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FORBIDDEN SUBGRAPHS FOR HAMILTONICITY OF 1-TOUGH GRAPHS

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Abstract

A graph G is said to be 1-tough if for every vertex cut S of G, the number of components of $G - S$ does not exceed $|S|$. Being 1-tough is an obvious necessary condition for a graph to be hamiltonian, but it is not sufficient in general. We study the problem of characterizing all graphs H such that every 1-tough H-free graph is hamiltonian. We almost obtain a complete solution to this problem, leaving $H = K_1 \cup P_4$ as the only open case.

Keywords: forbidden subgraph, 1-tough graph, H-free graph, hamiltonian graph.

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1. Introduction

We use Bondy and Murty [5] for terminology and notation not defined here and consider finite simple graphs only.

We start by presenting some of the relevant terminology and notation. Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. When no confusion can arise, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. For a subgraph F of G, we set $N_H(F) = \bigcup_{v \in V(F)} N_H(v)$.

Let $\omega(G)$ denote the number of components of the graph G. Adopting the terminology of [8], a connected graph G is said to be 1-*tough* if for every vertex cut S of G, $\omega(G - S) \leq |S|$. Note that every complete graph is 1-tough. A graph G is *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing every vertex of G.

Clearly, every hamiltonian graph is 1-tough and has at least 3 vertices. In the following, we use $\mathcal{G}_H$ to denote the set of hamiltonian graphs, and $\mathcal{G}_T$ to denote the set of 1-tough graphs on at least 3 vertices. Thus $\mathcal{G}_H \subset \mathcal{G}_T$. In this note, we give a number of sufficient conditions involving forbidden induced subgraphs for a 1-tough graph to be hamiltonian.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or the subgraph of G induced by $V(G')$). For a given graph H, we say that G is H-*free* if G does not contain an induced subgraph isomorphic to H. If we impose the condition that G has to be H-free, then H is called a *forbidden subgraph* for G, and the condition of being H-free is called a *forbidden subgraph condition.* Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

For hamiltonicity and related properties, many researchers have studied forbidden subgraph conditions, and for many cases full characterizations have been obtained. A good example article and an inspiration for much of the work done in this direction is [9]. For hamiltonicity, it is well-known and almost trivial to prove that the only connected forbidden subgraph that can guarantee that every connected H-free graph on at least three vertices is hamiltonian is $H = P_3$ (yielding complete graphs only). Hence, for general connected graphs such conditions involving only one forbidden subgraph are not interesting. Characterizations of all forbidden pairs for hamiltonicity and related properties are far from trivial, and the early results were first obtained in the Ph.D. work of Bedrossian [4]. In case of hamiltonicity, it can be observed that many of the nonhamiltonian graph families that show the necessity of forbidding certain subgraphs are not 1-tough. Therefore, it is natural to ask whether such characterizations alter much if one imposes that all graphs under consideration are 1-tough, a stronger property than

2-connectedness. As we will show, even the case with one forbidden subgraph turns out to be nontrivial. This gives little hope for characterizing all such pairs for hamiltonicity of 1-tough graphs. On the other hand, for 2-connected graphs all (non-trivial) forbidden pairs for hamiltonicity contain the claw (an induced $K_{1,3}$), and it is known that every 2-connected claw-free graph is 1-tough [11]. This makes it even more interesting to answer the question which forbidden graphs H guarantee that 1-tough H-free graphs are hamiltonian.

In the sequel we are trying to answer this question by characterizing for which graphs H every 1-tough H-free graph is hamiltonian. We almost establish a full characterization, leaving just one open case. In the next two sections we prove the following results. In the sequel, $H \cup F$ denotes the disjoint union of two vertex-disjoint graphs H and F, and we use the shorthand notation $H \cup F$-free instead of $(H \cup F)$-free.

Theorem 1. *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

The case with $K_1 \cup P_3$ was independently proved in a recent paper due to Nikoghosyan [12], where the case with P_4 was conjectured, and the nonhamiltonian $K_1 \cup K_2$-free graphs were characterized. The case with P_4 was proved back in the 1970s by Jung [10], where P_4-free graphs were studied 'under disguise' as D^*-graphs. These graphs are also known under other names, e.g., within algorithmic graph theory as cographs (since the complement of a P_4-free graph is also P_4-free).

Theorem 2. *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

Note that every induced subgraph of $K_1 \cup P_4$ is either $K_1 \cup P_4$ itself, or an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. By the above two theorems, the only graph for which we do not know whether forbidding it can ensure a graph in $\mathcal{G}_T$ to be hamiltonian is $K_1 \cup P_4$. We pose this as an open problem, but it appeared as a conjecture in [12].

Problem 1. Is every $K_1 \cup P_4$-free 1-tough graph on at least three vertices hamiltonian?

This question seems to be very hard to answer, even if we impose a higher toughness. Let us give a bit more background and some references to conclude the discussion, before we start presenting the proofs of the above results.

Back in the early 1970s, in a seminal paper due to Chvátal [8] it was conjectured that there exists a positive real number t_0 such that every t_0-tough graph on at least three vertices is hamiltonian. Here a graph G is *t-tough* if $t \cdot \omega(G-S) \leq |S|$

for every cut set $S \subset V(G)$, so the larger the value of t, the stricter the requirement. More than 40 years later, the graph theory community has still not solved or refuted this conjecture of Chvátal, but different groups of researchers obtained a lot of insight and many results related to this conjecture. (See, e.g., the survey papers [2] and [6].) So, the conjecture is still open for general graphs (and we now know that t_0 must be at least 9/4 by a result in [3]). For several specific graph classes the conjecture has been validated, with a recent result (that is not covered in [2]) showing that 25-tough $2K_2$-free graphs on at least three vertices are hamiltonian [7]. It is conjectured in [12] that t-tough $2K_2$-free graphs with $t > 1$ are hamiltonian, but this again seems very difficult to prove. Other known results show that sufficient conditions for guaranteeing hamiltonicity can be improved considerably if we impose a high toughness, e.g., the results on minimum degree conditions in [1].

In the light of the discussion, it is interesting to consider the following weaker version of Problem 1.

Problem 2. Is the general conjecture of Chvátal true for $K_1 \cup P_4$-free graphs?

Some of the partial results in [7] can be proved for $K_1 \cup P_4$-free graphs, but a similar approach fails for solving the general problem.

2. Preliminaries

Before we present our proofs of the above two theorems, we introduce some additional terminology. In fact, we will use this terminology to formulate a slightly stronger statement that immediately implies Theorem 1.

A *path partition* of a graph G is a spanning subgraph of G each component of which is a path. Note that every graph G has at least one path partition (the edgeless graph on $V(G)$). We define the *path partition number* of G, denoted by $\pi(G)$, as

$$\pi(G) = \begin{cases} 0, & \text{if } G \in \mathcal{G}_H; \\ \min\{\omega(\mathcal{P}) : \mathcal{P} \text{ is a path partition of } G\}, & \text{otherwise.} \end{cases}$$

Alternatively, $\pi(G)$ is the minimum number of edges we have to add to G to turn it into a hamiltonian graph, except for degenerate cases. Note that $\pi(K_1) = \pi(K_2) = 1$ and $\pi(2K_1) = 2$.

The concept of the *scattering number* of a connected non-complete graph was introduced by Jung [10], and is defined as $s(G) = \max\{\omega(G - S) - |S| : S \subset V(G) \text{ and } \omega(G - S) > 1\}$. We extend this concept to general graphs, as follows. If G is 1-tough and has at least 3 vertices, then let $s(G) = 0$. Otherwise let

$s(G) = \max\{\omega(G-S) - |S| : S \subset V(G)\}$, so here the set S is not necessarily a vertex cut of G (and not necessarily nonempty).

$$s(G) = \begin{cases} 0, & \text{if } G \in \mathcal{G}_T; \\ \max\{\omega(G-S) - |S| : S \subset V(G)\}, & \text{otherwise.} \end{cases}$$

Note that $s(K_1) = s(K_2) = 1$ and $s(2K_1) = 2$.

If a graph G is hamiltonian, then G is 1-tough and has at least 3 vertices. This implies that every hamiltonian graph has scattering number 0. In fact, it is easy to prove the following result.

Theorem 3. *For every graph G, $\pi(G) \geq s(G)$.*

Proof. If G has only one or two vertices, then the result is trivially true. Next, we assume that G has at least 3 vertices. If G is 1-tough, then by definition $s(G) = 0$ and $\pi(G) \geq s(G)$. Henceforth, we assume that G is not 1-tough. This implies that G is not hamiltonian.

Let $\mathcal{P}$ be a path partition of G such that $\omega(\mathcal{P}) = \pi(G)$ and let S be a subset of $V(G)$ such that $\omega(G-S) - |S| = s(G)$. If $S = \emptyset$, then $G - S = G$; and if $S \neq \emptyset$, then $G - S$ is disconnected. In any case, $G - S$ is not hamiltonian.

Clearly $\mathcal{P} - S$ is a path partition of $G - S$, and the removal of any vertex in $\mathcal{P}$ can increase the number of components by at most one (each time we remove the next vertex). Thus we have

$$\pi(G-S) \leq \omega(\mathcal{P}-S) \leq \omega(\mathcal{P}) + |S| = \pi(G) + |S|.$$

On the other hand, it is easy to see that $\pi(G-S) \geq \omega(G-S) = s(G) + |S|$. This implies that $s(G) + |S| \leq \pi(G) + |S|$ and $s(G) \leq \pi(G)$. ■

Instead of Theorem 1, we are going to prove the following stronger result.

Theorem 4. *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$, and let G be an R-free graph. Then $\pi(G) = s(G)$.*

The case with P_4 has been proved independently in [10].

We first complete this section by introducing some additional terminology and notation, and by stating a simple folklore result we will use throughout. We supply a proof for convenience.

Let G be a graph and let C be a cycle of G with a given orientation as a directed cycle. For a vertex v on C, v^+ denotes its immediate successor, and v^- its immediate predecessor on C, in the given orientation. If u, v are two vertices on C, $\overrightarrow{C}[u,v]$ (and $\overleftarrow{C}[v,u]$) denotes the path from u to v along C in the direction given by the orientation, and $[u,v]$ denotes the set of vertices in $\overrightarrow{C}[u,v]$.

Let C be a cycle of a graph G, and let u, v be two distinct vertices in $V(C)$. We say that u and v are *attached* (*to* C) if there is a path from u to v with all

internal vertices disjoint from $V(C)$ (we stipulate that a vertex is not attached to itself). We set $A^C(G) = \{uv : u \text{ and } v \text{ are attached to } C\}$. Note that $uv \in A^C(G)$ if and only if $uv \in E(G)$ or there is a component H of $G - C$ such that $u, v \in N_C(H)$.

Lemma 5. *Let G be a graph, C a longest cycle in G, and H a component of $G - C$.*

(1) *If $u \in N_C(H)$, then $u^+, u^- \notin N_C(H)$.*
(2) *If $u, v \in N_C(H)$, then $u^+v^+, u^-v^- \notin A^C(G)$.*

Proof. (1) Assume that $u^+ \in N_C(H)$. Let P be a path from u to u^+ of length at least 2 with all internal vertices in H. Then $C' = (C - uu^+) \cup P$ (with the obvious meaning, slightly abusing the notation) is a cycle longer than C, a contradiction. The second assertion can be proved similarly.

(2) Assume that $u^+v^+ \in A^C(G)$. Let P be a path from u to v of length at least 2 with all internal vertices in H, and P' be a path from u^+ to v^+ with all internal vertices disjoint from C. By (1), $u^+, v^+ \notin N_C(H)$. Thus P and P' are vertex-disjoint. Then $C' = (C - \{uu^+, vv^+\}) \cup P \cup P'$ is a cycle longer than C. The second assertion can be proved similarly. ■

In the next section we present a proof of Theorem 2, and in the final section we prove Theorem 4.

3. Proof of Theorem 2

In order to prove Theorem 2, suppose that R is a graph on at least three vertices. Assuming R is not an induced subgraph of $K_1 \cup P_4$, we will derive at a contradiction by exhibiting classes of nonhamiltonian 1-tough graphs that are R-free.

We will first prove that our assumption implies that R contains one of the graphs in $\mathcal{H}$ as an induced subgraph, where

$$\mathcal{H} = \{C_3, C_4, C_5, K_{1,3}, 2K_2, 4K_1\}.$$

To prove this, first assume that R contains a cycle. In this case, let C be a shortest cycle of R. Then either C is an induced copy of C_3, C_4 or C_5, or C contains an induced copy of $2K_2$. So in this case R clearly contains a graph of $\mathcal{H}$ as an induced subgraph.

Next assume that R contains no cycles. If R has a vertex with degree at least 3, then R clearly contains an induced copy of $K_{1,3} \in \mathcal{H}$. Thus we may assume that every component of R is a path. If R contains at least two non-trivial components, then two edges in distinct components form an induced copy of

$2K_2 \in \mathcal{H}$. Thus we may assume that there is at most one non-trivial component in R. If R contains a path with at least 5 vertices, then it again contains an induced copy of $2K_2$. Thus we may assume that the non-trivial component of R is a path with at most 4 vertices. If R has at least 4 components, or R has 3 components one of which is a path with at least 3 vertices, then R contains an induced copy of $4K_1 \in \mathcal{H}$. Thus we are left with the case that $\omega(R) \leq 3$ and if $\omega(R) = 3$, then the non-trivial component of R has at most two vertices. We distinguish the following cases to complete the proof of our claim.

If R is connected, then R is a path with at most 4 vertices, hence an induced subgraph of $K_1 \cup P_4$. If $\omega(R) = 2$, then one of the components of R is trivial and the other component is a path with at most 4 vertices, hence R is an induced subgraph of $K_1 \cup P_4$. If $\omega(R) = 3$, then two of the components of R are trivial and the third component is a path with at most 2 vertices. Also in this case R is an induced subgraph of $K_1 \cup P_4$. This completes the proof of our claim that R contains one of the graphs in $\mathcal{H} = \{C_3, C_4, C_5, K_{1,3}, 2K_2, 4K_1\}$ as an induced subgraph.

To complete the proof of Theorem 2 it suffices to show that an R-free 1-tough graph is not necessarily hamiltonian for $R \in \mathcal{H}$. For this purpose we constructed some graph families, the members of which are 1-tough but not hamiltonian (see Figure 1).

The members of the class G_1 consist of two disjoint odd cycles $x_1x_2 \cdots x_kx_1$ and $z_1z_2 \cdots z_kz_1$ on $k \geq 5$ vertices with connecting paths $x_iy_iz_i$ of length two between corresponding vertices of the two cycles. Members of this class are clearly C_3-free and C_4-free, and it is easy to check that they are 1-tough and nonhamiltonian.

The members of the class G_2 consist of two disjoint complete subgraphs on $k \geq 3$ vertices with three connecting vertex-disjoint paths of length two between the two subgraphs. Members of this class are clearly C_5-free and $K_{1,3}$-free, and it is easy to check that they are 1-tough and nonhamiltonian.

The members of the class G_3 consist of a complete subgraph H on $k \geq 3$ vertices $x_1, x_2, \ldots, x_k$, and an additional k vertices $y_1, y_2, \ldots, y_k$ such that $N_H(y_i) = \{x_i\}$, and an additional universal vertex z that is adjacent to all x_i and all y_i. Members of this class are clearly $2K_2$-free, and it is easy to check that they are 1-tough and nonhamiltonian.

The members of the class G_4 consist of three disjoint complete subgraphs on $k \geq 3$ vertices and the additional edges of two vertex-disjoint triangles that each contain exactly one vertex of each of the complete subgraphs. Members of this class are clearly $4K_1$-free, and it is easy to check that they are 1-tough and nonhamiltonian.

Together, the above graphs cover all cases. This completes the proof of Theorem 2.

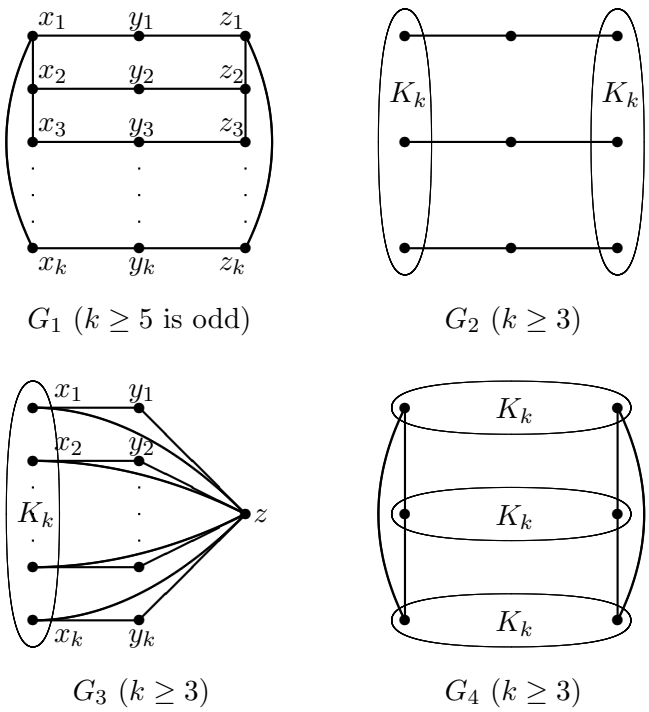

Figure 1. Four families of 1-tough nonhamiltonian graphs.

4. Proof of Theorem 4

Note that we only need to prove the statement of Theorem 4 for the cases $R = P_4$, $R = K_1 \cup P_3$ and $R = 2K_1 \cup K_2$. So, we let G be a P_4-free, $K_1 \cup P_3$-free or $2K_1 \cup K_2$-free graph, and we are going to prove that $\pi(G) = s(G)$.

The result is trivially true if $|V(G)| \leq 2$, and we proceed by induction on $|V(G)| \geq 3$. If G is disconnected, then it is not difficult to deduce that

$$\pi(G) = \sum \{\max\{1, \pi(H)\} : H \text{ is a component of } G\}, \text{ and}$$
$$s(G) = \sum \{\max\{1, s(H)\} : H \text{ is a component of } G\}.$$

Thus we can complete the proof by applying the induction hypothesis to each component of G.

Henceforth, we may assume that G is connected. If G is hamiltonian, then G is 1-tough and $\pi(G) = s(G) = 0$. Thus we may assume that G is not hamiltonian. By Theorem 3, $\pi(G) \geq s(G)$. So it is sufficient to prove that $\pi(G) \leq s(G)$. We distinguish the three cases that $R = P_4$, $R = K_1 \cup P_3$ or $R = 2K_1 \cup K_2$, and

we prove that $\pi(G) \leq s(G)$ in all cases by in each case first proving a number of claims.

The case $R = P_4$.

Claim 6. *There is a vertex cut S of G such that $|S| < |V(G-S)|$.*

Proof. Otherwise, G is $\lceil n/2 \rceil$-connected, and thus G is hamiltonian by Dirac's theorem, a contradiction. □

Now let S be a smallest vertex cut of G, so in particular S has the property of the above claim.

Claim 7. *Every vertex of S is adjacent to every vertex of $V(G-S)$.*

Proof. Clearly, the choice of S implies that for every vertex $x \in S$ and every component H of $G-S$, x is adjacent to at least one vertex in H; otherwise $S \setminus \{x\}$ is a vertex cut, contradicting the choice of S. Suppose that $xy \notin E(G)$ for some $y \in V(G-S)$. Let H be the component of $G-S$ containing y, let P be a shortest path from y to x with all internal vertices in H and let y' be a neighbor of x in a component of $G-S$ other than H. Then Pxy' is an induced path with at least 4 vertices, contradicting that G is P_4-free. □

Let $s = |S|$. By Claim 1, $s < |V(G-S)|$. Let $S = \{x_1, x_2, \ldots, x_s\}$. Let $\mathcal{P}$ be a path partition of $G-S$ such that $\omega(\mathcal{P}) = \pi(G-S)$ (note that $G-S$ is not hamiltonian since it is disconnected). If $\omega(\mathcal{P}) \leq s$, then we can remove $s - \omega(\mathcal{P})$ edges from $\mathcal{P}$ and get a path partition $\mathcal{P}'$ such that $\omega(\mathcal{P}') = s$. Let Q_i, $1 \leq i \leq s$, denote the paths of $\mathcal{P}'$, and let y_i, y_i' denote the end vertices of Q_i. We use the edges that are guaranteed by Claim 2 to obtain the Hamilton cycle $C = y_1 Q_1 y_1' x_1 y_2 Q_2 y_2' x_2 \cdots y_s Q_s y_s' x_s y_1$, a contradiction.

Thus we henceforth assume that $\omega(\mathcal{P}) > s$, and we let $k = \pi(G-S) = \omega(\mathcal{P})$. We denote by Q_i, $1 \leq i \leq k$, the paths of $\mathcal{P}$, and by y_i, y_i' the end vertices of Q_i. Let $P' = y_1 Q_1 y_1' x_1 y_2 Q_2 y_2' x_2 \cdots y_s Q_s y_s' x_s y_{s+1} Q_{s+1} y_{s+1}'$. Then $\mathcal{P}' = P' \cup \bigcup_{i=s+2}^{k} Q_i$ is a path partition of G. This implies that $\pi(G) \leq \omega(\mathcal{P}') = k - s$.

By the induction hypothesis, $s(G-S) = \pi(G-S) = k$. Let S' be a subset of $V(G-S)$ such that $\omega((G-S)-S') - |S'| = k$. Let $S'' = S' \cup S$. Note that

$$\omega(G-S'') - |S''| = \omega((G-S)-S') - |S'| - |S| = k - s.$$

This implies that $s(G) \geq k - s$, hence that $\pi(G) \leq s(G)$.

The case $R = K_1 \cup P_3$.

Claim 8. *If S is a vertex cut of G, then every component of $G-S$ is a clique.*

Proof. If there is a component H of $G-S$ that is not a clique, then H contains an induced copy of P_3. Such a P_3 in H and a vertex in a different component of $G-S$ together form an induced copy of $K_1 \cup P_3$, contradicting that G is $K_1 \cup P_3$-free. $\square$

We first deal with the case that G has a cut vertex x. In this case, let $k = \omega(G-x)$, let H_i, $1 \le i \le k$, denote the components of $G-x$, let y_i be a neighbor of x in H_i, and let Q_i be a Hamilton path of H_i starting from y_i (using Claim 3). Let $P' = Q_1 y_1 x y_2 Q_2$. Then $\mathcal{P} = P' \cup \bigcup_{i=3}^{k} Q_i$ is a path partition of G. This implies that $\pi(G) \le k-1$. Noting that $s(G) \ge \omega(G-x) - 1 = k-1$, we conclude that $\pi(G) \le s(G)$.

Next we assume that G is 2-connected. Using that G is nonhamiltonian, let C be a longest cycle of G, let H be a component of $G-C$, and let $S = \{x_1, x_2, \ldots, x_s\}$ be the neighborhood of H on C such that the vertices x_i appear in this order along C.

Note that S is a vertex cut of G. By Claim 3, each component of $G-S$ is a clique. In particular, H and $[x_i^+, x_{i+1}^-]$, $1 \le i \le s$, are cliques, where $x_{s+1} = x_1$ (indices are taken modulo s).

Claim 9. *Let y, y' be two vertices in $V(C) \backslash N_C(H)$. Then $yy' \in A^C(G)$ if and only if $y, y' \in [x_i^+, x_{i+1}^-]$ for some i, $1 \le i \le s$.*

Proof. Since $[x_i^+, x_{i+1}^-]$ is a clique, by definition, for any two vertices y, y' in $[x_i^+, x_{i+1}^-]$ we have $yy' \in A^C(G)$. This completes the proof of the 'if' part of the assertion.

Now assume that $y \in [x_i^+, x_{i+1}^-]$, $y' \in [x_j^+, x_{j+1}^-]$, where $1 \le i < j \le s$. By Lemma 1, $x_i^+ x_j^+ \notin E(G)$. This implies that x_i^+, x_j^+, and thus y, y', are in distinct components of $G-S$ (using Claim 3). Hence, $yy' \notin A^C(G)$. This completes the proof of the 'only-if' part of the assertion. $\square$

Claim 10. *For every vertex $v \in V(G) \setminus V(C)$, $N_C(v) \subset S$.*

Proof. If $v \in V(H)$, then the statement is trivially true. Now we assume, to the contrary, that there are $v \in V(H')$ and $y \in V(C) \backslash S$ such that $vy \in E(G)$, where H' is a component of $G-S$ other than H. Without loss of generality, let $y \in [x_i^+, x_{i+1}^-]$.

Note that $[x_i^+, x_{i+1}^-] \cup \{v\}$ is contained in a common component of $G-S$. If $y \ne x_i^+$, then by Claim 3, $vy^- \in E(G)$, and it is obvious that there is a cycle longer than C, a contradiction. Thus we may assume that $y = x_i^+$, and similarly, $y^+ = x_{i+1}$.

By Lemma 1, we obtain that $x_i, x_{i+1} \notin N_C(H')$. Using Claim 4, we get that $N_C(H') \subset (S \cup \{y\}) \backslash \{x_i, x_{i+1}\}$. Since G is 2-connected, there is a vertex $x_j \in N_C(H') \cap S$. This implies that $|S| \ge 3$ and $x_i^- x_{i+1}^+ \notin E(G)$.

On the other hand, since $N_C(H')$ is a vertex cut and x_i^-, x_{i+1}^+ are connected by a path with internal vertices in $V(H) \cup \{x_i, x_{i+1}\}$, we know that x_i^-, x_{i+1}^+ are in a common component of $G - N_C(H')$. By Claim 3, we conclude that $x_i^- x_{i+1}^+ \in E(G)$, a contradiction. □

Let $k = \omega(G - V(C))$, and let H_i, $1 \le i \le k$, be the components of $G - V(C)$, where $H_1 = H$. Note that H_i is a clique by Claim 3. Let y_1 be a neighbor of x_1 in H_1, let Q_1 be a Hamilton path of H_1 starting from y_1, let $P' = Q_1 y_1 x_1 \overrightarrow{C}[x_1, x_1^-]$, and let Q_i, $2 \le i \le k$, be a Hamilton path of H_i. Thus $\mathcal{P} = P' \cup \bigcup_{i=2}^k Q_i$ is a path partition of G. This implies that $\pi(G) \le k$.

On the other hand, noting that H_i, $1 \le i \le k$, and the subgraphs induced by $[x_i^+, x_{i-1}^-]$, $1 \le i \le s$, are all the components of $G - S$, this implies that

$$s(G) \ge \omega(G - S) - |S| = k + s - s = k.$$

Thus we conclude that $\pi(G) \le s(G)$.

The case $R = 2K_1 \cup K_2$.

Claim 11. *If S is a vertex cut of G such that $\omega(G - S) \ge 3$, then every component of $G - S$ is trivial.*

Proof. Otherwise, an edge in a non-trivial component and two vertices in another two distinct components induce a $2K_1 \cup K_2$. □

We assume, to the contrary, that $\pi(G) > s(G)$, and we will reach a contradiction in all cases.

Claim 12. *G is 2-connected and nonhamiltonian, or there exists a (pendant) edge $x_0 y_0$ such that $G - y_0$ is 2-connected and nonhamiltonian.*

Proof. Recall that G is nonhamiltonian. If G is 2-connected, then the result is trivially true. So we assume that G has a cut vertex x_0.

If $\omega(G - x_0) = k \ge 3$, then by Claim 6, G is isomorphic to $K_{1,k}$. Note that $\pi(K_{1,k}) = k - 1$ and $s(K_{1,k}) = k - 1$ $(k \ge 3)$. Thus we may assume $\omega(G - x_0) = 2$.

If the two components of $G - x_0$ are both non-trivial, then we claim that each component of $G - x_0$ is a clique; otherwise two nonadjacent vertices in one component and an edge in another component induce a $2K_1 \cup K_2$, a contradiction. Let H_i, $i = 1, 2$, be the two components of $G - x_0$, let y_i be a neighbor of x_0 in H_i, and let Q_i be a Hamilton path of H_i starting from y_i. Then $P = Q_1 y_1 x_0 y_2 Q_2$ is a Hamilton path of G, which implies that $\pi(G) = 1$. Clearly $s(G) \ge \omega(G - x_0) - 1 = 1$. Thus we conclude that $\pi(G) \le s(G)$, a contradiction. Hence, we may assume there is an isolated vertex y_0 in $G - x_0$.

If the component of $G - x_0$ not containing y_0 is trivial, then G is isomorphic to P_3. Note that $\pi(P_3) = 1$ and $s(P_3) = 1$. Thus we conclude that there are at least three vertices in $G - y_0$.

Next we assume that $G-y_0$ has a cut vertex x_1 (clearly x_0 is not a cut vertex of $G-y_0$). Since x_1 is also a cut vertex of G, by similar arguments as above, we get that there is an isolated vertex y_1 in $G-x_1$. In particular, $x_0y_1 \notin E(G)$.

If G consists of the four vertices x_0, y_0, x_1, y_1, then G is isomorphic to P_4. Note that $\pi(P_4) = 1$ and $s(P_4) = 1$. Thus we conclude that there are at least three vertices in $G - \{x_0, x_1\}$.

Note that y_0 and y_1 are both isolated vertices of $G - \{x_0, x_1\}$. This implies that $\omega(G-\{x_0, x_1\}) \geq 3$. By Claim 6, every component of $G-\{x_0, x_1\}$ is trivial.

Let $V(G-\{x_0, x_1\}) = \{y_0, y_1, y_2, y_3, \ldots, y_k\}$, where $k+1 = \omega(G-\{x_0, x_1\})$. If $y_ix_0 \notin E(G)$ for some i, $2 \leq i \leq k$, then the subgraph induced by $\{x_0, y_0, y_1, y_i\}$ is a $2K_1 \cup K_2$, a contradiction. Thus we have that $y_ix_0 \in E(G)$, and similarly $y_ix_1 \in E(G)$. Let $P = y_0x_0y_2x_1y_1$. Then $\mathcal{P} = P \cup \bigcup_{i=3}^{k} y_i$ is a path partition of G. This implies that $\pi(G) \leq k-1$. On the other hand,

$$s(G) \geq \omega(G - \{x_0, x_1\}) - |\{x_0, x_1\}| = k+1-2 = k-1.$$

Thus we have $\pi(G) \leq s(G)$, a contradiction.

Hence, we conclude that $G - y_0$ is 2-connected. This proves the first half of the second part of Claim 7. If $G - y_0$ is hamiltonian, then there is a Hamilton path P in $G - y_0$ starting from x_0. Thus $P' = y_0x_0P$ is a Hamilton path of G. This implies that $\pi(G) = 1$. Clearly $s(G) \geq 1$, so we get that $\pi(G) \leq s(G)$, a contradiction. □

If we are in the second case of Claim 7, we use x_0y_0 to denote the pendant edge such that $G - y_0$ is 2-connected. By Claim 7, G contains a cycle. Let C be a longest cycle of G.

Claim 13. *C is a dominating cycle (i.e., all edges of G have at least one end vertex on C).*

Proof. Let H be an arbitrary component of $G - V(C)$. If H is non-trivial, then H has at least two neighbors on C (note that if H is non-trivial, and if H contains y_0 for the pendant edge x_0y_0, then H will also contain x_0). Let y_1y_2 be an edge in H and let $x_1, x_2 \in N_C(H)$. By Lemma 1, $x_1^+, x_2^+ \notin N_C(H)$ and $x_1^+x_2^+ \notin E(G)$. Then the subgraph induced by $\{x_1^+, x_2^+, y_1, y_2\}$ is a $2K_1 \cup K_2$, a contradiction. □

Note that Claim 8 implies that if x_0y_0 is the pendant edge, then $x_0 \in V(C)$.

By Claims 7 and 8, there is an isolated vertex y_1 of $G - V(C)$ which is not incident with the pendant edge, so with $d(y_1) \geq 2$ neighbors on C. Let x_1 be a neighbor of y_1 on C. A subpath of C with two end vertices adjacent to y_1 and all internal vertices nonadjacent to y_1 is called a *simple y_1-segment* of C. Thus C is divided by $N(y_1)$ into $d(y_1)$ simple y_1-segments. By Lemma 1, all these segments have length at least 2. We assume an orientation on C and order all vertices according to this orientation. So, if we use indices for the vertices of the segments, they are increasing in accordance with the orientation.

Claim 14. *Let $P = z_0z_1z_2\cdots z_rz_{r+1}$ be a simple y_1-segment. Then r is odd, $x_1^+z_i \notin A^C(G)$ for every odd i, and $x_1^+z_j \in E(G)$ for every even j, $1 \le i, j \le r$.*

Proof. We first deal with the case that $x_1^{++} \in N(y_1)$, i.e., the length of the simple y_1-segment containing x_1^+ is 2. Then, by definition, the statement holds for this segment. We are going to verify that the statement holds for the segments P that do not contain x_1^+.

By Lemma 1, $x_1^+z_1, x_1^+z_r \notin A^C(G)$. Now we will prove that for any i, $1 \le i \le r-1$, if $x_1^+z_i \in E(G)$, then $x_1^+z_{i+1} \notin A^C(G)$; and if $x_1^+z_i \notin A^C(G)$, then $x_1^+z_{i+1} \in E(G)$. If $x_1^+z_i \in E(G)$ and $x_1^+z_{i+1} \in A^C(G)$, then let P' be a path from x_1^+ to z_{i+1} with all internal vertices disjoint from C. Then $C' = \overrightarrow{C}[x_1^{++}, z_i]z_ix_1^+P'z_{i+1}\overrightarrow{C}[z_{i+1}, x_1]x_1y_1x_1^{++}$ is a longer cycle than C, a contradiction. If $x_1^+z_i \notin A^C(G)$ and $x_1^+z_{i+1} \notin E(G)$, then the subgraph induced by $\{y_1, x_1^+, z_i, z_{i+1}\}$ is a $2K_1 \cup K_2$, a contradiction.

By using the above arguments repeatedly, we get that $x_1^+z_i \notin A^C(G)$ for every odd i, and $x_1^+z_j \in E(G)$ for every even j, $1 \le i, j \le r$. Since $x_1^+z_r \notin E(G)$, r is odd.

We next deal with the case that $x_1^{++} \notin N(y_1)$. We first assume that x_1^+ is not contained in the segment P. If P has length 2, then we are done by Lemma 1. So we assume that the length of P is at least 3.

By Lemma 1, $x_1^+z_1 \notin A^C(G)$. If $x_1^{++}z_1 \notin E(G)$, then the subgraph induced by $\{y_1, z_1, x_1^+, x_1^{++}\}$ is a $2K_1 \cup K_2$, a contradiction. So we conclude that $x_1^{++}z_1 \in E(G)$. Now we will prove that for any i, $1 \le i \le r-1$, if $x_1^+z_i \in E(G)$, then $x_1^+z_{i+1} \notin A^C(G)$; and if $x_1^+z_i \notin A^C(G)$, then $x_1^+z_{i+1} \in E(G)$. If $x_1^+z_i \in E(G)$ and $x_1^+z_{i+1} \in A^C(G)$, then $z_i \ne z_1$. Let P' be a path from x_1^+ to z_{i+1} with all internal vertices disjoint from C. Then $C' = \overrightarrow{C}[x_1^{++}, z_0]z_0y_1x_1\overleftarrow{C}[x_1, z_{i+1}]z_{i+1}P'x_1^+z_i\overleftarrow{C}[z_i, z_1]z_1x_1^{++}$ is a cycle longer than C, a contradiction. If $x_1^+z_i \notin A^C(G)$ and $x_1^+z_{i+1} \notin E(G)$, then the subgraph induced by $\{y_1, x_1^+, z_i, z_{i+1}\}$ is a $2K_1 \cup K_2$, a contradiction. Thus we get that $x_1^+z_i \notin A^C(G)$ for every odd i, and $x_1^+z_j \in E(G)$ for every even j, $1 \le i, j \le r$.

If $x_1^+z_r \in E(G)$, then $C' = \overrightarrow{C}[x_1^{++}, z_0]z_0y_1z_{r+1}\overrightarrow{C}[z_{r+1}, x_1^+]x_1^+z_r\overleftarrow{C}[z_r, z_1]z_1x_1^{++}$ is a cycle longer than C, a contradiction. This implies that $x_1^+z_r \notin E(G)$ and r is odd.

Finally we deal with the case that $x_1^+ \in V(P)$ (i.e., $z_1 = x_1^+$ and $z_2 = x_1^{++}$). Since we already have $x_1^+z_1 \notin A^C(G)$ and $x_1^+z_2 \in E(G)$, we will prove that for any i, $2 \le i \le r-1$, if $x_1^+z_i \in E(G)$, then $x_1^+z_{i+1} \notin A^C(G)$; and if $x_1^+z_i \notin A^C(G)$, then $x_1^+z_{i+1} \in E(G)$. Note that $x_1^+z_{r+1}^+ \notin A^C(G)$. If $x_1^{++}z_{r+1}^+ \notin E(G)$, then the subgraph induced by $\{y_1, z_{r+1}^+, x_1^+, x_1^{++}\}$ is a $2K_1 \cup K_2$, a contradiction. So we conclude that $x_1^{++}z_{r+1}^+ \in E(G)$. If $x_1^+z_i \in E(G)$ and $x_1^+z_{i+1} \in A^C(G)$, then let P' be a path from x_1^+ to z_{i+1} with all internal vertices disjoint from C. Then $C' = \overrightarrow{C}[x_1^{++}, z_i]z_ix_1^+P'z_{i+1}\overrightarrow{C}[z_{i+1}, z_{r+1}]z_{r+1}y_1x_1\overleftarrow{C}[x_1, z_{r+1}^+]z_{r+1}^+x_1^{++}$ is a cycle

longer than C, a contradiction. If $x_1^+z_i \notin A^C(G)$ and $x_1^+z_{i+1} \notin E(G)$, then the subgraph induced by $\{y_1, x_1^+, z_i, z_{i+1}\}$ is a $2K_1 \cup K_2$, a contradiction. Thus we get that $x_1^+z_i \notin A^C(G)$ for every odd i, and $x_1^+z_j \in E(G)$ for every even j, $1 \le i, j \le r$.

Now we will prove that $x_1^+z_r \notin E(G)$ (in order to show that the segment has an odd number of internal vertices). We assume that $x_1^+z_r \in E(G)$. If $z_{r+1}^{++} \in N(y_1)$, then $C' = \overrightarrow{C}[x_1^{++}, z_r]z_rx_1^+\overleftarrow{C}[x_1^+, z_{r+1}^{++}]z_{r+1}^{++}y_1z_{r+1}z_{r+1}^+x_1^{++}$ is a cycle longer than C, a contradiction. If $z_{r+1}^{++} \notin N(y_1)$, then $x_1^+z_{r+1}^{++} \in E(G)$ and $C' = \overrightarrow{C}[x_1^{++}, z_r]z_rx_1^+z_{r+1}^{++}\overrightarrow{C}[z_{r+1}^{++}, x_1]x_1y_1z_{r+1}z_{r+1}^+\ x_1^{++}$ is a cycle longer than C, a contradiction. $\square$

By Claim 9, the cycle C has an even length, and x_1^+ is not attached to at least one vertex, say z_1, of $V(C)\backslash N(y_1)$. If $x_1^+y_2 \in E(G)$ for some $y_2 \in V(G) \setminus V(C)$, then the subgraph induced by $\{y_1, z_1, x_1^+, y_2\}$ is a $2K_1 \cup K_2$. Thus we conclude that $N(x_1^+) \subset V(C)$.

Let $S = N(y_1) \cup N(x_1^+)$. Then S consists of $s = |V(C)|/2$ alternating vertices on C (i.e., no pair is adjacent on C, but pairs might be adjacent in G). Clearly S is a vertex cut and y_1, x_1^+ are two isolated vertices in $G - S$. This implies that $G - S$ has at least three components. By Claim 6, every component of $G - S$ is trivial.

Let $V(G) \setminus V(C) = \{y_1, y_2, \ldots, y_k\}$, where $k = \omega(G - V(C))$. Let $P' = y_1x_1\overrightarrow{C}[x_1, x_1^-]$. Then $\mathcal{P} = P' \cup \bigcup_{i=2}^k y_i$ is a path partition of G. This implies that $\pi(G) \le k$.

On the other hand, every vertex in $(V(G) \setminus V(C)) \cup (V(C) \setminus S)$ is a trivial component of $G - S$, which implies that

$$s(G) \ge \omega(G - S) - |S| = |S| + k - |S| = k.$$

Thus we have $\pi(G) \le s(G)$, our final contradiction, completing the proof of Theorem 4.

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