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NICHE HYPERGRAPHS

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Abstract

If $D = (V, A)$ is a digraph, its *niche hypergraph* $N\mathcal{H}(D) = (V, \mathcal{E})$ has the edge set $\mathcal{E} = \{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v \in V : e = N_D^-(v) \vee e = N_D^+(v)\}$. Niche hypergraphs generalize the well-known niche graphs (see [11]) and are closely related to competition hypergraphs (see [40]) as well as double competition hypergraphs (see [33]). We present several properties of niche hypergraphs of acyclic digraphs.

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1. Introduction and Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered in the following may have isolated vertices but no multiple edges; loops are allowed only in digraphs. With $N_D^-(v)$, $N_D^+(v)$, $d_D^-(v)$ and $d_D^+(v)$ we denote the in-neighborhood, out-neighborhood, in-degree and out-degree of $v \in V(D)$, respectively. In standard terminology we follow Bang-Jensen and Gutin [4].

In 1968, Cohen [12] introduced the *competition graph* $C(D) = (V, E)$ of a digraph $D = (V, A)$ representing a food web of an ecosystem. Here the vertices correspond to the species and different vertices are connected by an edge if and only if they compete for a common prey, i.e.,

$$E = E(C(D)) = \{\{v_1, v_2\} \mid v_1 \neq v_2 \wedge \exists\, w \in V : v_1 \in N_D^-(w) \wedge v_2 \in N_D^-(w)\}.$$

Surveys of the large literature around competition graphs (and its variants) can be found in [17, 26]; for (a selection of) recent results see [13, 20, 22, 23, 24, 27, 29, 30].

Meanwhile the following variants of $C(D)$ are investigated: The *common-enemy graph* $CE(D)$ (see [26]) of D is the competition graph of the digraph obtained by reversing all the arcs of D, that is,

$$E(CE(D)) = \{\{v_1, v_2\} \mid v_1 \neq v_2 \wedge \exists\, w \in V : v_1 \in N_D^+(w) \wedge v_2 \in N_D^+(w)\},$$

the *double competition graph* or *competition common-enemy graph* $DC(D)$ (see [18, 25, 34, 39, 44]) is defined by $E(DC(D)) = E(C(D)) \cap E(CE(D))$ and the *niche graph* $N(D)$ (see [1, 2, 3, 6, 7, 8, 9, 10, 11, 14, 15, 16, 19, 32, 36, 37, 38]) is defined by $E(N(D)) = E(C(D)) \cup E(CE(D))$.

In 2004, the concept of competition hypergraphs was introduced by Sonntag and Teichert [40]. The *competition hypergraph* $C\mathcal{H}(D)$ of a digraph $D = (V, A)$ has the vertex set V and the edge set

$$\mathcal{E}(C\mathcal{H}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v \in V : e = N_D^-(v)\}.$$

Clearly, for many digraphs this hypergraph concept includes considerably more information than the competition graph. For further investigations see [21, 28, 31, 35, 40, 41, 42, 43]. As a second hypergraph generalization, recently Park and Sano [33] investigated the *double competition hypergraph* $DC\mathcal{H}(D)$ of a digraph $D = (V, A)$, which has the vertex set V and the edge set

$$\mathcal{E}(DC\mathcal{H}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v_1, v_2 \in V : e = N_D^+(v_1) \cap N_D^-(v_2)\}.$$

Our paper is a third step in this direction; we consider the *niche hypergraph* $N\mathcal{H}(D)$ of a digraph $D = (V, A)$, again with the vertex set V and the edge set

$$\mathcal{E}(N\mathcal{H}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v \in V : e = N_D^-(v) \vee e = N_D^+(v)\}.$$

Figure 1 illustrates the three types of hypergraphs, $C\mathcal{H}(D)$, $DC\mathcal{H}(D)$ and $N\mathcal{H}(D)$.

Let I_k denote a set of k isolated vertices. Cable *et al.* [11] defined the *niche number*, $\hat{n}_g(G)$, of an undirected graph G as the smallest number of isolated vertices k such that $G \cup I_k$ is the niche graph of an acyclic digraph. Analogously we consider the *niche number* $\hat{n}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ as the smallest $k \in \mathbb{N}$ such that $\mathcal{H} \cup I_k$ is the niche hypergraph of an acyclic digraph D; if no such $k \in \mathbb{N}$ exists, we define $\hat{n}(\mathcal{H}) = \infty$. Each graph G can be considered as a 2-uniform hypergraph, i.e., both $\hat{n}_g(G)$ and $\hat{n}(G)$ are well defined; later we will see that sometimes $\hat{n}_g(G) \neq \hat{n}(G)$ is possible.

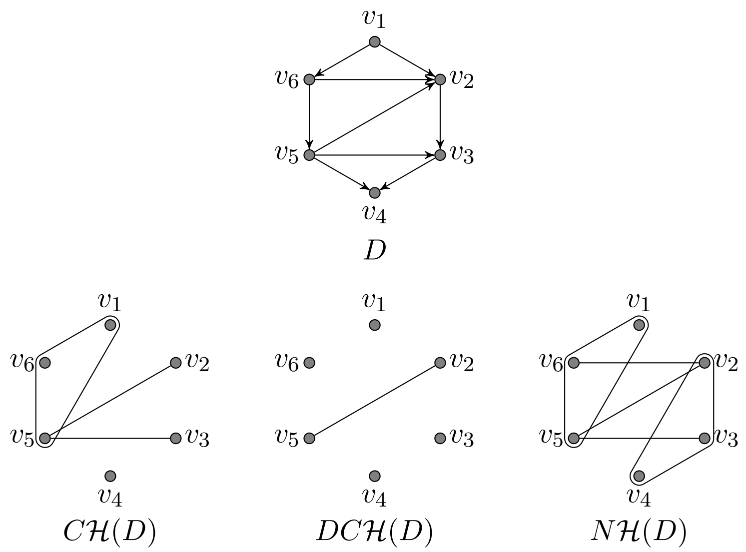

Figure 1. A digraph D and the hypergraphs $C\mathcal{H}(D)$, $DC\mathcal{H}(D)$ and $N\mathcal{H}(D)$.

Having a look at the numerous results for niche graphs and the niche numbers of graphs in the following we prove several properties of niche hypergraphs and the niche numbers of hypergraphs. Note that in most of the results the generating digraph D of $N\mathcal{H}(D)$ is assumed to be acyclic.

2. Tools

If $M = (m_{ij})$ denotes the adjacency matrix of the digraph $D = (V, A)$, then the *row hypergraph* $Ro\mathcal{H}(M)$ has the vertex set $V = \{v_1, \ldots, v_n\}$ and the edge set

$$\mathcal{E}(Ro\mathcal{H}(M)) = \{\{v_{i_1}, \ldots, v_{i_k}\} \mid k \geq 2 \wedge \exists\, j \in \{1, \ldots, n\}\, \forall\, i \in \{1, \ldots, n\} : m_{ij} = 1 \leftrightarrow i \in \{i_1, \ldots, i_k\}\}.$$

This notion implies that the competition hypergraph $C\mathcal{H}(D)$ is the row hypergraph $Ro\mathcal{H}(M)$ (see [40]). If we consider the *column hypergraph* $Co\mathcal{H}(M)$ having again the vertex set V and the edge set

$$\mathcal{E}(Co\mathcal{H}(M)) = \{\{v_{j_1}, \ldots, v_{j_l}\} \mid l \geq 2 \wedge \exists\, i \in \{1, \ldots, n\}\, \forall\, j \in \{1, \ldots, n\} : m_{ij} = 1 \leftrightarrow j \in \{j_1, \ldots, j_l\}\}$$

Lemma 1 immediately follows.

Lemma 1. *Let $D = (V, A)$ be a digraph with adjacency matrix M. Then the niche hypergraph $N\mathcal{H}(D)$ is the union of the row hypergraph $Ro\mathcal{H}(M)$ and the column hypergraph $Co\mathcal{H}(M)$, that is, $V(N\mathcal{H}(D)) = V(Ro\mathcal{H}(M)) = V(Co\mathcal{H}(M)) = V$ and*

$$\mathcal{E}(N\mathcal{H}(D)) = \mathcal{E}(Ro\mathcal{H}(M)) \cup \mathcal{E}(Co\mathcal{H}(M)).$$

Further, we remember a well-known property of acyclic digraphs (see [4]).

Lemma 2. *A digraph D is acyclic if and only if its vertices can be labeled such that the adjacency matrix of D is strictly lower triangular (i.e., D has an acyclic ordering).*

Analogously to Bowser and Cable [7], we say that an acyclic digraph D is *niche minimal* for a hypergraph $\mathcal{H}$ with $\hat{n}(\mathcal{H}) < \infty$ if $N\mathcal{H}(D) = \mathcal{H} \cup I_k$ and $\hat{n}(\mathcal{H}) = k$.

Lemma 3. *Let $\mathcal{H}$ be a hypergraph with $\hat{n}(\mathcal{H}) = k < \infty$ and D an acyclic digraph with $N\mathcal{H}(D) = \mathcal{H} \cup I_k$. Then for all $v \in V(\mathcal{H}) \cup I_k$,*

- *$N^+_D(v) \cap I_k \neq \emptyset$ implies $N^+_D(v) = \{w\}$ for some $w \in I_k$, and*
- *$N^-_D(v) \cap I_k \neq \emptyset$ implies $N^-_D(v) = \{w\}$ for some $w \in I_k$.*

Proof. Assume there is a vertex $v \in V(\mathcal{H}) \cup I_k$ with either $N^+_D(v) = \{w\} \cup V^*$ or $N^-_D(v) = \{w\} \cup V^*$, where $w \in I_k$ and $\emptyset \neq V^* \subseteq V(\mathcal{H}) \cup (I_k \setminus \{w\})$. Then w is adjacent in $\mathcal{H} \cup I_k$ to all vertices of V^*, a contradiction to $w \in I_k$. ∎

The following is the hypergraph version of a result of Bowser and Cable [7] (Lemma 2.1).

Lemma 4. *Let $\mathcal{H}$ be a hypergraph with $\hat{n}(\mathcal{H}) = k < \infty$ and D an acyclic digraph with $N\mathcal{H}(D) = \mathcal{H} \cup I_k$. Then for all $w \in I_k$, $N^-_D(w) = \emptyset$ or $N^+_D(w) = \emptyset$.*

Proof. Assume $k \geq 1$ and there is a vertex $w \in I_k$ with $N^+_D(w) = \{v^+_1, \ldots, v^+_s\} \neq \emptyset$ and $N^-_D(w) = \{v^-_1, \ldots, v^-_t\} \neq \emptyset$. Then $N^-_D(w) \cap N^+_D(w) = \emptyset$ because D is acyclic. Lemma 3 yields $N^-_D(v^+_i) = N^+_D(v^-_j) = \{w\}$ for $i = 1, \ldots, s$, $j = 1, \ldots, t$. Now consider the digraph D' with $V(D') = V(D) \setminus \{w\}$ and $A(D') = (A(D) \setminus A_1) \cup A_2$, where

$$A_1 = \{(w, v^+_i) \mid i \in \{1, \ldots, s\}\} \cup \{(v^-_j, w) \mid j \in \{1, \ldots, t\}\} \text{ and}$$
$$A_2 = \{(v^-_1, v^+_i) \mid i \in \{1, \ldots, s\}\} \cup \{(v^-_j, v^+_1) \mid j \in \{1, \ldots, t\}\}.$$

We obtain $N^-_D(w) = N^-_{D'}(v^+_1)$, $N^+_D(w) = N^+_{D'}(v^-_1)$ and hence $N\mathcal{H}(D') = \mathcal{H} \cup I_{k-1}$, a contradiction to $\hat{n}(\mathcal{H}) = k$. ∎

If e is an edge of $N\mathcal{H}(D)$ and v is a vertex of D such that $N_D^-(v) = e$ or $N_D^+(v) = e$, we call v a *generating vertex of* e. Particularly, if e is generated by the in-neighborhood (out-neighborhood) of v, i.e., e belongs to $Ro\mathcal{H}(M)$ ($Co\mathcal{H}(M)$) we denote e as an α-edge e^α (β-*edge* e^β). For the maximum and minimum edge cardinality in a hypergraph $\mathcal{H}$ we write $\bar{d}(\mathcal{H})$ and $\underline{d}(\mathcal{H})$, respectively; $\bar{d}^\alpha(\mathcal{H})(\bar{d}^\beta(\mathcal{H}))$ denotes the maximum cardinality of an α-edge (β-edge). Analogously $d_\mathcal{H}(v)$ ($d^\alpha_\mathcal{H}(v)$,$d^\beta_\mathcal{H}(v)$) is the degree (α-degree, β-degree) of $v \in V(\mathcal{H})$, hence $d_\mathcal{H}(v) \leq d^\alpha_\mathcal{H}(v) + d^\beta_\mathcal{H}(v)$ (note that $<$ appears in this inequality if there exists an edge being both an α-edge and a β-edge). With $\mathcal{H}(v)$ we denote the subhypergraph of $\mathcal{H}$ containing all hyperedges incident to v, i.e., $\mathcal{H}(v)$ has the edge set $\mathcal{E}(\mathcal{H}(v)) = \{e \mid e \in \mathcal{E}(\mathcal{H}) \wedge v \in e\}$.

Lemma 5. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\hat{n}(\mathcal{H}) = k < \infty$ and D an acyclic digraph with $N\mathcal{H}(D) = \mathcal{H} \cup I_k$. Then for every $v \in V$ the following holds.*

(i) *If $d^\alpha_\mathcal{H}(v) = b \geq 2$ ($d^\beta_\mathcal{H}(v) = b \geq 2$), then there is a β-edge (an α-edge) $\hat{e} \in \mathcal{E}$ consisting of at least the b generating vertices for the α-edges (β-edges) adjacent to v and $v \notin \hat{e}$.*

(ii) $d^\alpha_\mathcal{H}(v) \leq \bar{d}^\beta(\mathcal{H})$, $d^\beta_\mathcal{H}(v) \leq \bar{d}^\alpha(\mathcal{H})$.

(iii) $d_\mathcal{H}(v) \leq \bar{d}^\alpha(\mathcal{H}) + \bar{d}^\beta(\mathcal{H}) \leq 2\bar{d}(\mathcal{H})$.

Proof. (i) Let $e^\alpha_1, \ldots, e^\alpha_b \in \mathcal{E}$ be the pairwise distinct α-edges with $v \in e^\alpha_i$, $i = 1, \ldots, b$. Then v belongs in D to the in-neighborhoods of at least b pairwise distinct vertices of $V \backslash \{v\}$ generating $e^\alpha_1, \ldots, e^\alpha_b \in \mathcal{E}$. Hence $|N_D^+(v)| \geq b$ and v generates a β-edge e^β with $|e^\beta| \geq b$. The acyclicity of D yields $v \notin e^\beta$. The result in parentheses follows analogously and the statements (ii) and (iii) are direct conclusions of statement (i). ∎

3. Structural Properties

Characterizations of competition hypergraphs and double competition hypergraphs of acyclic digraphs can be found in [40] and [33], respectively. In the following we give a necessary condition for a hypergraph $\mathcal{H}$ to be a niche hypergraph of an acyclic digraph.

Theorem 6. *If a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with n vertices is a niche hypergraph of an acyclic digraph $D = (V, A)$, then its vertices can be labeled $v_1, \ldots, v_n$ and there is a partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ with $\mathcal{E}_1 = \{e_1, \ldots, e_s\}$, $\mathcal{E}_2 = \{e'_{n-t+1}, \ldots, e'_n\}$ such that*

$$\begin{aligned} &\forall\, i \in \{1, \ldots, n\}\ \forall\, j \in \{1, \ldots, s\}\ \forall\, k \in \{n-t+1, \ldots, n\} : \\ &\qquad (v_i \in e_j \to i > j) \wedge (v_i \in e'_k \to i < k). \end{aligned} \tag{1}$$

Proof. Suppose $\mathcal{H} = N\mathcal{H}(D)$ for some acyclic digraph D. By Lemma 2 there is a vertex labeling $v_1, \ldots, v_n$ of $V(D)$ which generates a strictly lower triangular adjacency matrix M of D. From Lemma 1 it follows $N\mathcal{H}(D) = Ro\mathcal{H}(M) \cup Co\mathcal{H}(M)$. Further the α-edges $e_1, \ldots, e_s$ (β-edges $e'_{n-t+1}, \ldots, e'_n$) of $\mathcal{H}$ correspond to columns $1 \le a_1 < \cdots < a_s < n-1$ (rows $2 < b_1 < \cdots < b_t \le n$) and thus we have

$$(v_i \in e_j \to i > a_j \ge j) \quad \text{and} \quad (v_i \in e'_k \to i < b_k \le k). \qquad \blacksquare$$

To show that condition (1) in Theorem 6 is not sufficient, we consider a hypergraph $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{\mathcal{E}})$ with $|\tilde{V}| = n \ge 4$, $\tilde{\mathcal{E}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ such that $\tilde{e}_1 = \tilde{V} \backslash \{x\}$, $\tilde{e}_2 = \tilde{V} \backslash \{y\}$ with $x, y \in \tilde{V}$, $x \ne y$. The postulate that $\tilde{\mathcal{H}} = N\mathcal{H}(\tilde{D})$ for some acyclic digraph $\tilde{D}$ fulfills (1) implies the following facts.

For each acyclic ordering $(v_1, \ldots, v_n)$ of $\tilde{V}$ it follows that $\{x, y\} = \{v_1, v_n\}$; without loss of generality let $x = v_1$ and $y = v_n$. Further we obtain for the adjacency matrix $\tilde{M} = (\tilde{m}_{ij})$ of $\tilde{D}$ the values $\tilde{m}_{i,1} = 1$ for $i = 2, \ldots, n$ and $\tilde{m}_{n,j} = 1$ for $j = 1, \ldots, n-1$. Using the notations of Theorem 6, the edges $e_1 = \tilde{e}_1$, $e'_n = \tilde{e}_2$ correspond to the first column and the last row in $\tilde{M}$, respectively. The existence of $\tilde{e}_3$ implies $\tilde{m}_{ij} = 1$ for some $1 < j < i < n$. This yields the existence of $\tilde{e}', \tilde{e}'' \in \tilde{\mathcal{E}}$ such that $e_2 = \tilde{e}'$ is an α-edge with $\{v_i, v_n\} \subseteq \tilde{e}'$ and $e'_{n-1} = \tilde{e}''$ is a β-edge with $\{v_1, v_j\} \subseteq \tilde{e}''$, $\tilde{e}' \ne \tilde{e}''$ and $\{\tilde{e}_1, \tilde{e}_2\} \cap \{\tilde{e}', \tilde{e}''\} = \emptyset$, a contradiction to $|\tilde{\mathcal{E}}| = 3$.

Lemma 7. *Let $\mathcal{H} = N\mathcal{H}(D)$ be the niche hypergraph of an acyclic digraph D with n vertices. Then we have*

$$|\mathcal{E}(\mathcal{H})| \le 2n - 4, \tag{2}$$

$$\bar{d}(\mathcal{H}) \le n - 1, \tag{3}$$

$$\sum_{e \in \mathcal{E}(\mathcal{H})} |e| \le n(n-1) - 2. \tag{4}$$

Proof. The boundaries are direct conclusions from Lemmata 1 and 2. Note that equalities hold in (2)–(4) if D is the transitive tournament. $\blacksquare$

Cable *et al.* [11] proved for graphs with finite niche number the following theorem.

Theorem 8 [11]. *If G is K_{m+1}-free and $\hat{n}_g(G) = k < \infty$, then G has maximum degree at most $2m(m-1)$.*

Taking into consideration that edges in niche hypergraphs correspond to cliques in niche graphs, we obtain a similar result for hypergraphs.

Theorem 9. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\bar{d}(\mathcal{H}) = p$ and $\hat{n}(\mathcal{H}) = k < \infty$. Then we have for every $v \in V$*

$$d_{\mathcal{H}}(v) \le 2p \ \wedge \ |V(\mathcal{H}(v))| \le 2p(p-1) + 1. \tag{5}$$

Proof. Let $\mathcal{H} \cup I_k$ be the niche hypergraph of an acyclic digraph D. From $\bar{d}(\mathcal{H}) = p$ it follows $d^+_D(v) \leq p$ and $d^-_D(v) \leq p$ for each $v \in V$ and therefore $d^\alpha_{\mathcal{H}}(v) \leq p$ and $d^\beta_{\mathcal{H}}(v) \leq p$. Thus we have $d_{\mathcal{H}}(v) \leq d^\alpha_{\mathcal{H}}(v) + d^\beta_{\mathcal{H}}(v) \leq 2p$.

Furthermore there are at most $2p$ edges in $\mathcal{H}$ containing a fixed vertex and each of these edges contains at most $p-1$ vertices different from v; this yields the second part of (5). ■

This theorem yields strong restrictions for graphs $\tilde{G}$ that (considered as 2-uniform hypergraphs) have a finite niche number $\hat{n}(\tilde{G})$. For those $\tilde{G}$, from $\bar{d}(\tilde{G}) = 2$ it follows that $d_{\tilde{G}}(v) \leq 4$ for every vertex v of $\tilde{G}$.

Corollary 10. *For the complete graph $\mathcal{H} = K_n$ and the wheel $\mathcal{H} = W_n$ with $n \geq 6$ vertices it holds $\hat{n}(\mathcal{H}) = \infty$.*

Now we consider the niche number of the disjoint union of graphs and hypergraphs, respectively. In this context Bowser and Cable [7] proved the following theorem.

Theorem 11 [7]**.** *If $G_1, \ldots, G_r$ are graphs such that $\hat{n}_g(G_i) \leq 2$ for $1 \leq i \leq r$ and G is the disjoint union of these graphs, then $\hat{n}_g(G) \leq 2$.*

The disjoint union $\mathcal{H}$ of hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_r$ with $\hat{n}(\mathcal{H}_i) = k_i < \infty$ for $1 \leq i \leq r$ has a finite niche number $\hat{n}(\mathcal{H}) = k < \infty$, too. This follows from the more detailed result for $r = 2$, which generalizes Theorem 11.

Theorem 12. *Let $\mathcal{H}_1, \mathcal{H}_2$ be hypergraphs with $\hat{n}(\mathcal{H}_i) = k_i < \infty$ for $i = 1, 2$ and $\tilde{k} = \min\{k_1, k_2\}$. Then for the disjoint union $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ it holds*

$$\hat{n}(\mathcal{H}) \leq \begin{cases} k_1 & \text{if } k_2 = 0, \\ k_2 & \text{if } k_1 = 0, \\ k_1 + k_2 - \lceil \frac{\tilde{k}}{2} \rceil - 1 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. For $i = 1, 2$ let D_i be niche minimal acyclic digraphs with $N\mathcal{H}(D_i) = \mathcal{H}_i \cup I_{k_i}$ and $I_{k_1} \cap I_{k_2} = \emptyset$. Observe that by Lemma 1 for every digraph D it holds $N\mathcal{H}(D) = N\mathcal{H}(D^T)$, where D^T arises from D by reversing the directions of all arcs.

If $k_1 = 0$ or $k_2 = 0$, we obtain (6) immediately; now assume $k_1 \geq 1$ and $k_2 \geq 1$. For $i = 1, 2$ let A^+_i and A^-_i contain those vertices $w_i \in I_{k_i}$ with $N^+_{D_i}(w_i) = \emptyset$ and $N^-_{D_i}(w_i) = \emptyset$, respectively. Then Lemma 4 yields $I_{k_i} = A^+_i \cup A^-_i$ for $i = 1, 2$. Now assume (maybe by considering D^T_i instead of D_i) that $|A^+_1| \geq |A^-_1|$ and $|A^+_2| \leq |A^-_2|$. Then A^+_1 and A^-_2 contain at least $t = \lceil \frac{\tilde{k}}{2} \rceil \geq 1$ vertices $\{w^+_1, \ldots, w^+_t\}$ and $\{w^-_1, \ldots, w^-_t\}$, respectively. Let D' be the digraph which arises from the disjoint union $D_1 \cup D_2$ by identifying the vertices w^+_i and w^-_i for $i = 1, \ldots, t$.

Then $N\mathcal{H}(D') = \mathcal{H}_1 \cup \mathcal{H}_2 \cup I_{k_1+k_2-t}$. There are $t \geq 1$ vertices $w_1, \ldots, w_t \in I_{k_1+k_2-t}$ with $N^+_{D'}(w_i) \neq \emptyset$ and $N^-_{D'}(w_i) \neq \emptyset$. Hence, by Lemma 4, D' is not niche minimal ($\hat{n}(\mathcal{H}) < k_1 + k_2 - t = k_1 + k_2 - \lceil \frac{\tilde{k}}{2} \rceil$) and this yields (6). ∎

Our next result shows that the niche number does not increase if we add vertices belonging to exactly one edge.

Lemma 13. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\hat{n}(\mathcal{H}) = k < \infty$. Let $\mathcal{H}' = (V', \mathcal{E}')$ arise from $\mathcal{H}$ by adding a vertex $v' \notin V$ to exactly one edge $e \in \mathcal{E}$, i.e., $V' = V \cup \{v'\}$ and $\mathcal{E}' = (\mathcal{E} \backslash \{e\}) \cup \{e \cup \{v'\}\}$. Then we have $\hat{n}(\mathcal{H}') \leq \hat{n}(\mathcal{H})$.*

Proof. Let $D = (V, A)$ be a generating acyclic digraph such that $N\mathcal{H}(D) = \mathcal{H} \cup I_k$ and $e \in \mathcal{E}$.

First, consider the case that e is either an α-edge or a β-edge. Without loss of generality let e be an α-edge (otherwise take D^T). Further let $v \in V \cup I_k$ be a vertex with $e = N^-_D(v)$ and consider $D' = (V', A')$ with $V' = V \cup \{v'\}$ and $A' = A \cup \{(v', v)\}$. Then $N\mathcal{H}(D') = \mathcal{H}' \cup I_k$, i.e., $\hat{n}(\mathcal{H}') \leq k$.

Secondly, let e be (additionally) a β-edge with generating vertex $\tilde{v} \in (V \cup I_k) \backslash \{v\}$. Then we have to add (additionally) the arc $(\tilde{v}, v')$ in D'. In this case v' can be placed between $\tilde{v}$ and the first vertex of e in the acyclic ordering of D, and D' is acyclic too.

Note that D' can be constructed similarly if more than one vertex generates the α-edge e (the β-edge e). ∎

Note that for the hypergraph $\mathcal{H}'$ in Lemma 13 sometimes $\hat{n}(\mathcal{H}') < \hat{n}(\mathcal{H})$ may be possible. This case appears if there is a vertex $\hat{v} \in I_k$, $\hat{v} \neq v'$ which generates an α-edge $e^\alpha \neq e$ in $\mathcal{H}$. Because of $N^-_{D'}(v') = \emptyset$ this edge e^α can be generated by v' in $\mathcal{H}'$ (if this operation does not produce cycles in the resulting digraph). Then $\hat{v}$ can be deleted in I_k.

Cable *et al.* [11] asked for a general upper bound for the niche number $\hat{n}_g(G)$ of a graph G and proved

$$\hat{n}_g(G) \leq |V(G)|. \tag{7}$$

Two years later Bowser and Cable [7] improved this result and showed

$$\hat{n}_g(G) \leq \frac{2}{3}|V(G)|. \tag{8}$$

It seems to be difficult to find a hypergraph result corresponding to (8), but the next theorem is a generalization of (7) for hypergraphs.

Theorem 14. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\hat{n}(\mathcal{H}) = k < \infty$. Then we have*

$$\hat{n}(\mathcal{H}) \leq \frac{2}{\underline{d}(\mathcal{H})}|V|. \tag{9}$$

Proof. Let $D = (V \cup I_k, A)$ be an acyclic digraph such that $N\mathcal{H}(D) = \mathcal{H} \cup I_k$. Clearly, (9) is true for $k = 0$. Assume $k \geq 1$ and consider an arbitrary $w \in I_k$. Because w generates exactly one edge of $\mathcal{H}$ (see Lemma 4) we have $d^+_D(w) \geq \underline{d}(\mathcal{H})$ or $d^-_D(w) \geq \underline{d}(\mathcal{H})$. Hence we obtain for the set S of all arcs of A connecting vertices of V to vertices of I_k the bound $|S| \geq k\,\underline{d}(\mathcal{H})$. On the other hand Lemma 3 yields $|S| \leq 2|V|$ and both inequalities imply (9). ∎

4. The Niche Number for Special Classes of Hypergraphs

The complete graph K_n is an example for distinct niche numbers $\hat{n}_g$ and $\hat{n}$. Cable *et al.* [11] proved

$$\hat{n}_g(K_n) = 1 \text{ for } n \geq 2. \tag{10}$$

Considering K_n as a hypergraph, we obtain the following result.

Theorem 15. *For $\mathcal{H} = K_n$ we have*

$$\hat{n}(K_n) = \infty \textit{ for } n \geq 3. \tag{11}$$

Proof. For $n \geq 6$ the result follows from Corollary 10. Now assume that for $n \in \{3, 4, 5\}$ there is a $k \in \mathbb{N}$ such that $K_n \cup I_k = N\mathcal{H}(D)$ for some acyclic digraph D.

First we consider the case $\mathcal{H} = K_5$. Because of $\bar{d}^\alpha(\mathcal{H}) = \bar{d}^\beta(\mathcal{H}) = 2$, by Lemma 5 we obtain $d^\alpha_\mathcal{H}(v) = d^\beta_\mathcal{H}(v) = 2$ for every $v \in V(\mathcal{H})$. Now Lemma 3 implies that no edge of $\mathcal{H} = K_5$ can be generated by an isolated vertex from I_k, i.e., $k = 0$ and the ten generating vertices belong to $V(\mathcal{H})$. Hence D must be the tournament T_5 with $d^+_{T_5}(v) = d^-_{T_5}(v) = 2$ (for all $v \in V$), which is not acyclic.

For $\mathcal{H} = K_4$ we obtain by Lemma 5 two possible cases for the distribution of α-edges and β-edges in K_4.

(i) There are three α-edges and three β-edges in K_4. Obviously, for each α-edge (β-edge) e_i we find another α-edge (β-edge) $e_j \neq e_i$ with $e_i \cap e_j \neq \emptyset$. Again by Lemma 3 we obtain that none of the α-edges (β-edges) can be generated by an isolated vertex, i.e., D has four vertices and the in-degree (out-degree) of three vertices is two. This leads necessarily to cycles of length two, i.e., D is not acyclic.

(ii) Without loss of generality, there are four α-edges and two β-edges in K_4. Because the α-edges form a cycle, none of them can be generated by an isolated vertex. This results in a subgraph of D with the vertex set $V(K_4)$ where each vertex has in-degree two. Hence D is no acyclic digraph D with $N\mathcal{H}(D) = K_4 \cup I_k$.

For $\mathcal{H} = K_3$ there are two edges of the same type, say two α-edges $\{v_1, v_2\}$, $\{v_1, v_3\}$. Then Lemma 5(i) yields that $\{v_2, v_3\}$ is a β-edge containing the generating vertices of $\{v_1, v_2\}$ and $\{v_1, v_3\}$, which is not possible for an acyclic digraph D. ∎

Note that Theorem 15 does not imply that K_n is a forbidden subgraph for niche hypergraphs (see Figure 2).

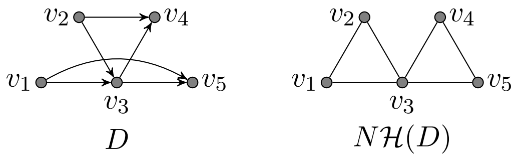

Figure 2. A niche hypergraph containing two copies of K_3.

Cable *et al.* [11] considered the following (infinite) class of graphs with infinite niche number $\hat{n}_g$: A *nova* arises from a star $K_{1,m}$, $m \geq 3$ by replacing each edge e_i by a clique Cl_i with at least two vertices such that all these cliques have exactly one vertex in common.

A *hypernova* is a hypergraph obtained from a nova by replacing each clique Cl_i by a hyperedge $\tilde{e}_i$ with the same vertex set: $\tilde{e}_i = V(Cl_i)$. Corresponding to the result mentioned above, we obtain the following theorem.

Theorem 16. *If $\mathcal{H} = (V, \mathcal{E})$ is a hypernova, then $\hat{n}(\mathcal{H}) = \infty$.*

Proof. Let $z \in V$ be the central vertex in $\mathcal{H}$. Because of $|\mathcal{E}| \geq 3$ it follows $d^{\alpha}_{\mathcal{H}}(z) \geq 2$ or $d^{\beta}_{\mathcal{H}}(z) \geq 2$. Now assume $\hat{n}(\mathcal{H}) = k < \infty$; then Lemma 5(i) yields the existence of an edge $e \in \mathcal{E}$ not containing z, a contradiction. ■

Next we consider paths; for graphs P_n the following result is known.

Theorem 17 [11]**.** *If P_n is a path with n vertices, then $\hat{n}_g(P_n) = 0$ for $n \geq 3$ and $\hat{n}_g(P_2) = 1$.*

As a generalization of paths, *linear hyperpaths* $\mathcal{P}_m$ with $m \geq 1$ edges (which were first introduced as *chains* by Berge [5]) are defined as follows:

$$\begin{aligned} V(\mathcal{P}_m) &= \bigcup_{i=1}^{m} \{v_1^i, \ldots, v_{d_i}^i\} \text{ and } \mathcal{E}(\mathcal{P}_m) = \{e_1, \ldots, e_m\}, \text{ where } |e_i| = d_i \geq 2 \\ &\text{and } e_i = \left\{v_1^i, \ldots, v_{d_i-1}^i, v_{d_i}^i = v_1^{i+1}\right\} \text{ for } i = 1, \ldots, m-1 \qquad (12) \\ &\text{as well as } e_m = \{v_1^m, \ldots, v_{d_m}^m\}. \end{aligned}$$

Theorem 18. *If $\mathcal{P}_m = (V, \mathcal{E})$ is a linear hyperpath, then $\hat{n}(\mathcal{P}_1) = 1$ and $\hat{n}(\mathcal{P}_m) = 0$ for $m \geq 2$.*

Proof. We use the notations from (12). For $m = 1$ consider the acyclic digraph $D = (V, A)$ with $V = \{w, v_1^1, \ldots, v_{d_1}^1\}$ and $A = \{(w, v_j^1) \mid j \in \{1, \ldots, d_1\}\}$. Then $N\mathcal{H}(D) = \mathcal{P}_1 \cup \{w\}$ and therefore $\hat{n}(\mathcal{P}_1) \leq 1$. The assumption $\hat{n}(\mathcal{P}_1) = 0$

cannot be true because in the generating acyclic digraph of $\mathcal{P}_1$ the existence of a generating vertex $w \notin e_1$ for the only edge e_1 is necessary.

Next we construct for even $m \geq 2$ an acyclic digraph D with $N\mathcal{H}(D) = \mathcal{P}_m$. The subdigraph $D_1 = (V_1, A_1)$ with $V_1 = e_1 \cup e_2$ and $A_1 = \{(i, v^2_{d_2}) \mid i \in e_1\} \cup \{(v^1_1, j) \mid j \in \{v^2_1, \ldots, v^2_{d_2-1}\}\}$ generates $\mathcal{P}_2$ (with the α-edge e_1 and the β-edge e_2). We repeat the procedure and construct subdigraphs $D_3, D_5, \ldots, D_{m-1}$ generating the hyperedges e_3 and e_4, e_5 and $e_6, \ldots, e_{m-1}$ and e_m, respectively. The union of these subdigraphs yields the wanted acyclic digraph D which generates $\mathcal{P}_m$.

For $m \geq 2$, m odd we construct the edges $e_1, \ldots, e_{m-1}$ as described above; again let $D = (V, A)$ be the resulting digraph of this procedure. Then the acyclic digraph $D' = (V', A')$ with $V' = V \cup \{v^m_2, \ldots, v^m_{d_m}\}$ and $A' = A \cup \{(v^{m-1}_1, k) \mid k \in e_m\}$ generates $\mathcal{P}_m$, i.e., $N\mathcal{H}(D') = \mathcal{P}_m$. ■

The situation becomes more complicated for cycles C_n; Cable *et al.* [11] showed

$$\hat{n}_g(C_n) = \begin{cases} 0 & \text{for } n = 7, n \geq 9, \\ 1 & \text{for } n = 3, 8, \\ 2 & \text{for } n = 4, 5, 6. \end{cases} \tag{13}$$

If we suppose $m \geq 2$ and identify the vertices $v^1_1 = v^m_{d_m}$ in (12), we obtain as a generalization of C_n the *linear hypercycle* $\mathcal{C}_m$ (introduced by Berge [5] as the *cycle* of length m). We can show that $\hat{n}(\mathcal{C}_m) = 0$ for $m = 3$ and $\bar{d}(\mathcal{C}_3) \geq 3$, $m = 4$ with at least two edges e_i,e_j with $|e_i| \geq 3$ and $|e_j| \geq 3$, $m = 7$ and $m \geq 9$ by (laboriously) constructing the corresponding generating digraphs D. However, these partial results are unsatisfactory but they lead to the following conjecture.

Conjecture 19. *If $\mathcal{C}_m$ is a linear hypercycle, then $\hat{n}(\mathcal{C}_m) = 0$ for $\underline{d}(\mathcal{C}_m) \geq 3$.*

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