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# VERTEX COLORINGS WITHOUT RAINBOW SUBGRAPHS

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**Abstract**

Given a coloring of the vertices of a graph $G$, we say a subgraph is rainbow if its vertices receive distinct colors. For a graph $F$, we define the $F$-upper chromatic number of $G$ as the maximum number of colors that can be used to color the vertices of $G$ such that there is no rainbow copy of $F$. We present some results on this parameter for certain graph classes. The focus is on the case that $F$ is a star or triangle. For example, we show that the $K_3$-upper chromatic number of any maximal outerplanar graph on $n$ vertices is $\lfloor n/2 \rfloor + 1$.

**Keywords:** coloring, rainbow, monochromatic, forbidden, path.

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## 1. Introduction

Given a coloring of the vertices of a graph $G$, we say a subgraph is *rainbow* if its vertices receive distinct colors. For a graph $F$, we refer to a (not necessarily proper) vertex coloring of $G$ without rainbow copies of $F$ as a *no-rainbow-$F$ coloring* (*valid coloring* for short); we define the *$F$-upper chromatic number* of $G$ as the maximum number of colors that can be used in a valid coloring. We denote this maximum by $NR_F(G)$. A valid coloring is *optimal* if it uses exactly $NR_F(G)$ colors.

There are many papers on the edge-coloring version, where the parameter is called the *anti-Ramsey number*. Note that this parameter is also exactly 1 less than the *rainbow number*, which is the minimum number of colors such that every edge-coloring of $G$ with at least that many colors produces a rainbow $F$. For the edge-coloring case, most studied is the situation that $G$ is complete and

$F$ is a cycle, clique, tree, or matching. For example, a *Gallai-coloring* is an edge-coloring of the complete graph without a rainbow triangle [10, 11]. For a survey of anti-Ramsey theory, see [7].

In contrast, not much has been written about the vertex-coloring case. There are two papers on avoiding rainbow *induced* subgraphs: [1] and [12]. More recently, the special case where $F$ is $P_3$ was considered by Bujtás *et al.* [3] (under the name 3-*consecutive upper chromatic number*), and then the case where $F$ is $K_{1,k}$ was considered by Bujtás *et al.* [2] (under the name *star*-[$k$] *upper chromatic number*). Besides these, a related question that has been studied is coloring embedded graphs with no rainbow *faces*, see for example [5, 13].

Let us mention that there are also results on colorings that avoid both rainbow and monochromatic subgraphs, see for example [4, 8, 9]. Further, graph colorings without rainbow (monochromatic) copies of subgraph $F$ can be considered as hypergraph colorings without rainbow (monochromatic) hyperedges where the hyperedges are the copies of $F$. Thus they fall within the theory of mixed hypergraphs introduced by Voloshin [14].

In this paper, we investigate the $F$-upper chromatic number for certain graph classes. We proceed as follows. In Section 2 we present some basic observations. Then in Section 3 we consider the case that $F$ is a path on three vertices, in Section 4 the case that $F$ is a triangle, and in Section 5 the case that $F$ is the star $K_{1,r}$.

## 2. Preliminaries

Bujtás *et al.* [2] observed the following when $F$ is a star, but the results hold in general:

- For fixed $F$, the parameter is monotonic: if $H$ is a spanning subgraph of $G$, then $NR_F(G) \leq NR_F(H)$.
- If $F$ is connected and $G$ is disconnected, then $NR_F(G)$ is the sum of the $NR_F$'s of the components of $G$.
- The chromatic spectrum has no gaps: $G$ has a coloring without a rainbow $F$ using $k$ colors for $1 \leq k \leq NR_F(G)$. Simply take the optimal coloring and successively merge color classes.
- $NR_F(G) = |V(G)|$ if and only if $G$ is $F$-free.
- $NR_F(G) \geq |F| - 1$, provided $G$ has that many vertices.

For a natural lower bound, one can define an $F$-*bi-cover* of a graph as a set of vertices that contains at least two vertices from every copy of $F$. It follows that one can obtain a no-rainbow-$F$ coloring by giving all vertices in an $F$-bi-cover the same color and giving all other vertices unique colors. For example, if $G$ is a connected graph of order at least 3, then a $P_3$-bi-cover is the complement

of a packing. (A *packing* is a set of vertices at pairwise distance at least 3; the *packing number* $\rho(G)$ is the maximum size of a packing.) The lower bound $NR_{P_3}(G) \geq \rho(G) + 1$ follows. A vertex cover is an $F$-bi-cover for any connected non-star graph $F$. In this case we have $NR_F(G) \geq \beta(G)+1$ (where $\beta(G)$ denotes the independence number) provided $G$ is not empty.

A related idea can sometimes provide an upper bound. We say that a set $S$ *bi-covers* a subgraph $H$ if at least two vertices of $H$ are in $S$. For positive integer $s$, define $b_F(s)$ to be the maximum number of copies of $F$ that can be bi-covered by using a set of size $s$. Note that $b_F(1) = 0$.

**Proposition 1.** *Suppose that graph $G$ of order $n$ contains $f$ copies of $F$ and that $b_F(s) \leq a(s-1)$ for all $s$. Then $NR_F(G) \leq n - f/a$.*

***Proof.*** Consider a no-rainbow-$F$ coloring. Say one uses $k$ colors, being used $s_1, \ldots, s_k$ times respectively. Then $k = n - \sum_{i=1}^{k}(s_i - 1)$. Since every copy of $F$ has to be bi-covered by some color class, $\sum_{i=1}^{k} b_F(s_i) \geq f$. It follows that $k \leq n - f/a$. ■

## 3. FORBIDDEN $P_3$

The parameter $NR_{P_3}(G)$ can also be thought of as the maximum number of colors in a coloring such that each vertex sees at most one color other than its own.

### 3.1. Fundamentals

There are two natural lower bounds.

**Observation 2.** (a) *For a graph $G$, $NR_{P_3}(G) \geq \mathrm{diam}(G)/2 + 1$.*
(b) *For any nonempty graph $G$, $NR_{P_3}(G) \geq \rho(G) + 1$.*

***Proof.*** (a) Let $x$ be a vertex of eccentricity $\mathrm{diam}(G)$, and color each vertex $v$ by $\lceil d(x,v)/2 \rceil$, where $d(x,v)$ denotes the distance from $x$ to $v$.

(b) See the previous section. (Give every vertex in a maximum packing a unique color, and give all other vertices the same color.) ■

Bujtás *et al.* [3] showed a partial converse to the first bound.

**Proposition 3** [3]**.** *If $G$ has diameter 2, then $NR_{P_3}(G) = 2$.*

For example, it is well known that the random graph $G(n, \frac{1}{2})$ almost surely has diameter 2 and so $NR_{P_3}(G(n, \frac{1}{2})) = 2$ almost surely. Similarly, a rooks graph (the cartesian product of cliques) has diameter 2 and so it has $P_3$-upper chromatic number 2.

For an upper bound, one can consider a spanning tree, since the parameter is monotonic. Bujtás *et al.* [3] showed that there is a formula for the parameter on a tree $T$.

**Theorem 4** [3]**.** *For a tree $T$, it holds that $NR_{P_3}(T)$ is one more than the matching number of $T$.*

By taking a spanning tree of $G$ we get

**Corollary 5.** *For a connected graph $G$ of order $n$, $NR_{P_3}(G) \leq \lfloor n/2 \rfloor + 1$.*

Note that every other connected $F$ with at least three vertices yields a maximum of $n$, since one can construct a connected $F$-free graph $G$ with $n$ vertices (for example, either the star or the path or both). It is natural to ask what graphs achieve equality in Corollary 5.

The *corona* $\mathrm{cor}(G)$ of a graph $G$ is the graph obtained from $G$ by adding, for each vertex $v$ in $G$, a new vertex $v'$ and the edge $vv'$. The new vertices are called the *leaves* of the corona. Note that equality occurs in Theorem 4, Corollary 5 and Observation 2(b), for coronas.

**Observation 6.** *If $G$ is connected, then $NR_{P_3}(\mathrm{cor}(G)) = |G| + 1$.*

***Proof.*** The lower bound follows from Observation 2(b). That is, give all the leaves unique colors and give all the original vertices of $G$ the same color. The upper bound is from Corollary 5. ■

Note that it does not follow that the color classes must be connected in an optimal coloring. For example, Figure 1 gives a graph that has $NR_{P_3}(G) = 4$, which is uniquely attained by giving each white vertex a unique color and all black vertices the same color.

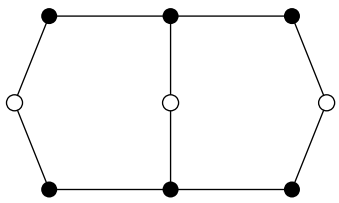

Figure 1. A graph whose optimal no-rainbow-$P_3$ coloring has a disconnected color class.

### 3.2. Complexity

We next show that calculating $NR_{P_3}(G)$ is NP-hard. We will need the following construction. For graph $G$, define graph $M(G)$ by adding, for every vertex $v$ in $G$, a new vertex $v'$ adjacent to $v$, and adding edges to make $C = \{ v' : v \in V(G) \}$ a clique.

**Observation 7.** *For any graph $G$ it holds that $NR_{P_3}(M(G)) = \rho(G) + 1$.*

***Proof.*** Note that $\rho(M(G)) = \rho(G)$. So the lower bound follows from Observation 2(b). To prove the upper bound, consider a coloring of $M(G)$ with no rainbow $P_3$. Note that the clique $C$ contains at most two colors. There are two cases.

First, consider that $C$ contains two colors. Note that for every vertex $v$ in $V(G)$, there is a vertex $w'$ such that $v'$ and $w'$ receive different colors. It follows that $v$ receives one of the two colors in $C$. That is, the coloring uses two colors.

Second, consider that $C$ contains only one color, say red. Let $v$ and $w$ be vertices of $V(G)$ such that neither is red and they have different colors. Then they cannot be adjacent, since that would make $vww'$ rainbow, nor can they have a common neighbor $x$, since $x$ would see three colors. It follows that if we take one vertex of each non-red color, we obtain a packing. That is, the number of non-red colors is at most $\rho(G)$, as required. ■

As a consequence it follows that computing $NR_{P_3}(G)$ is NP-hard, since computing the packing number is NP-hard.

In contrast, Bujtás *et al.* [3] showed that determining whether a graph $G$ has $NR_{P_3}(G) = 3$ or $NR_{P_3}(G) = 4$ is solvable in polynomial time.

### 3.3. Graph families and operations

#### 3.3.1. Clones

In general, if $v$ and $w$ have the same neighbors (themselves excluded), then $NR_F(G) \geq NR_F(G - v)$, since one can take any coloring of $G - v$ and give $v$ the same color as $w$. But we have equality for $F = P_3$.

**Observation 8.** *Assume vertices $v$ and $w$ are such that $N(v) \setminus \{w\} = N(w) \setminus \{v\} \neq \emptyset$. Then $NR_{P_3}(G) = NR_{P_3}(G - v)$.*

***Proof.*** Consider a valid coloring of $G$. Let $x$ be any common neighbor of $v$ and $w$. If $v$ and $w$ have different colors, then $x$ must have the same color as one of them. If $x$ has the same color as $v$, then the coloring restricted to $G - v$ is a valid coloring with every color of $G$. If $x$ has the same color as $w$, then the coloring restricted to $G - w$ is a valid coloring with every color of $G$. Note that $G - w = G - v$ and so the conclusion follows. ■

#### 3.3.2. Maximal outerplanar graphs

We now consider avoiding rainbow $P_3$ in maximal outerplanar graphs. The minimum value of $NR_{P_3}(G)$ for an outerplanar graph of order $n$ is obtained by the fan (having value 2). The maximum value for a maximal outerplanar graph of order $n$ is given by the following.

**Theorem 9.** *The maximum value of $NR_{P_3}(G)$ for a maximal outerplanar graph $G$ of order $n \geq 3$ is $\lfloor n/3 \rfloor + 1$.*

***Proof.*** We prove the lower bound by the following construction: start with a cycle $v_1v_2 \cdots v_nv_1$. For $1 \leq i \leq \lfloor n/3 \rfloor$, assign $v_{3i}$ a distinct color. Then use one additional color for all the remaining vertices, and add edges between them until we have a maximal outerplanar graph $G$. Clearly, exactly $\lfloor n/3 \rfloor + 1$ colors are used and there is no rainbow $P_3$.

We prove the upper bound by induction on $n$. It suffices to show that $NR_{P_3}(G) \leq n/3 + 1$. It is easy to verify the result for $n = 3$. For larger $n$, the outer cycle of $G$ has a chord.

*Case* 1. There is a chord, say $uv$, with different colors on its ends. Say the removal of $\{u, v\}$ from $G$ yields components with vertex sets $V_1$ and $V_2$. Let $G_i$ be the subgraph of $G$ induced by the vertices $V_i \cup \{u, v\}$. Note that $G_i$ is a maximal outerplanar graph. By the induction hypothesis, $G_i$ has at most $|G_i|/3+1$ colors. But $G_1$ and $G_2$ share two colors. So the total number of colors in $G$ is at most $(|V_1| + 2)/3 + 1 + (|V_2| + 2)/3 + 1 - 2 = (n + 2)/3 < n/3 + 1$.

*Case* 2. Every chord is monochromatic. Since the chords induce a connected subgraph of $G$, it follows that all the vertices with degree at least 3 in $G$ have the same color, say red. Let $X$ be the set consisting of one vertex of each remaining color.

Since the vertices with degree 2 are independent, it follows that $X$ is independent. Further, vertices $x_1$ and $x_2$ of $X$ cannot have a common neighbor, since that vertex would be red and we would have a rainbow $P_3$. It follows that $|X| \leq \rho(C_n)$, and so the total number of colors in $G$ is at most $\lfloor n/3 \rfloor + 1$. ■

Note that there are maximal outerplanar graphs where $NR_{P_3}(G) \neq \rho(G) + 1$.

### 3.3.3. Cubic graphs

We consider now avoiding rainbow $P_3$ in cubic graphs. It is unclear what the minimum and maximum values are. Here is computer data.

| order | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|
| min | 2 | 2 | 2 | 2 | 3 | 3 | 4 | 4 |
| max | 2 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |

For the minimum, computer search shows that, for $n \leq 18$, the minimum value of the parameter is one more than the minimum value of the packing number. However, it is unclear what the asymptotics of the packing number are. Favaron [6] showed that $\rho(G) \geq n/8$ for a cubic graph $G$ of order $n$ other than

the Petersen graph, but it is unclear if this bound is sharp in general. Furthermore, it is unclear under what circumstances a graph has parameter equal to the packing number lower bound.

We consider next the maximum value of the parameter for cubic graphs of order $n$.

**Theorem 10.** *For any connected cubic graph $G$ on $n \geq 6$ vertices, $NR_{P_3}(G) \leq 2n/5$.*

***Proof.*** We extend a counting idea suggested in [2]. For a color $c$, define $\mathrm{CN}(c)$ as the number of closed neighborhoods that $c$ is in. (Equivalently, the number of vertices dominated by a vertex of color $c$.) Let $A = \sum_c \mathrm{CN}(c)$; that is, $A$ is the number of pairs $(c, v)$ where $c$ is a color that occurs in $N[v]$. The requirement of no rainbow $P_3$ means that each closed neighborhood has at most 2 colors in it, and so $A \leq 2n$. To prove the theorem, it suffices to show that the average value of $\mathrm{CN}(c)$ is at least 5.

Since $G$ is cubic, it is immediate that $\mathrm{CN}(c) \geq 4$ for all colors $c$. Call a color $c$ *sparse* if $\mathrm{CN}(c) = 4$. Say vertex $v$ has color $c$. Then all other vertices with color $c$, if any, must be neighbors $u$ of $v$ such that $N[u] = N[v]$. Since the graph is not $K_4$, it follows that there are at most two vertices with color $c$. The remaining neighbors of $v$ (which are also the neighbors of the other vertex of color $c$, if any) must be the same color, say $c'$.

We claim that $\mathrm{CN}(c') \geq 6$. By connectivity there is a vertex $w$ that is not in $N[v]$ but has a neighbor $x$ in $N[v]$. Since $N[w]$ does not contain color $c$ but $vxw$ is a $P_3$, it follows that $x$ and $w$ are both color $c'$. Since $n \geq 6$, there must be a vertex that is not in $N[v] \cup \{w\}$ and is adjacent to a vertex of color $c'$ in $N[v] \cup \{w\}$. So $\mathrm{CN}(c') \geq 6$. In particular, $\mathrm{CN}(c) + \mathrm{CN}(c') \geq 10$.

Now, suppose that the same color surrounds multiple sparse colors. Say, we have $c_1, \ldots, c_b$ such that $c_1' = \cdots = c_b' = d$. Then we claim that $\mathrm{CN}(d) \geq 4b$. This follows by noting that the $N[v_i]$ are disjoint if $v_i$ has color $c_i$, and all of $N[v_i]$ is dominated by a vertex of color $d$. It follows that $\mathrm{CN}(d) + \sum_i \mathrm{CN}(c_i) \geq 8b \geq 5(b+1)$, since $b \geq 2$.

So, by partitioning the sparse colors into sets based on the surrounding color, it follows that the average value of $\mathrm{CN}(c)$ is at least 5, whence the result. ■

The computer data verifies that the maximum value is $\lfloor 2n/5 \rfloor$ for $6 \leq n \leq 18$. However, the bound in Theorem 10 might not be sharp in general. Let $H$ be the graph of order 5 obtained from $K_4$ by subdividing one edge. Let $I_0$ be built from two copies of $H$ by adding an edge joining the vertices of degree 2. Computer confirms that for $n = 10$ this is the unique extremal graph. In general, let graph $I_j$ be the cubic graph built from two copies of $H$ by adding $j$ copies of $K_4 - e$ between the copies of $H$. The graph $I_1$ is extremal for $n = 14$, but not unique. Similarly

$I_2$ is extremal for $n = 18$. But there is one other extremal graph: take three copies of $H$ and one copy of $K_3$ and add edges to make a connected cubic graph, see Figure 2. It can be checked that $NR_{P_3}(I_j)$ is $3n/8 + O(1)$. It is unclear if this is best possible.

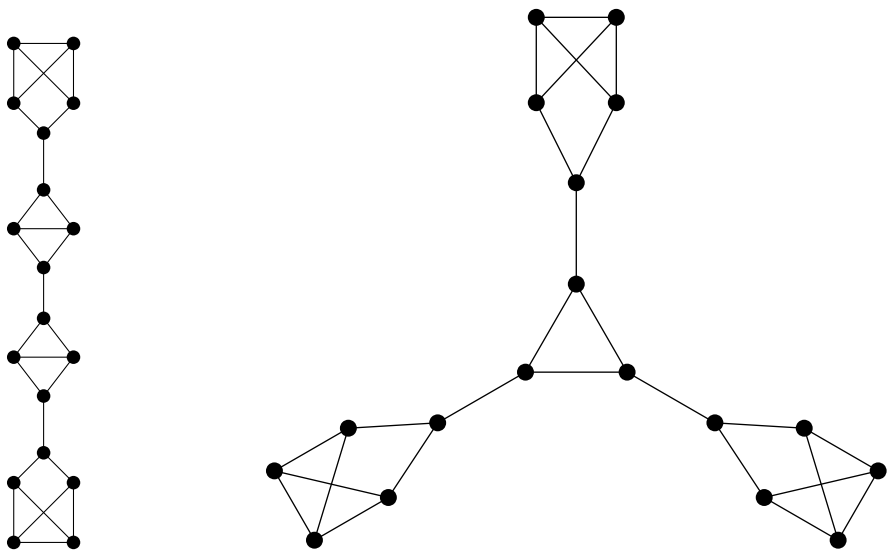

Figure 2. The two cubic graphs of order 18 with maximum $NR_{P_3}$.

## 4. Forbidden Triangles

We consider now colorings that forbid a rainbow copy of the other connected graph on three vertices, a triangle. That is, we consider colorings where every triangle has a monochromatic edge.

We saw earlier that $NR_{K_3}(G) \geq \beta(G) + 1$, provided $G$ is nonempty. In particular, we note that if every edge of the graph is in a triangle, then a subset $S$ is a $K_3$-bi-cover if and only if $S$ is a vertex cover. Note that (in contrast with Gallai colorings), when $R$ is complete the optimal coloring of a graph has every color class connected. (For suppose color red is disconnected; then change the vertices in one red component to a new color pink. There cannot be a red vertex and a pink vertex together in a clique, since the pink vertex and red vertex were not adjacent.)

One can again investigate the minimum and maximum values of the parameter for graphs of fixed order in particular classes. For example, the extremal values of $NR_{K_3}(G)$ for cubic graphs $G$ of order $n$ are straightforward. The maximum is $n$, achieved by a triangle-free graph. The minimum is $2n/3$, achieved by a cubic graph with $n/3$ disjoint triangles.

### 4.1. Maximal outerplanar graphs

Perhaps surprisingly, the value of $NR_{K_3}(G)$ for a maximal outerplanar graph $G$ of fixed order does not depend on the structure of $G$.

**Theorem 11.** *Let $G$ be a maximal outerplanar graph of order $n$. Then it holds that $NR_{K_3}(G) = \lfloor n/2 \rfloor + 1$.*

***Proof.*** We prove the upper bound $n/2+1$ by induction on $n$. It is easy to verify the result for $n = 3$. For larger $n$, the outer cycle of $G$ has a chord.

The first case is that there is a chord, say $uv$, with different colors on its ends. Say the removal of $\{u, v\}$ from $G$ yields components with vertex sets $V_1$ and $V_2$. Let $G_i$ be the subgraph of $G$ induced by the vertices $V_i \cup \{u, v\}$. Note that each $G_i$ is a maximal outerplanar graph. But $G_1$ and $G_2$ share two colors. So, by the induction hypothesis, the total number of colors in $G$ is at most $(|V_1| + 2)/2 + 1 + (|V_2| + 2)/2 + 1 - 2 = n/2 + 1$.

The second case is that every chord is monochromatic. Since the chords induce a connected subgraph of $G$, it follows that all the vertices with degree at least 3 in $G$ have the same color, say red. Since the vertices with degree 2 are independent, it follows that the number of colors in $G$ is at most $n/2 + 1$.

We prove the lower bound by induction. The result is true for $n \leq 4$; so assume $n \geq 5$. Note that the weak dual of $G$ is a tree $T$ of order $n - 2$ and maximum degree at most 3. Let $b$ be a penultimate vertex on a longest path in $T$. There are two cases.

The first case is that $b$ has degree 2, with leaf neighbor $a$. Say $b$ lies in triangle $xyz$ of $G$ and $a$ in triangle $xyu$, with vertex $y$ of degree 3. Then let $G' = G - \{u, y\}$. It is maximal outerplanar and consider a valid coloring $\phi$ of $G'$. Then $\phi$ can be extended to a valid coloring of $G$ by giving $u$ a new color and giving $y$ the same color as $x$. The lower bound follows by induction.

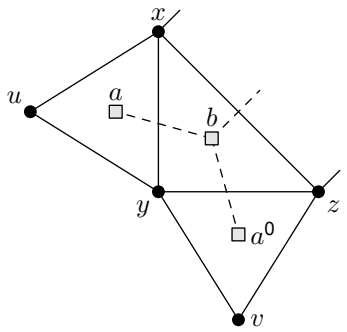


Figure 3. Part of a maximal outerplanar graph and its weak dual.

The second case is that $b$ has degree 3, with leaf neighbors $a$ and $a'$. (See Figure 3.) Say vertex $b$ lies in triangle $xyz$ of $G$, vertex $a$ in triangle $xyu$ and vertex $a'$ in triangle $yzv$. Then let $G' = G - \{u, v\}$. It is maximal outerplanar and consider any valid coloring $\phi$ of $G'$. We need to show how to introduce one new color. If vertex $y$ has the same color as either $x$ or $z$, then this is immediate. So assume $y$ has a different color to both $x$ and $z$. Since triangle $xyz$ is not rainbow, this means that $x$ and $z$ have the same color. Then one can proceed by recoloring $y$ to be the same color as $x$ and $z$, and then giving both $u$ and $v$ unique colors. It follows that $NR_{K_3}(G) \geq \lfloor n/2 \rfloor + 1$, as required. ■

Note that the above result does not extend to 2-trees.

### 4.2. Rooks graphs

Define $R_m$ as the rooks graph given by the cartesian product $K_m \square K_m$. The following is probably known.

**Observation 12.** *Consider a coloring of the rooks graph $R_m$ such that every row and column contains at most two colors. Then the number of colors used is at most* $\max\{4, m+1\}$.

***Proof.*** Suppose first that there is both a row and a column that are monochromatic, say red. If red does not appear elsewhere, then the rest of the graph is monochromatic, while if red does appear elsewhere, the bound follows by induction. So we may assume that every row say contains exactly two colors; say row $i$ has $A_i$ and $B_i$ for $1 \leq i \leq m$.

Suppose two rows, say $i$ and $j$, have disjoint colors. Then every column contains one color of $\{A_i, B_i\}$ and one of $\{A_j, B_j\}$ and thus the total number of colors used is at most 4. So we may assume that every pair of rows share a color. If we construct an auxiliary graph $H$ with the colors as nodes and join two nodes if they are together in some row, then this means that every pair of edges in $H$ share an end-node. Thus, $H$ is either a star or a triangle. The former means there is one color that occurs in every row, which means at most $m+1$ colors total; and the latter means at most three colors total. ■

**Theorem 13.**

$$NR_{K_3}(R_m) = \begin{cases} 4, & \text{if } m = 2, \\ m+1, & \text{if } m \geq 3. \end{cases}$$

***Proof.*** For $m = 2$, the rooks graph has no triangle, whence the result. In general, $m+1$ is a lower bound by the independence number bound. The upper bound follows from Observation 12. ■

### 4.3. Complexity

It is straightforward to show that the parameter is NP-hard. For example, one can reduce from the independence number as follows.

**Observation 14.** *Consider the graph $G'$ obtained by adding one new vertex adjacent to all vertices of $G$. Then $NR_{K_3}(G') = \beta(G) + 1$.*

***Proof.*** The lower bound follows as before. For the upper bound, say the dominating vertex has color red. Then for every other color, choose one vertex; let $S$ be the resultant set. Then $S$ is independent, and so $|S| \leq \beta(G)$. ■

## 5. FORBIDDEN STARS

We consider here the star $K_{1,r}$. The parameter $NR_{K_{1,r}}(G)$ is equal to the maximum number of colors in a coloring such that each vertex sees at most $r-1$ colors other than its own. This parameter was studied by Bujtás *et al.* [2]. They showed that

**Theorem 15** [2]**.** (a) *For a graph $G$ of order $n$ and minimum degree $\delta$, it holds that $NR_{K_{1,r}}(G) \leq nr/(\delta+1)$.*

(b) *For a graph $G$ of order $n$ and vertex cover number $\alpha_0$, it holds that $NR_{K_{1,r}}(G) \leq 1 + (r-1)\alpha_0$.*

(c) *For a graph $G$ of domination number $\gamma$, it holds that $NR_{K_{1,r}}(G) \leq r\gamma$.*

As an example of a specific result, it was shown in [8] that $NR_{K_{1,r}}(G) = 2(r-1)$ for the complete bipartite graph $G = K_{m,m}$ when $m \geq r \geq 2$.

Bujtás *et al.* [2] ask: when is $NR_{K_{1,r}}(G) = r$? They showed that $G$ having diameter at most 2 is necessary (e.g. for stars) but not sufficient.

### 5.1. Trees

We show first that Theorem 4 generalizes to all stars. Indeed, it is true for any forbidden rainbow subgraph.

**Theorem 16.** *For a tree $T$ and any connected graph $F$, $NR_F(T)$ equals 1 more than the maximum number of edges in an $F$-free subgraph of $T$.*

***Proof.*** Let $H$ be an arbitrary $F$-free subgraph of $T$. Let $T'$ be the spanning subgraph of $T$ with the edges of $H$ removed. By giving each component of $T'$ a different color, we get a valid coloring of $T$: every rainbow subgraph of $T$ is a subgraph of $H$. Since the number of colors used equals 1 more than the number of edges in $H$ and $H$ is arbitrary, $NR_F(T)$ is at least 1 more than the maximum number of edges in an $F$-free subgraph of $T$.

Conversely, take an optimal coloring of $T$. Let $B$ be the set of edges whose ends have different colors. Consider an edge $e = uv$ in $B$; say $u$ is red and $v$ is blue. If red appears in the component of $T - e$ containing $v$, recolor all red vertices in that component with color blue. (Note that this does not decrease the number of colors.) If this recoloring increases the number of colors in some copy of $F$, then that copy must now contain both a red vertex and a newly-blue vertex; but by connectivity that means it must also contain $v$, and thus is not rainbow. That is, we may assume that all color classes are connected. Then the set $B$ induces an $F$-free subgraph of $T$. And the number of colors in $T$ equals the number of components of $T - B$, which is $|B| + 1$. It follows that $NR_F(T)$ is at most 1 more than the maximum number of edges in an $F$-free subgraph of $T$. ■

### 5.2. Specific results for forbidden $K_{1,3}$

#### 5.2.1. Rooks graphs

For a rooks graph, there is a connection between forbidding stars and cliques.

**Observation 17.** *For any rooks graph $R_m$, it holds that*

$$NR_{K_{1,r}}(R_m) \leq \max\{r, NR_{K_r}(R_m)\}.$$

***Proof.*** The requirement for no rainbow $K_{1,r}$ is that every vertex sees at most $r$ colors including its own. Suppose some row of $R_m$ contains $r$ colors. Then every vertex in that row sees the same $r$ colors. That is, the graph is $r$-colored. So we may assume that each row and column contains at most $r - 1$ colors. That is, there is no rainbow $K_r$. ■

It follows that

**Theorem 18.**

$$NR_{K_{1,3}}(R_m) = \begin{cases} 4, & \text{if } m = 2, \\ m + 1, & \text{if } m \geq 3. \end{cases}$$

***Proof.*** For $m = 2$, the rooks graph has no $K_{1,3}$ nor $K_3$, whence the result. In general, $m + 1$ is a lower bound by giving each vertex on one diagonal a unique color and coloring all other vertices the same. The upper bound follows from Theorem 13 and Observation 17. ■

#### 5.2.2. Cubic graphs

We consider here the question of forbidding rainbow $K_{1,3}$ in cubic graphs. Let $G$ be a cubic graph. By Theorem 15(a) it holds that $NR_{K_{1,3}}(G) \leq 3n/4$, where $n$ is the order. As observed in [8], equality can be obtained by taking disjoint

copies of $K_4 - e$ and adding edges to make a connected cubic graph. Here is the resulting graph.

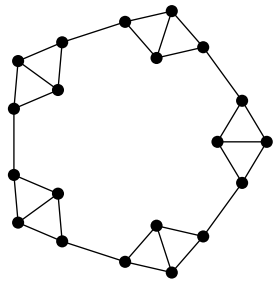

Figure 4. The cubic graph of order 20 with maximum $NR_{K_{1,3}}$.

For $n < 10$ it can readily be shown that the minimum value of $NR_{K_{1,3}}(G)$ of order $n$ is $n/2 + 1$. But we conjecture the following.

**Conjecture 19.** *For $n \geq 10$, the minimum value of $NR_{K_{1,3}}(G)$ over all cubic graphs $G$ of order $n$ is $n/2$.*

Computer data confirms this conjecture for $n \leq 14$. Note that if a cubic graph $G$ has a matching, then indeed $NR_{K_{1.3}}(G) \geq n/2$, by giving, for each edge of the matching, the two matched vertices the same color. If the conjecture is true, then the prism is an extremal graph when $n$ is not a multiple of 4, as we now show.

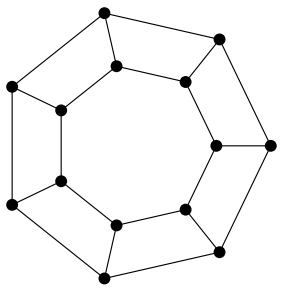

Figure 5. A cubic graph of order 14 with minimum $NR_{K_{1,3}}$.

We will need the following observations about bi-covers.

**Observation 20.** *For the ladder $P_m \square K_2$, it holds that $b_{K_{1,3}}(s) \leq 2(s-1)$.*

***Proof.*** The result is by induction. Think of the $P_m$ as the rows and the $K_2$ as columns. Consider a set $S$ of vertices. If $S$ is contained within one of the columns, then the result is immediate. So assume $S$ includes vertices from at least two columns.

Let $S'$ be the vertices of $S$ in the leftmost column that $S$ occupies. Then by going through the cases, one can check that the number of copies of $K_{1,3}$ that are bi-covered by $S$ but not by $S \backslash S'$, is at most $2|S'|$. The bound follows by induction. ■

**Observation 21.** *For the prism $C_m \square K_2$, it holds that $b_{K_{1,3}}(s) \leq 2(s-1)$ provided $s < 2m/3$. Further if $m$ is odd, then $b_{K_{1,3}}(s) \leq 2s-1$ for all $s$.*

***Proof.*** Consider a set $S$ of $s$ vertices. If there are two consecutive $K_2$-fibers without a vertex of $S$, then the result follows from Observation 20. Further, if there are three consecutive $K_2$-fibers with only one vertex of $S$ between them, say $v$, then we can remove vertex $v$, apply the above observation, and noting that $v$ can contribute to the bi-cover of at most 2 copies, again obtain the result. So we may assume that every three consecutive $K_2$-fibers contain at least two vertices of $S$; in particular, $s \geq 2m/3$.

Now, note that since the graph is cubic, every vertex of $S$ lies in exactly 4 copies of $K_{1,3}$. So it is immediate that $b_{K_{1,3}}(s) \leq 4s/2 = 2s$. So suppose that $S$ bi-covers exactly $2s$ copies of $K_{1,3}$. Then by the calculation, it must be that $S$ covers each of the $2s$ copies exactly twice, and covers no other copy at all.

In particular, consider a vertex $v$ in $S$. Since $v$ dominates itself, one of its neighbors must be in $S$, say $w$. There are two cases. If $vw$ is a $K_2$-fiber, then since neither $v$ not $w$ is triple dominated, it follows that neither adjacent $K_2$-fiber contains a vertex of $S$. But since both these fibers are dominated, it follows that in the next $K_2$-fibers, both vertices are in $S$. By repeated application of this, it follows that every alternate fiber contains two vertices of $S$. This is only possible if $m$ is even.

The second case is that $vw$ lies within a $C_m$-fiber. Then by similar reasoning, no other vertex in $N(v) \cup N(w)$ is in $S$. But they dominate the other vertex of their $K_2$-fibers. So in the two adjacent $K_2$-fibers, the vertex not in $N(v) \cup N(w)$ is in $S$. Since that vertex is doubly dominated by $S$, it follows that its neighbor outside $N(v) \cup N(w)$ is in $S$. By repeated application of this, it follows that $S$ consists of one vertex from each $K_2$-fiber and that $S$ induces a matching. This is only possible if $m$ is a multiple of four. ■

**Theorem 22.** *For $m \geq 3$,*

$$NR_{K_{1,3}}(C_m \square K_2) = \begin{cases} m+1, & \text{if } m \text{ is even or } m = 3, \\ m, & \text{if } m \text{ is odd and } m \geq 5. \end{cases}$$

***Proof.*** For $C_3 \square K_2$, color two vertices in one triangle red and two vertices in another triangle green, and then give the other two vertices unique colors. For $C_m \square K_2$ when $m$ is even, color every alternate $K_2$-fiber red and then give the

remaining $m$ vertices unique colors. For $C_m \square K_2$ when $m$ is odd, give each $K_2$-fiber a different color.

It remains to prove the upper bound. The upper bound for $m = 3$ is straightforward; so assume $m > 3$.

We note first that if it were true that $b_{K_{1,3}}(s) \leq 2(s-1)$, then an upper bound of $m$ would follow from Proposition 1. Indeed, by the proof of that proposition, that bound follows provided every color class bi-covers at most $2(c-1)$ copies of $K_{1,3}$, where $c$ is the number of times that color is used.

So assume some color, say red used $c$ times, bi-covers more than $2(c-1)$ copies of $K_{1,3}$. By Observation 21, red is used at least $2m/3$ times. If there was another such color, then the total number of colors would be at most $2m - 2(2m/3) + 2 = 2m/3 + 2$, which is less than $m + 1$ (since $m > 3$), and the result follows. So we may assume that red is the only such color.

Assume first that $m$ is even. Then by the argument in Observation 21, it holds that red bi-covers at most $2c$ copies of $K_{1,3}$. By repeating the proof of Proposition 1, it follows that at most $m + 1$ colors are used, and so the result follows. If $m$ is odd, then by Observation 21, it holds that red bi-covers at most $2c - 1$ copies of $K_{1,3}$. By repeating the proof of Proposition 1, it follows that at most $m + 1/2$ colors are used, and so by integrality the result follows. ■

### 5.2.3. Maximal outerplanar graphs

The minimum value of $NR_{K_{1,3}}(G)$ for an outerplanar graph of order $n$ is obtained by the fan (having value 3). For the maximum, we need to restrict to maximal outerplanar graphs. It is unclear what the maximum value is. Here is computer data:

| order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| max | 3 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 9 |

Figure 6 shows the unique graphs achieving the maximum for $n = 7$ and $n = 9$. The black vertices are all colored the same and each white vertex gets a unique color.

However, while this data suggests that the maximum is $n/2 + O(1)$, that is not correct. It is possible to construct maximal outerplanar graphs where $NR_{K_{1,3}} = 4n/7 + 1$. Let $s \geq 2$. Start with a cycle $\mathcal{C}$ of $3s$ vertices and partition the vertex set into copies of $P_3$. For each copy $abc$, introduce vertices $d$, $e$, $f$, and $g$, and add edges $ad$, $ed$, $ae$, $be$, $bf$, $fg$, $cf$, and $cg$. Finally, add edges incident with the cycle $\mathcal{C}$ to make it a triangulation. Let $M_s$ denote the resultant graph. For example, $M_3$ is shown in Figure 7. The graph $M_s$ can be colored by giving all the vertices on the cycle $\mathcal{C}$ the same color, and all other vertices unique colors. This shows that $NR_{K_{1,3}}(M_s) \geq 4|M_s|/7 + 1$. We omit the details here but it can be verified that $NR_{K_{1,3}}(M_s) = 4|M_s|/7 + 1$.

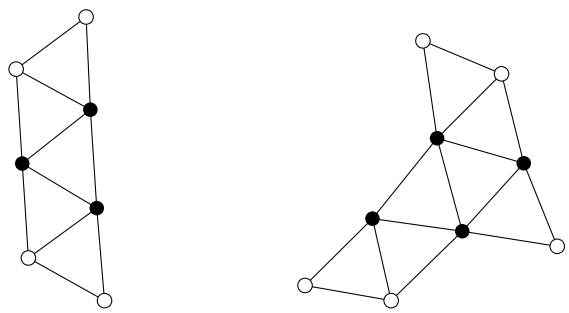

Figure 6. Maximal outerplanar graphs with maximum $NR_{K_{1,3}}$.

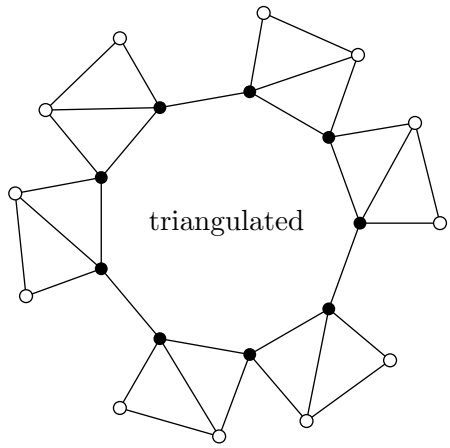


Figure 7. The maximal outerplanar graph $M_3$ with $NR_{K_{1,3}} = 13$.

## 6. Conclusion

We have considered colorings without rainbow stars or cliques. Besides the specific open problems and conjectures presented here, a future direction of research would be colorings without other rainbow subgraphs, say trees, cycles, or bicliques. One avenue that looks interesting is coloring grids and other products while forbidding rainbow subgraphs.

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