcement problem ( $k$ RR)**

**Instance:** A nonempty graph  $G$ , and two positive integers  $k \geq 2$  and  $t \geq 1$ .

**Question:** Is  $r_{rk}(G) \leq t$ ?

**Theorem 5.** *For  $k \geq 2$ , the  $k$ -rainbow reinforcement problem is NP-complete for bipartite graphs.*

**Proof.** We show the NP-hardness of the  $k$ -rainbow reinforcement by transforming the 3-SAT to it in polynomial time. Let  $U = \{u_1, u_2, \dots, u_n\}$  and  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$  be an arbitrary instance of the 3-SAT problem. We construct a bipartite graph  $G$  and an integer  $t$  such that  $\mathcal{C}$  is satisfiable if and only if  $r_{rk}(G) \leq t$ . The bipartite graph  $G$  is constructed as follows. For  $i = 1, 2, \dots, n$ , let  $H_i$  be a graph with  $V(H_i) = \{u_i, \bar{u}_i, b_i, d_i\} \cup \{c_{ij}, e_{ij} : j = 1, 2, \dots, k+1\}$  and  $E(H_i) = \{u_i d_i, \bar{u}_i b_i\} \cup \{c_{ij} e_{ij}, c_{ij} d_i, c_{ij} b_i, e_{ij} u_i, e_{ij} \bar{u}_i : j = 1, 2, \dots, k+1\}$ . Figure 1 shows the graph  $H_i$  for  $k = 2$ . Corresponding to each variable  $u_i \in U$ , we associate a graph  $H_i$ .

Corresponding to each clause  $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$ , associate a single vertex  $c_j$  and add the edge-set  $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ . Finally, add a star  $K_{1, k-1}$  with central vertex  $s$  and leaves  $s_1, \dots, s_{k-1}$ , and join  $s$  to each vertex  $c_j$  with  $1 \leq j \leq m$ . Let  $G$  be the resulting graph. Set  $t = 1$ . Let  $f$  be a  $\gamma_{rk}(G)$ -function. We show that  $\sum_{v \in V(H_i)} |f(v)| \geq 2k$  for  $i = 1, 2, \dots, n$ . Let  $i \in \{1, 2, \dots, n\}$ . If  $|f(c_{ij})| = |f(e_{ij})| = 0$  for all  $j = 1, 2, \dots, k+1$ , then clearly  $\sum_{v \in V(H_i)} |f(v)| \geq$

$2k + 2 > 2k$ . Thus without loss of generality assume that  $|f(c_{i1})| \neq 0$ . Then  $|f(d_i)| + |f(b_i)| + |f(e_{i1})| \geq k$ . If  $|f(e_{il})| = 0$  for some  $l \in \{1, 2, \dots, k+1\}$ , then  $|f(c_{il})| + |f(u_i)| + |f(\bar{u}_i)| \geq k$ , and so  $\sum_{v \in V(H_i)} |f(v)| \geq 2k$ . We thus assume that  $|f(e_{il})| \neq 0$  for  $l = 1, 2, \dots, k+1$ . Then  $\sum_{v \in V(H_i)} |f(v)| \geq (|f(d_i)| + |f(b_i)| + \sum_{j=1}^{k+1} |f(e_{ij})|) \geq 2k$ , as desired. Since  $|f(s)| + \sum_{j=1}^{k-1} |f(s_i)| + \sum_{j=1}^m |f(c_j)| \geq k$ , we obtain  $\gamma_{rk}(G) = w(f) \geq 2kn + k$ . On the other hand  $f_1$  defined on  $V(G)$ , by  $f_1(s) = f_1(u_i) = f_1(b_i) = \{1, 2, \dots, k\}$  for  $i = 1, 2, \dots, n$ , and  $f_1(u) = \emptyset$  otherwise, is a  $k$ -rainbow dominating function of weight  $2kn + k$ . Hence  $\gamma_{rk}(G) = 2kn + k$ .

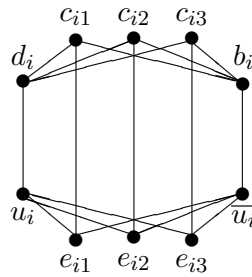


Figure 1. The graph  $H_i$  for  $k = 2$ .

We show that  $\mathcal{C}$  is satisfiable if and only if  $r_{rk}(G) = 1$ . Assume that  $\mathcal{C}$  is satisfiable. Let  $t' : U \rightarrow \{T, F\}$  be a satisfying truth assignment for  $\mathcal{C}$ . We construct a subset  $D$  of vertices of  $G$  as follows. If  $t'(u_i) = T$ , then we put the vertices  $u_i$  and  $b_i$  in  $D$ ; if  $t'(u_i) = F$ , then put the vertices  $\bar{u}_i$  and  $d_i$  in  $D$ . Clearly,  $|D| = 2n$ . Now  $f_2$  defined on  $V(G)$  by  $f_2(u) = \{1, 2, \dots, k\}$  if  $u \in D$ ,  $f_2(s) = f_2(s_i) = \{1\}$  for  $i = 1, 2, \dots, k-1$  and  $f_2(u) = \emptyset$  otherwise is a  $\gamma_{rk}(G)$ -function, while  $f_3$  defined on  $V(G)$  by  $f_3(u) = \{1, 2, \dots, k\}$  if  $u \in D$ ,  $f_3(s) = f_3(s_i) = \{1\}$  for  $i = 1, 2, \dots, k-2$ , and  $f_3(u) = \emptyset$  otherwise is a  $k$ RDF for  $G + xs_{k-1}$ , where  $x \in D \cap V(H_1)$ . Thus  $r_{kr}(G) = 1$ . Conversely, assume that  $r_{rk}(G) = 1$ . Thus there is an edge  $e \in E(\bar{G})$  such that  $\gamma_{rk}(G + e) < 2kn + k$ . Let  $g$  be a  $\gamma_{rk}(G + e)$ -function. Suppose that  $\sum_{v \in V(H_i)} |g(v)| \leq 2k - 1$ , for some  $i \in \{1, 2, \dots, n\}$ . Then there is an integer  $l$  such that  $c_{il}$  or  $e_{il}$  is not  $k$ -rainbow dominated by  $g$ , a contradiction. Thus  $\sum_{v \in V(H_i)} |g(v)| \geq 2kn$ , for each  $i \in \{1, 2, \dots, n\}$ . Since  $|g(s)| + \sum_{i=1}^{k-1} |g(s_i)| \geq k - 1$ , we obtain  $\sum_{v \in V(H_i)} |g(v)| = 2kn$ , for each  $i \in \{1, 2, \dots, n\}$ . If  $g(u_i) = g(\bar{u}_i) = \{1, 2, \dots, k\}$  for some  $i$ , then  $c_{ij}$  is not  $k$ -rainbow dominated by  $g$ , for  $j = 1, 2, \dots, k+1$ , a contradiction. Thus  $|\{u_i, \bar{u}_i\} \cap V_{12\dots k}^g| \leq 1$ . Since  $\sum_{v \in V(H_i)} |g(v)| = 2k$  for each  $i \in \{1, 2, \dots, n\}$ , and  $w(g) \leq 2kn + k - 1$ , we obtain  $w(g) = 2kn + k - 1$ ,  $\sum_{j=1}^m |g(c_j)| = 0$ , and  $|g(s)| \neq k$ . Thus any vertex of  $\{c_1, c_2, \dots, c_m\}$  is dominated by a vertex in  $\{u_i, \bar{u}_i\}$ , for some  $i \in \{1, 2, \dots, n\}$ .

Let  $t' : U \rightarrow \{T, F\}$  be a mapping defined by  $t'(u_i) = T$  if  $u_i \in V_{12\dots k}^g$



and  $t'(u_i) = F$  if  $\overline{u_i} \in V_{12\dots k}^g$ . For each  $j \in \{1, 2, \dots, m\}$ , there is an integer  $i \in \{1, 2, \dots, n\}$  such that  $c_j$  is dominated by  $V_{12\dots k}^g \cap \{u_i, \overline{u_i}\}$ . Assume that  $u_i \in V_{12\dots k}^g$ , and  $c_j$  is dominated by  $u_i$ . By the construction of  $G$  the literal  $u_i$  is in the clause  $C_j$ . Then  $t'(u_i) = T$ , which implies that the clause  $C_j$  is satisfied by  $t'$ . Next assume that  $\overline{u_i} \in V_{12\dots k}^g$ , and  $c_j$  is dominated by  $\overline{u_i}$ . By the construction of  $G$  the literal  $\overline{u_i}$  is in the clause  $C_j$ . Then  $t'(u_i) = F$ . Thus,  $t'$  assigns  $\overline{u_i}$  the truth value  $T$ , that is,  $t'$  satisfies the clause  $C_j$ . Hence  $\mathcal{C}$  is satisfiable.

Since the construction of the  $k$ -rainbow reinforcement instance is straightforward from a 3-SAT instance, the size of the  $k$ -rainbow reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. ■

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### REFERENCES

- [1] M. Blidia, M. Chellali and L. Volkmann, *Some bounds on the  $p$ -domination number in trees*, Discrete Math. **306** (2006) 2031–2037.  
doi:10.1016/j.disc.2006.04.010
- [2] J.R.S. Blair, W. Goddard, S.T. Hedetniemi, S. Horton, P. Jones and G. Kubicki, *On domination and reinforcement numbers in trees*, Discrete Math. **308** (2008) 1165–1175.  
doi:10.1016/j.disc.2007.03.067
- [3] B. Brešar, M.A. Henning and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008) 213–225.
- [4] B. Brešar and T.K. Šumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007) 2394–2400.  
doi:10.1016/j.dam.2007.07.018
- [5] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann,  *$k$ -domination and  $k$ -independence in graphs: A survey*, Graphs Combin. **28** (2012) 1–55.  
doi:10.1007/s00373-011-1040-3
- [6] X.-G. Chen, D.-X. Ma and L. Sun, *On total restrained domination in graphs*, Czechoslovak Math. J. **55** (2005) 165–173.  
doi:10.1007/s10587-005-0012-2
- [7] J. Cyman and J. Raczek, *On the total restrained domination number of a graph*, Australas. J. Combin. **36** (2006) 91–100.
- [8] G.S. Domke and R.C. Laskar, *The bondage and reinforcement numbers of  $\gamma_f$  for some graphs*, Discrete Math. **167/168** (1997) 249–259.  
doi:10.1016/S0012-365X(97)00232-X

- [9] J.E. Dunbar, T.W. Haynes, U. Teschner and L. Volkmann, *Bondage, insensitivity, and reinforcement*, in: T.W. Haynes, S.T. Hedetniemi and P.J. Slater (Ed(s)), *Domination in Graphs: Advanced Topics* (Marcel Dekker, New York, 1998) 471–489.
- [10] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, in: Y. Alavi and A.J. Schwenk (Ed(s)), *Graph Theory with Applications to Algorithms and Computer Science* (Wiley, New York, 1985) 283–300.
- [11] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, San Francisco, 1979).
- [12] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, Inc., New York, 1998).
- [13] M.A. Henning, N. Jafari Rad and J. Raczek, *A note on total reinforcement in graphs*, *Discrete Appl. Math.* **159** (2011) 1443–1446.  
doi:10.1016/j.dam.2011.04.024
- [14] M.A. Henning and A. Yeo, *Total Domination in Graphs* (Springer Verlag, New York, 2013).
- [15] F.-T. Hu and M.Y. Sohn, *The algorithmic complexity of bondage and reinforcement problems in bipartite graphs*, *Theoret. Comput. Sci.* **535** (2014) 46–53.  
doi:10.1016/j.tcs.2014.04.005.
- [16] F.-T. Hu and J.-M. Xu, *On the complexity of the bondage and reinforcement problems*, *J. Complexity* **28** (2012) 192–201.  
doi:10.1016/j.jco.2011.11.001
- [17] N. Jafari Rad and L. Volkmann, *Total restrained reinforcement in graphs*, *AKCE Int. J. Graphs Comb.* **13** (2016) 16–21.  
doi:10.1016/j.akcej.2016.02.003
- [18] J. Kok and C.M. Mynhardt, *Reinforcement in graphs*, *Congr. Numer.* **79** (1990) 225–231.
- [19] Y. Lu, F.-T. Hu and J.-M. Xu, *On the p-reinforcement and the complexity*, *J. Comb. Optim.* **29** (2015) 389–405.  
doi:10.1007/s10878-013-9597-9
- [20] D. Rautenbach and L. Volkmann, *New bounds on the k-domination number and the k-tuple domination number*, *Appl. Math. Lett.* **20** (2007) 98–102.  
doi:10.1016/j.aml.2006.03.006
- [21] R.S. Shaheen, *Bounds for the 2-domination number of toroidal grid graphs*, *Int. J. Comput. Math.* **86** (2009) 584–588.  
doi:10.1080/00207160701690284
- [22] N. Sridharan, M.D. Elias and V.S.A. Subramanian, *Total reinforcement number of a graph*, *AKCE Int. J. Graphs Combin.* **4** (2007) 197–202.
- [23] C. Tong, X. Lin, Y. Yang and M. Luo, *2-rainbow domination of generalized Petersen graphs  $P(n, 2)$* , *Discrete Appl. Math.* **157** (2009) 1932–1937.  
doi:10.1016/j.dam.2009.01.020

- [24] Y. Wu and N. Jafari Rad, *Bounds on the 2-rainbow domination number of graphs*, Graphs Combin. **29** (2013) 1125–1133.  
doi:10.1007/s00373-012-1158-y
- [25] Y. Wu and H. Xing, *Note on 2-rainbow domination and Roman domination in graphs*, Appl. Math. Lett. **23** (2010) 706–709.  
doi:10.1016/j.aml.2010.02.012
- [26] G. Xu, *2-rainbow domination in generalized Petersen graphs  $P(n, 3)$* , Discrete Appl. Math. **157** (2009) 2570–2573.  
doi:10.1016/j.dam.2009.03.016
- [27] J.H. Zhang, H.L. Liu and L. Sun, *Independence bondage number and reinforcement number of some graphs*, Trans. Beijing Inst. Tech. **23** (2003) 140–142.

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