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K_3-WORM COLORINGS OF GRAPHS: LOWER CHROMATIC NUMBER AND GAPS IN THE CHROMATIC SPECTRUM

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Abstract

A K_3-WORM coloring of a graph G is an assignment of colors to the vertices in such a way that the vertices of each K_3-subgraph of G get precisely two colors. We study graphs G which admit at least one such coloring. We disprove a conjecture of Goddard *et al.* [Congr. Numer. 219 (2014) 161–173] by proving that for every integer $k \geq 3$ there exists a K_3-WORM-colorable graph in which the minimum number of colors is exactly k. There also exist K_3-WORM colorable graphs which have a K_3-WORM coloring with two colors and also with k colors but no coloring with any of $3, \ldots, k-1$ colors. We also prove that it is NP-hard to determine the minimum number of colors, and NP-complete to decide k-colorability for every $k \geq 2$ (and remains intractable even for graphs of maximum degree 9 if $k = 3$). On the other hand, we prove positive results for d-degenerate graphs with small d, also including planar graphs.

Keywords: WORM coloring, lower chromatic number, feasible set, gap in the chromatic spectrum.

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1. Introduction

In a vertex-colored graph, a subgraph is *monochromatic* if its vertices have the same color, and it is *rainbow* if its vertices have pairwise different colors. Given two graphs F and G, an F-*WORM coloring* of G is an assignment of colors to its vertices such that no subgraph isomorphic to F is monochromatic or rainbow.

This notion was introduced recently in [9] by Goddard, Wash, and Xu. As noted in [9], however, for some types of F some earlier results due to Bujtás *et al.* [3, 4] imply upper bounds on the possible number of colors in F-WORM colorings of graphs G. The name 'F-WORM' comes as the abbreviation of 'WithOut a Rainbow or Monochromatic subgraph isomorphic to F'.

If G has at least one F-WORM coloring, then $W^-(G, F)$ denotes the minimum number of colors and $W^+(G, F)$ denotes the maximum number of colors in an F-WORM coloring of G; they are termed the *F-WORM lower* and *upper chromatic number*, respectively. Moreover, the *F-WORM feasible set* $\Phi_W(G, F)$ of G is the set of those integers s for which G admits an F-WORM coloring with exactly s colors. In general, we say that G has a *gap* at k in its F-WORM chromatic spectrum, if $W^-(G, F) < k < W^+(G, F)$ but G has no F-WORM coloring with precisely k colors. Otherwise, if $\Phi_W(G, F)$ contains all integers between $W^-(G, F)$ and $W^+(G, F)$, we say that the F-WORM feasible set (or the F-WORM chromatic spectrum) of G is *gap-free*.

We shall not mention later in each assertion, but it should be emphasized that the values $W^-(G, F)$ and $W^+(G, F)$ are defined only for F-WORM-colorable graphs. Hence, wherever W^- or W^+ appears in the text, it is assumed that the graph in question is colorable.

As one can see, four fundamental problems arise in this context: testing whether G is F-WORM colorable, computing $W^-(G, F)$, computing $W^+(G, F)$, and determining $\Phi_W(G, F)$.

1.1. Results

In this paper we focus on the case of $F = K_3$, i.e., K_3-WORM colorings of graphs. It is clear that K_5 has no K_3-WORM coloring. Moreover, $W^-(G, K_3) = 1$ and $W^+(G, K_3) = n$ are valid for all triangle-free n-vertex graphs G (and only for them), and any number of colors between 1 and n can occur in this case. Therefore, the interesting examples are the graphs whose clique number equals 3 or 4.

Goddard, Wash, and Xu [8] proved that $W^-(G, K_3) \leq 2$ holds for outerplanar graphs and also for cubic graphs. They conjectured that every K_3-WORM-colorable graph admits a K_3-WORM coloring with two colors ([8, Conjecture 1]). Our Theorem 4 disproves this conjecture in a wide sense, showing that the minimum number of colors in K_3-WORM-colorable graphs can be arbitrarily large.

It was proved in [9] that there exist graphs with gaps in their P_3-WORM chromatic spectrum. In [8], the authors remark that for trees the K_3-WORM chromatic spectrum is trivially gap-free (as noted above, it is clearly so for all triangle-free graphs), and they ask whether this is true for every K_3-WORM colorable graph. Our constructions presented in Section 3 show the existence of graphs H_k which have $W^-(H_k, K_3) = 2$ and $W^+(H_k, K_3) \geq k$, but the feasible set $\Phi_W(G, K_3)$ contains no element from the range $[3, k-1]$. Further types of constructions (applying a different kind of methodology) and a study of the K_3-WORM upper chromatic number will be presented in our follow-up paper [6].

Goddard, Wash, and Xu proved that the decision problem whether a generic input graph admits a K_3-WORM coloring is NP-complete ([8, Theorem 3]). We consider complexity issues related to the determination of $W^-(G, K_3)$. In Section 4, we show that it is NP-hard to distinguish between graphs which are K_3-WORM-colorable with three colors and those needing precisely four as minimum. This hardness is true already on the class of graphs with maximum degree 9. Additionally, we prove that for every $k \geq 4$, the decision problem whether $W^-(G, K_3) \leq k$ is NP-complete already when restricted to graphs with a sufficiently large but bounded maximum degree. Deciding K_3-WORM 2-colorability is hard, too, but so far we do not have a bounded-degree version of this result. We also prove that the algorithmic problem of deciding if the K_3-WORM chromatic spectrum is gap-free is intractable.

In Section 5 we deal with 4-colorable graphs and some subclasses. It was observed by Ozeki [15] that the property of being K_3-WORM 2-colorable is valid for planar graphs; and his argument directly extends to any graph of chromatic number at most 4. As a property stronger than 4-colorability, a graph is 3-*degenerate* if each of its non-empty subgraphs contains a vertex of degree at most 3. We point out that every 3-degenerate graph has a gap-free K_3-WORM chromatic spectrum. For graphs of maximum degree 3, a formula for $W^+(G, K_3)$ can also be given.

We conclude the paper with several open problems and conjectures in Section 6.

2. Mixed Bi-Hypergraphs

The notion of mixed hypergraph was introduced by Voloshin in the 1990s [16, 17]. A detailed overview of the theory is given in the monograph [18], for up to date information see also [7] and [19]. Many open problems in the area are surveyed in [2] and [5]. In the present context the relevant structures will be what are called 'mixed bi-hypergraphs'.[1]

A *mixed bi-hypergraph* $\mathcal{H}$ is a pair $(X, \mathcal{B})$, where X is the vertex set and $\mathcal{B}$ is a set system over X. A (feasible) coloring of $\mathcal{H}$ is a mapping $\varphi : X \to \mathbb{N}$ such that each $B \in \mathcal{B}$ contains two vertices with a common color and also contains two vertices with distinct colors. In other words, no hyperedges are rainbow or monochromatic.

[1]In the literature of mixed hypergraphs the term simply is 'bi-hypergraph'. Since here our main subject is a different structure class, we will emphasize that it is a *mixed* bi-hypergraph.

For a given mixed bi-hypergraph $\mathcal{H}$, four fundamental questions arise in a very natural way.

Colorability. Does $\mathcal{H}$ admit any coloring?

Lower chromatic number. If $\mathcal{H}$ is colorable, what is the minimum number $\chi(\mathcal{H})$ of colors in a coloring?

Upper chromatic number. If $\mathcal{H}$ is colorable, what is the maximum number $\overline{\chi}(\mathcal{H})$ of colors in a coloring?

Feasible set. If $\mathcal{H}$ is colorable, what is the set $\Phi(\mathcal{H})$ of integers s such that $\mathcal{H}$ admits a coloring with exactly s colors?

The next observation shows that mixed hypergraph theory provides a proper and very natural general framework for the study of F-WORM colorings.

Proposition 1. *Let F be a given graph. For any graph G on a vertex set V, let $\mathcal{H} = (X, \mathcal{B})$ be the mixed bi-hypergraph in which $X = V$, and $\mathcal{B}$ consist of those vertex subsets of cardinality $|V(F)|$ which induce a subgraph containing F in G. Then:*

(i) *G is F-WORM-colorable if and only if $\mathcal{H}$ is colorable.*
(ii) $W^-(G, F) = \chi(\mathcal{H})$.
(iii) $W^+(G, F) = \overline{\chi}(\mathcal{H})$.
(iv) $\Phi_W(G, F) = \Phi(\mathcal{H})$.

Proof. By the definitions, an assignment $\varphi : V \to \mathbb{N}$ is an F-WORM coloring of G if and only if it is a feasible coloring of the mixed bi-hypergraph $\mathcal{H}$. Then, the statements (i)–(iv) immediately follow. ■

A similar bijection between 'WORM edge colorings' of K_n and the colorings of a mixed bi-hypergraph defined in a suitable way on the edge set of K_n was observed by Voloshin in an e-mail correspondence to us in 2013 [20].

Due to the strong correspondence above, it is meaningful and reasonable to adopt the terminology of mixed hypergraphs to the study of WORM colorings.

3. Large W^- and Gap in the Chromatic Spectrum

In several proofs of this paper we will use the following notion and notation. For a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$, the strong product

$$H = G \boxtimes K_2$$

is obtained from G by replacing each vertex v_i with two adjacent vertices x_i, y_i and each edge $v_i v_j$ with a copy of K_4 on the vertex set $\{x_i, y_i, x_j, y_j\}$.

Lemma 2. *Let G be a connected and triangle-free graph and let $H = G \boxtimes K_2$. Then, every K_3-WORM coloring φ of H is one of the following two types:*

(i) *for each $v_i \in V(G)$, the vertices x_i and y_i receive the same color, and if v_i and v_j are adjacent in G then $\varphi(x_i) \neq \varphi(x_j)$; or*

(ii) *φ uses two colors and for each $v_i \in V(G)$, the vertices x_i and y_i receive different colors.*

Moreover, if G has at least one edge, $W^-(H, K_3) = 2$, and H has a K_3-WORM coloring with precisely s colors for an $s \geq 3$ if and only if $\chi(G) \leq s \leq |V(G)|$.

Proof. Since G is triangle-free, each triangle of H is inside a copy of K_4 originating from an edge of G. Thus, the K_3-WORM colorings of H are precisely those vertex colorings in which

($*$) each copy K of K_4 gets exactly two colors such that each of them appears on exactly two vertices of this K.

First, assume that x_1 and y_1 have the same color in the K_3-WORM coloring φ. If a vertex v_j is adjacent to v_1 then, by ($*$), the only way in a K_3-WORM coloring is to assign x_j and y_j to the same color which is different from the color of $\{x_1, y_1\}$. This property of monochromatic pairs propagates along paths, therefore each pair $\{x_i, y_i\}$ ($1 \leq i \leq n$) is monochromatic whenever G is connected, and for every edge $v_i v_j \in E(G)$ the colors $\varphi(x_i)$ and $\varphi(x_j)$ are different.

On the other hand, if x_1 and y_1 have distinct colors, and a vertex v_j is adjacent to v_1, then again by ($*$), the only way in a K_3-WORM coloring is to assign $\{x_j, y_j\}$ to the same pair of colors. Then, if G is connected, precisely two colors are used in the entire graph.

A K_3-WORM coloring of H is easily obtained by assigning color 1 to all vertices x_i and color 2 to all vertices y_i. Hence, $W^-(H, K_3) = 2$. Further, if $\chi(G) \leq s \leq |V(G)|$ then G has a proper coloring ϕ which uses precisely s colors. Assigning the color $\phi(v_i)$ to the vertices x_i and y_i, yields a K_3-WORM coloring of H with exactly s colors. This completes the proof of the lemma. ■

Theorem 3. *The feasible sets of K_3-WORM-colorable graphs may contain arbitrarily large gaps.*

Proof. For an integer $k \geq 4$, consider a connected triangle-free graph G_k whose chromatic number equals k. It is well known[2] that such a graph exists for each positive k. By Lemma 2, the K_3-WORM feasible set of the graph $H_k = G_k \boxtimes K_2$ is

$$\{2\} \cup \left\{ s \mid k \leq s \leq \frac{|V(H_k)|}{2} \right\},$$

which contains a gap of size $k - 3$. ■

[2]The existence is known for over a half century, by explicit constructions and also by applying the probabilistic method; see e.g. [10, Section 1.5] for references.

Theorem 4. *For every $k \geq 3$ there exists a graph F_k such that $W^-(F_k, K_3) = k$.*

Proof. We start with a connected triangle-free graph G_k whose chromatic number is equal to k. Let H_k be again $G_k \boxtimes K_2$, as above. We define F_k as the graph obtained from three vertex-disjoint copies H_k^i of H_k $(i = 1, 2, 3)$ by the following three identifications of vertices:

$$x_1^1 = y_1^2, \qquad x_1^2 = y_1^3, \qquad x_1^3 = y_1^1.$$

This graph is K_3-WORM-colored if and only if so is each H_k^i and moreover the triangle $\{x_1^1, x_1^2, x_1^3\}$ gets precisely two colors.

Suppose, without loss of generality, that x_1^1 and x_1^2 get color 1, and x_1^3 gets color 2. Then, according to Lemma 2, both H_k^1 and H_k^3 are colored entirely with $\{1, 2\}$. On the other hand, we have $\{x_1^2, y_1^2\} = \{x_1^1, x_1^2\}$, hence this vertex pair is monochromatic in color 1, therefore H_k^2 is colored according to a proper vertex coloring of H_k. Thus, the smallest possible number of colors equals the chromatic number k of G_k. ∎

We close this section with an example which shows that $W^-(G, K_3)$ can exceed 2 even when K_4 is not a subgraph of G. Note first that in every K_3-WORM 2-coloring of W_5 the 5-cycle contains a monochromatic edge and the center of the wheel gets the opposite color. Thus, making complete adjacencies between consecutive members of the sequence of two vertex-disjoint 5-cycles C^1, C^2 and further three independent vertices x, y, z in the order

$$x,\ C^1,\ y,\ C^2,\ z$$

the vertices x and z get the same color in every K_3-WORM coloring with two colors. Let us take two copies of this graph with ends x', z' and x'', z'', respectively; moreover, take a triangle $w_1w_2w_3$ and make the following identifications:

$$w_1 = x', \qquad w_2 = z' = z'', \qquad w_3 = x''.$$

Should this K_4-free 25-vertex graph G have a K_3-WORM 2-coloring ϕ, we should have $\phi(w_1) = \phi(w_2) = \phi(w_3)$ due to the construction of the copies, and

$$|\{\phi(w_1), \phi(w_2), \phi(w_3)\}| = 2$$

due to the triangle $w_1w_2w_3$. This contradiction implies $W^-(G, K_3) \geq 3$.

4. Algorithmic Complexity

In this section we consider two algorithmic problems: to determine the minimum number of colors, and to decide whether no gaps occur in the chromatic spectrum.

4.1. Lower chromatic number

Here we prove that the determination of $W^-(G, K_3)$ is NP-hard, and it remains hard even when the input is restricted to graphs with maximum degree 9. We give degree-restricted versions of such results for every number $k \geq 3$ of colors. At the end of the subsection we prove a theorem on 2-colorings, but without upper bound on vertex degrees.

More formally, we will consider the case $F = K_3$ of the following decision problem for every positive integer k.

F-WORM k-Colorability
Input: An F-WORM-colorable graph $G = (V, E)$.
Question: Is $W^-(G, F) \leq k$?

The membership of this problem in NP is obvious for every F and every k. To prove NP-completeness for $F = K_3$ and any $k \geq 3$, we will refer to our constructions from Section 3 and the following result of Maffray and Preissmann concerning the complexity of deciding whether a graph has a proper vertex coloring with a given number k of colors, which we shall refer to as Graph k-Colorability.

Theorem 5 [14]. (i) *The* Graph 3-Colorability *problem remains* NP*-complete when the input is restricted to the class of triangle-free graphs with maximum degree four.*

(ii) *For each $k \geq 4$, the* Graph k-Colorability *problem is* NP*-complete on the restricted class of triangle-free graphs with maximum degree $3 \cdot 2^{k-1} + 2k - 2$.*

By a closer look into the proof in [14] we see that this theorem is also valid if one restricts to *connected non-regular graphs* with that maximum degree.

Theorem 6. (i) *The decision problem of K_3-WORM 3-Colorability is NP-complete already on the class of graphs with maximum degree 9.*

(ii) *The decision problem of K_3-WORM k-Colorability is NP-complete for each $k \geq 4$ already on the class of graphs with maximum degree $3 \cdot 2^k + 4k - 3$.*

Proof. As noted above, the problems are clearly in NP. To prove (i), we reduce the Graph 3-Colorability problem on the class of triangle-free graphs to the problem of K_3-WORM 3-Colorability. Consider a generic input graph G' of the former problem with $\Delta(G) = 4$. Without loss of generality we can assume that G' is connected and non-regular. Hence attaching a pendant edge to a vertex of minimum degree we get a graph G without increasing the maximum degree, such that it has a degree-1 vertex v_0. Then, we define H to be the graph $G \boxtimes K_2$, as in Section 3. Observe that $\Delta(H) = 9$. In the next step, we take three vertex-disjoint copies H^1, H^2, and H^3 of H, and make the following three identifications of vertices, each of which originates from the vertex v_0 of G:

$$x_0^1 = y_0^2, \qquad x_0^2 = y_0^3, \qquad x_0^3 = y_0^1.$$

The maximum degree of the obtained graph F remains 9, as the vertices x_0^i and y_0^i had only degree 3 in H^i. By Lemma 2, and similarly to the proof of Theorem 4, we obtain that $\chi(G) = 3$ if and only if $W^-(F, K_3) = 3$. Thus, part (i) of Theorem 5 implies the NP-completeness of K_3-WORM 3-Colorability for graphs of maximum degree 9.

Part (ii) of our theorem follows from Theorem 5 (ii) by similar steps of reductions as discussed above. ■

The following result states that the case of two colors is already hard.

Theorem 7. *The decision problem of* K_3-WORM 2-Colorability *is* NP-*complete on* K_3-*WORM-colorable graphs.*

Proof. We apply reduction from the 2-colorability of 3-uniform hypergraphs; we denote by $\mathcal{H} = (X, \mathcal{F})$ a generic input of this problem. Hence, X is the vertex set of $\mathcal{H}$, and $\mathcal{F}$ is a family of 3-element subsets of X. It is NP-complete to decide whether there exists a proper 2-coloring of $\mathcal{H}$, that is a partition (X_1, X_2) of X such that each $F \in \mathcal{F}$ meets both X_1 and X_2 [12].

>From $\mathcal{H} = (X, \mathcal{F})$ we construct a graph $G = (V, E)$ such that $\mathcal{H}$ has a proper 2-coloring if and only if G has a K_3-WORM coloring with two colors. This correspondence between $\mathcal{H}$ and G will imply the validity of the theorem.

For each hyperedge $F \in \mathcal{F}$ of $\mathcal{H}$ and each vertex $x \in F$, we create a vertex $(x, F) \in V$ of G. If $F = \{x, x', x''\}$, then the vertices $(x, F), (x', F), (x'', F)$ will be mutually adjacent in G. Moreover, small gadgets will ensure that any two vertices $(x, F'), (x, F'') \in V$ with the same x get the same color whenever G is K_3-WORM-colored.

To ensure this, suppose that an x is incident with the hyperedges $F_1, \ldots, F_d$. Then, for any two edges F_i, F_{i+1} having consecutive indices in this set (where $1 \le i < d$), we take a graph $H(x, i)$ which is isomorphic to $K_5 - e$, and identify its two non-adjacent vertices — say y and z — with (x, F_i) and (x, F_{i+1}), respectively. We make this kind of extension for each pair (x, i) in such a way that the triangles $H(x, i) - y - z$ are mutually vertex-disjoint. Let G denote the graph obtained in this way.

Consider any of the gadgets $H = H(x, i)$; we shall abbreviate it as H. Every K_3-WORM coloring of H uses a color twice on $H - y - z$, therefore the second color of $H - y - z$ (which occurs just once there) must be repeated on y and on z as well, for otherwise $H - y$ or $H - z$ would violate the conditions of K_3-WORM coloring. Thus, all of $(x, F_1), \ldots, (x, F_d)$ sharing any x must have the same color. Consequently, every K_3-WORM coloring of the obtained graph G defines a proper vertex coloring of $\mathcal{H}$ in a natural way.

Conversely, if $\mathcal{H}$ is properly colored, we can assign the color of each $x \in X$ to all vertices of type (x, F) with the same x. Then, in each $H(x, i)$, the non-adjacent vertices y and z have the same color. Repeating this color on one vertex of $H(x,i) - y - z$ and assigning one different color to its remaining vertex pair we eventually obtain a K_3-WORM coloring of G. Moreover, if $\mathcal{H}$ is 2-colored, we do not need to introduce any further colors for G.

The two-way correspondence between the 2-colorings of $\mathcal{H}$ (if they exist) and the K_3-WORM colorings of G with two colors verifies the validity of the theorem. ■

4.2. The Chromatic Gap decision problem

The problem considered in this subsection is as follows.

F-WORM Chromatic Gap
Input: An F-WORM-colorable graph G.
Question: Does the F-WORM chromatic spectrum of G have a gap?

Here we prove

Theorem 8. *The K_3-WORM Chromatic Gap problem is NP-hard.*

Proof. Part (ii) of Lemma 2 yields that the K_3-WORM chromatic spectrum of the graph $G_k \boxtimes K_2$ is gap-free if and only if G_k has a proper vertex coloring with at most three colors. This property is NP-hard (actually NP-complete) to decide. ■

5. 4-Colorable and 3-Degenerate Graphs

Here we show that two of the four basic problems listed in Section 2 have a simple solution on 4-colorable graphs. Moreover, we prove further results on 3-degenerate graphs and on graphs of maximum degree 3.

The complete graph K_5 shows that not every 5-colorable graph is K_3-WORM-colorable. On the other hand, as we shall see, every 4-colorable graph is K_3-WORM-colorable, and this can be done by using only two colors. This was commented to us after our talk at the AGTAC 2015 conference by Kenta Ozeki; hence, the next proposition should be attributed to him [15].

Proposition 9. *Every 4-colorable graph G is K_3-WORM-colorable, and the lower chromatic number $W^-(G,K_3)$ is at most 2.*

Proof. If (V_1, V_2, V_3, V_4) is a vertex partition of G into four independent sets, then each of $V_1 \cup V_2$ and $V_3 \cup V_4$ meets all triangles of G. Hence, the two color classes $V_1 \cup V_2$ and $V_3 \cup V_4$ determine a K_3-WORM-coloring of G. ■

Remark 10. It follows from the Four Color Theorem and Proposition 9 that every planar graph is K_3-WORM colorable with (at most) two colors. We note that this can also be derived by a modification of the proof of [11, Theorem 2.1], without using the 4CT. In the quoted result, Kündgen and Ramamurthi prove

WORM 2-colorability of triangular faces of planar graphs; i.e., the condition is not required there for separating triangles. Planar graphs containing separating triangles can be handled recursively, using a local condition stronger than the global one given in [11]. We omit the details.

Next we prove that for 3-degenerate graphs not only $W^-(G,K_3) \le 2$ holds but there are no gaps in their K_3-WORM chromatic spectrum.

Proposition 11. *If G is a 3-degenerate graph, then G has a gap-free K_3-WORM chromatic spectrum.*

Proof. The proof proceeds by induction on the order of the graph. Consider a 3-degenerate graph G, and a vertex $v \in V(G)$ which has three neighbors, say a, b, and c. By the induction hypothesis, the graph G^- obtained by removing v and its incident edges has a gap-free chromatic spectrum. We show that G has a K_3-WORM coloring with exactly t colors for each $t \ge 2$ in the range $W^-(G^-,K_3) \le t \le W^+(G^-,K_3)$. To do this, we start with a t-coloring φ of G^-. First, assume that $\varphi(a)$, $\varphi(b)$, and $\varphi(c)$ are pairwise distinct. Then abc is not a triangle. If a,b,c induce a P_3, the color of its central vertex can be repeated on v. If a,b,c induce only one edge, say ab, then $\varphi(a)$ can be assigned to v. If a,b,c are pairwise non-adjacent, then v can get any of the t colors of G^-. Next, consider the case of $\varphi(a)=\varphi(b)$. If this color is different from $\varphi(c)$, then it is appropriate to define $\varphi(v)=\varphi(c)$. In the last case, $\{a,b,c\}$ is monochromatic and v can be assigned to any color which is different from $\varphi(a)$. This proves that G is K_3-WORM colorable with exactly t colors for each t with $t \ge 2$ and $W^-(G^-,K_3) \le t \le W^+(G^-,K_3)$.

Note that $W^-(G,K_3)=1$ if and only if G is triangle-free, and this implies gap-free spectrum; moreover observe that $W^+(G,K_3) \le W^+(G^-,K_3)+1$. By induction, we obtain that the statement holds for every 3-degenerate graph. ■

Suppose now that G has maximum degree 3. By Proposition 9 and 11 we know that G is K_3-WORM-colorable, has $W^-(G,K_3)=2$, and its chromatic spectrum is gap-free. Next, we show that $W^+(G,K_3)$ can be computed efficiently.

Let G^Δ be the graph obtained from G by removing all edges which are not contained in any triangles. This G^Δ can have the following types of connected components:

$$K_1, \qquad K_3, \qquad K_4-e, \qquad K_4.$$

For these four types of F, let us denote by $n_G(F)$ the number of components isomorphic to F in G^Δ.

Proposition 12. *If G has n vertices, and has maximum degree at most 3, then*

$$W^+(G,K_3) = n - n_G(K_3) - n_G(K_4-e) - 2n_G(K_4).$$

Moreover, $W^+(G,K_3)$ can be determined in $O(n)$ time.

Proof. A vertex coloring is a K_3-WORM coloring of G if and only if it is a K_3-WORM coloring of each connected component in G^{Δ}. Starting from the rainbow coloring of the vertex set, a K_3-WORM coloring with maximum number of colors needs:

- to decrease the number of colors from 3 to exactly 2 in a K_3 component,
- to make the pair of the two degree-3 vertices monochromatic in a $K_4 - e$ component,
- to reduce the number of colors from 4 to 2 in a K_4 component.

This proves the correctness of the formula on $W^+(G, K_3)$. Linear time bound follows from the fact that one can construct G^{Δ} and enumerate its components of the three relevant types in $O(n)$ steps in any graph of maximum degree at most 3. ■

>From the formula above, the following tight lower bounds can be derived; part (ii) was proved for cubic graphs by Goddard *et al.* in [8].

Corollary 13. *If G is a graph of order n and maximum degree* 3, *then*

(i) $W^+(G, K_3) \geq n/2$, *with equality if and only if* $G \cong \frac{n}{4} K_4$;

(ii) *if G does not have any K_4 components, then* $W^+(G, K_3) \geq 2n/3$, *with equality if and only if G contains $\frac{n}{3} K_3$ as a subgraph;*

(iii) *if G does not have any K_4 components, and each of its triangles shares an edge with another triangle, then* $W^+(G, K_3) \geq 3n/4$, *with equality if and only if G contains $\frac{n}{4}(K_4 - e)$ as a subgraph.*

Proof. The formula in Theorem 12 shows that the number of colors lost, when compared to the number of vertices, is 2 from 4 in K_4, 1 from 3 in K_3, and 1 from 4 in $K_4 - e$. ■

A notable particular case of (ii) is where $n \geq 5$ and G is connected. Moreover, since $K_4 - e$ has just two vertices of degree 2, contracting each copy of $K_4 - e$ in the extremal structure described in (iii) we obtain a collection of vertex-disjoint paths and cycles (where cycles of length 2 are also possible).

6. Concluding Remarks

We have solved several problems — some of them raised in [8] — concerning the K_3-WORM colorability and the corresponding lower chromatic number of graphs. Further properties of K_3-WORM feasible sets and the complexity of determining the upper chromatic number will be studied in the successor of this paper, [6].

Below we mention several problems which remain open. The first one proposes a strengthening of Theorem 4. Recall that at the end of Section 3 we gave an example of K_4-free graph with $W^-(G, K_3) = 3$.

Conjecture 14. *For every integer $k \geq 4$ there exists a K_3-WORM-colorable K_4-free graph G such that $W^-(G, K_3) = k$.*

The other problems deal with algorithmic complexity. We have proved that it is NP-hard to test whether $\Phi_W(G, K_3)$ is gap-free. On the other hand, after checking that G is not triangle-free, $n-2$ questions to an NP-oracle in parallel (asking in a non-adaptive manner whether the input graph G of order n admits a K_3-WORM coloring with exactly k colors, for $k = 2, 3, \ldots, n-1$) solves the problem, hence it is in the class Θ_2^p (see [13] for a nice introduction to Θ_2^p, or the last part of [1] for short comments on its properties). However, the exact status of the problem is unknown so far.

Problem 15. Is the decision problem K_3-WORM Chromatic Gap Θ_2^p-complete?

On several natural classes of graphs we do not even have a lower bound on the complexity of this problem.

Problem 16. What is the time complexity of deciding whether the K_3-WORM chromatic spectrum is gap-free, if the input is restricted to K_3-WORM-colorable

(i) K_4-free graphs, or
(ii) 4-colorable graphs, or
(iii) planar graphs?

It is not even known at the time of writing this paper whether the feasible sets of graphs from the above classes can contain any gaps.

Also, the classes of d-degenerate graphs for various values of d offer interesting questions.

Problem 17. (i) Can the value of $W^+(G, K_3)$ be determined in polynomial time on 3-degenerate graphs?
(ii) If the answer is yes, what is the smallest d such that the computation of $W^+(G, K_3)$ is NP-hard on the class of d-degenerate graphs?
(iii) Prove that a finite threshold value d with the property described in part (ii) exists.

Problem 18. Consider the class of graphs with maximum degree at most d.

(i) Is it NP-complete to decide whether $W^-(G, K_3) = 2$ if d is large enough?
(ii) What is the smallest d_k as a function of k such that the decision problem of $W^-(G, K_3) \leq k$ is NP-hard on the class of graphs with maximum degree d_k?
(iii) What is the smallest d for which it is NP-complete to decide whether a generic input graph of maximum degree at most d is K_3-WORM-colorable?

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References

[1] S. Arumugam, B.K. Jose, Cs. Bujtás and Zs. Tuza, *Equality of domination and transversal numbers in hypergraphs*, Discrete Appl. Math. **161** (2013) 1859–1867. doi:10.1016/j.dam.2013.02.009

[2] G. Bacsó, Cs. Bujtás, Zs. Tuza and V. Voloshin, *New challenges in the theory of hypergraph coloring*, in: Advances in Discrete Mathematics and Applications, Ramanujan Mathematical Society Lecture Notes Series **13** (2010) 45–57.

[3] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, Ch. Dominic and L. Pushpalatha, *Vertex coloring without large polychromatic stars*, Discrete Math. **312** (2012) 2102–2108. doi:10.1016/j.disc.2011.04.013

[4] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, M.S. Subramanya and Ch. Dominic, *3-consecutive C-colorings of graphs*, Discuss. Math. Graph Theory **30** (2010) 393–405. doi:10.7151/dmgt.1502

[5] Cs. Bujtás and Zs. Tuza, *Maximum number of colors: C-coloring and related problems*, J. Geom. **101** (2011) 83–97. doi:10.1007/s00022-011-0082-2

[6] Cs. Bujtás and Zs. Tuza, *K_n-WORM colorings of graphs: Feasible sets and upper chromatic number*, manuscript in preparation (2015).

[7] Cs. Bujtás, Zs. Tuza and V.I. Voloshin, *Hypergraph colouring*, Chapter 11 in: L.W. Beineke and R.J. Wilson, (Eds.), Topics in Chromatic Graph Theory, Encyclopedia of Mathematics and Its Applications **156** (Cambridge University Press, 2014), 230–254.

[8] W. Goddard, K. Wash and H. Xu, *WORM colorings forbidding cycles or cliques*, Congr. Numer. **219** (2014) 161–173.

[9] W. Goddard, K. Wash and H. Xu, *WORM colorings*, Discuss. Math. Graph Theory **35** (2015) 571–584. doi:10.7151/dmgt.1814

[10] T.R. Jensen and B. Toft, Graph Coloring Problems (Wiley-Interscience, 1995).

[11] A. Kündgen and R. Ramamurthi, *Coloring face-hypergraphs of graphs on surfaces*, J. Combin. Theory Ser. B **85** (2002) 307–337. doi:10.1006/jctb.2001.2107

[12] L. Lovász, *Coverings and colorings of hypergraphs*, Congr. Numer. **8** (1973) 3–12.

[13] D. Marx, *The complexity of chromatic strength and chromatic edge strength*, Comput. Complexity **14** (2006) 308–340.
doi:10.1007/s00037-005-0201-2

[14] F. Maffray and M. Preissmann, *On the NP-completeness of the k-colorability problem for triangle-free graphs*, Discrete Math. **162** (1996) 313–317.
doi:10.1016/S0012-365X(97)89267-9

[15] K. Ozeki, private communication, June 2015.

[16] V.I. Voloshin, *The mixed hypergraphs*, Comput. Sci. J. Moldova **1** (1993) 45–52.

[17] V.I. Voloshin, *On the upper chromatic number of a hypergraph*, Australas. J. Combin. **11** (1995) 25–45.

[18] V.I. Voloshin, Coloring Mixed Hypergraphs: Theory, Algorithms and Applications (Fields Institute Monographs 17, Amer. Math. Soc., 2002).

[19] V.I. Voloshin, *Mixed hypergraph coloring web site.*
http://spectrum.troy.edu/voloshin/mh.html

[20] V.I. Voloshin, private communication, November 2013.

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