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NORDHAUS-GADDUM-TYPE RESULTS FOR RESISTANCE DISTANCE-BASED GRAPH INVARIANTS

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Abstract

Two decades ago, resistance distance was introduced to characterize "chemical distance" in (molecular) graphs. In this paper, we consider three resistance distance-based graph invariants, namely, the Kirchhoff index, the additive degree-Kirchhoff index, and the multiplicative degree-Kirchhoff index. Some Nordhaus–Gaddum-type results for these three molecular structure descriptors are obtained. In addition, a relation between these Kirchhoffian indices is established.

Keywords: resistance distance, Kirchhoff index, additive degree-Kirchhoff index, multiplicative degree-Kirchhoff index, Nordhaus-Gaddum-type result.

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1. INTRODUCTION

As an alternative of the traditional shortest-path distance, the novel concept of resistance distance was introduced by Klein and Randić [12]. For a connected (molecular) graph G, they view G as an electrical network N by replacing each edge of G with a unit resistor. Then the resistance distance between any two vertices is defined to be the net effective resistance between the corresponding nodes in N. As the chemical communication (along edges) between two vertices in molecules is rather wavelike and the resistance distance is imagined to be more relevant when the communication is wave- or fluid-like, the utility of resistance distance in chemistry has been greatly emphasized. For example, resistance distance is used to construct new molecular structure descriptors [2, 7, 12], to characterize molecular cyclicity and centricity [1, 11], to characterize geometric structure [13], and to measure the centrality [10], etc. [3, 9, 17, 21, 18].

The oldest and most famous distance-based graph invariant is the Wiener index [16], which is defined as the sum of (shortest-path) distances between all pairs of vertices. Analogous to the Wiener index, the Kirchhoff index [1, 12] was defined as the sum of resistance distances between all pairs of vertices. Later, two modifications of the Kirchhoff index, the additive degree-Kirchhoff index [7] and the multiplicative degree-Kirchhoff index [2], which take the vertex degrees into consideration, were defined.

In [19, 20], some Nordhaus-Gaddum-type results for the Kirchhoff index were obtained. Motivated by their results, we consider Nordhaus-Gaddum-type results for the additive degree-Kirchhoff index [7] and the multiplicative degree-Kirchhoff index [2]. In addition, we give an improved Nordhaus-Gaddum-type result for the Kirchhoff index and establish a relation between these three Kirchhoffian indices.

For convenience, we introduce some notations. Throughout this paper G will denote a simple, undirected, connected molecular graph and the vertices of it will be labelled by $v_1, v_2, \ldots, v_n$. Let d_i be the degree of vertex v_i for $i = 1, 2, \ldots, n$. The minimum and maximum vertex degrees are denoted by δ and Δ, respectively. The shortest path distance between two vertices v_i and v_j is denoted by d_{ij}, whereas the resistance distance between v_i and v_j is denoted by r_{ij}. It is well known that $r_{ij} \leq d_{ij}$ with equality iff v_i and v_j are connected by only one path. Palacios and Renom [15] gave the following lower bound on the resistance distance:

$$r_{ij} \geq \begin{cases} \dfrac{d_i + d_j - 2}{d_i d_j - 1} & \text{if } v_i v_j \in E(G), \\ \dfrac{1}{d_i} + \dfrac{1}{d_j} & \text{if } v_i v_j \notin E(G). \end{cases} \tag{1}$$

The Kirchhoff index [1, 12] is the sum of resistance distances between all pairs of vertices of G, i.e.,

$$R(G) = \sum_{i<j} r_{ij}.$$

The Laplacian matrix of a graph G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the $(0,1)$-adjacency matrix of graph G. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ denote the eigenvalues of $L(G)$. They are usually called the Laplacian eigenvalues of G. The Kirchhoff index $R(G)$ can also be written in terms of Laplacian eigenvalues [8, 22] as

$$R(G) = n \sum_{k=1}^{n-1} \frac{1}{\mu_k}. \tag{2}$$

The additive degree-Kirchhoff index [7] is defined as

$$R^+(G) = \sum_{i<j} (d_i + d_j)\, r_{ij}. \tag{3}$$

The multiplicative degree-Kirchhoff index [2] is defined as

$$R^*(G) = \sum_{i<j} d_i\, d_j\, r_{ij}. \tag{4}$$

A Nordhaus-Gaddum-type result [14] is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. To obtain the Nordhaus-Gaddum-type result for $R^+(G)$ and $R^*(G)$, the following two graph invariants are used. One is the inverse degree of G [4],

$$ID(G) = \sum_{v_i \in V(G)} \frac{1}{d_i};$$

the other is the modified second Zagreb index of G [5],

$$M_2^*(G) = \sum_{v_i v_j \in E(G)} \frac{1}{d_i\, d_j}. \tag{5}$$

Now we study $R(G)$, $R^+(G)$ and $R^*(G)$ in more detail, especially Nordhaus-Gaddum-type results for them, which are given in terms of the number of vertices, the number of edges, minimum and maximum vertex degrees, the inverse degree and the modified second Zagreb index. The paper is organized as follows. In Section 2, we present a lower bound on $R(G)\, R(\overline{G})$, where $\overline{G}$ denotes the complement of G. In Section 3, we obtain Nordhaus-Gaddum-type results for the additive degree-Kirchhoff index. In Section 4, we give Nordhaus-Gaddum-type results for the multiplicative degree-Kirchhoff index. Finally, in Section 5, we find a relation between $R(G)$, $R^+(G)$ and $R^*(G)$ of a graph G.

2. Nordhaus-Gaddum-Type Results for the Kirchhoff Index

In this section and later, for convenience, we use $\overline{r}_{ij}$ (resp. $\overline{d}_{ij}$) to denote the resistance distance (resp. distance) between v_i and v_j in $\overline{G}$, use $\overline{d}_i$ to denote the degree of v_i in $\overline{G}$.

In [19], Yang *et al.* proved that

$$R(G)\,R(\overline{G}) > 4(n-1)^2. \tag{6}$$

We now give a new lower bound for $R(G)\,R(\overline{G})$ in terms of n. For this we need the following result:

Lemma 1 (Foster's Formula [6])**.** *Let G be a connected graph with n vertices. Then the sum of resistance distances between all pairs of adjacent vertices is equal to $n-1$, i.e.,*

$$\sum_{v_iv_j\in E(G)} r_{ij} = n-1. \tag{7}$$

Theorem 2. *Let G be a connected graph of order n with connected complement $\overline{G}$. Then*

$$R(G)\,R(\overline{G}) \geq (2n-1)^2.$$

Proof. By the arithmetic-harmonic mean inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{d_i}{n-d_i-1}\sum_{i=1}^{n}\frac{n-d_i-1}{d_i}\\
&=\left[-n+(n-1)\sum_{i=1}^{n}\frac{1}{n-d_i-1}\right]\left[-n+(n-1)\sum_{i=1}^{n}\frac{1}{d_i}\right]\\
&\geq\left[-n+(n-1)\frac{n^2}{n(n-1)-2m}\right]\left[-n+(n-1)\frac{n^2}{2m}\right]\\
&=\frac{2mn}{n(n-1)-2m}\,\frac{n^2(n-1)-2mn}{2m}=n^2.
\end{aligned} \tag{8}$$

One can easily see that

$$\frac{d_i}{n-d_i-1}+\frac{n-d_i-1}{d_i}\geq 2. \tag{9}$$

Using (7), we get

$$
\begin{aligned}
R(G)\,R(\overline{G}) &= \sum_{i<j} r_{ij} \sum_{i<j} \overline{r}_{ij} = \left(n-1+\sum_{d_{ij}\geq 2} r_{ij}\right)\left(n-1+\sum_{\overline{d}_{ij}\geq 2} \overline{r}_{ij}\right) \\
&= (n-1)^2 + (n-1)\left(\sum_{d_{ij}\geq 2} r_{ij} + \sum_{\overline{d}_{ij}\geq 2} \overline{r}_{ij}\right) + \sum_{d_{ij}\geq 2} r_{ij} \sum_{\overline{d}_{ij}\geq 2} \overline{r}_{ij} \\
&\geq (n-1)^2 + (n-1)\left(\sum_{d_{ij}\geq 2}\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big) + \sum_{\overline{d}_{ij}\geq 2}\Big(\frac{1}{n-d_i-1}\right. \\
&\quad \left. + \frac{1}{n-d_j-1}\Big)\right) + \sum_{d_{ij}\geq 2}\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big) \sum_{\overline{d}_{ij}\geq 2}\Big(\frac{1}{n-d_i-1}+\frac{1}{n-d_j-1}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (1)} \\
&= (n-1)^2 + (n-1)\sum_{i=1}^{n}\left(\frac{d_i}{n-d_i-1}+\frac{n-d_i-1}{d_i}\right) \\
&\quad + \sum_{i=1}^{n}\frac{d_i}{n-d_i-1}\sum_{i=1}^{n}\frac{n-d_i-1}{d_i} \\
&\geq (n-1)^2 + 2n\,(n-1) + n^2 = (2n-1)^2, \text{ by (8) and (9).}
\end{aligned}
$$

This completes the proof. ∎

Remark 3. Since

$$(2n-1)^2 > 4(n-1)^2,$$

our result is better than the previous result in (6).

3. NORDHAUS-GADDUM-TYPE RESULTS FOR THE ADDITIVE DEGREE-KIRCHHOFF INDEX

We now give a lower bound for $R^+(G) + R^+(\overline{G})$ in terms of n only.

Theorem 4. *Let G be a connected graph of order n with connected complement $\overline{G}$. Then*

$$R^+(G) + R^+(\overline{G}) > 4n(n-2).$$

Proof. From (3),

$$
\begin{aligned}
&R^+(G)+R^+(\overline{G}) = \sum_{i<j}\Big[(d_i+d_j)\,r_{ij}+(2n-d_i-d_j-2)\,\overline{r}_{ij}\Big]\\
&\geq \sum_{v_iv_j\in E(G)}\left[\frac{(d_i+d_j-2)(d_i+d_j)}{d_id_j-1}+\Big(\frac{1}{n-d_i-1}+\frac{1}{n-d_j-1}\Big)(2n-d_i-d_j-2)\right]\\
&+\sum_{i<j,\,d_{ij}\geq 2}\left[\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big)(d_i+d_j)+\frac{2n-d_i-d_j-4}{(n-d_i-1)(n-d_j-1)-1}(2n-d_i-d_j-2)\right]\\
&\hspace{25em}\text{by (1)}\\
&> \sum_{v_iv_j\in E(G)}\left[\frac{(d_i+d_j-2)(d_i+d_j)}{d_id_j}+\Big(\frac{1}{n-d_i-1}+\frac{1}{n-d_j-1}\Big)(2n-d_i-d_j-2)\right]\\
&+\sum_{i<j,\,d_{ij}\geq 2}\left[\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big)(d_i+d_j)+\frac{2n-d_i-d_j-4}{(n-d_i-1)(n-d_j-1)}(2n-d_i-d_j-2)\right]\\
&= \sum_{v_iv_j\in E(G)}\left[\frac{d_i}{d_j}+\frac{d_j}{d_i}+2-2\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big)+2+\Big(\frac{n-d_j-1}{n-d_i-1}+\frac{n-d_i-1}{n-d_j-1}\Big)\right]\\
&+\sum_{i<j,\,d_{ij}\geq 2}\left[\frac{d_i}{d_j}+\frac{d_j}{d_i}+2+\Big(\frac{n-d_j-1}{n-d_i-1}+\frac{n-d_i-1}{n-d_j-1}+2\Big)\right.\\
&\left.-2\Big(\frac{1}{n-d_i-1}+\frac{1}{n-d_j-1}\Big)\right]\geq 8m-2n+8\Big(\frac{n(n-1)}{2}-m\Big)-2n=4n(n-2),
\end{aligned}
$$

as $\dfrac{d_i}{d_j}+\dfrac{d_j}{d_i}\geq 2,\quad \dfrac{n-d_j-1}{n-d_i-1}+\dfrac{n-d_i-1}{n-d_j-1}\geq 2$ and $\displaystyle\sum_{v_iv_j\in E(G)}\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big)=n.$

This completes the proof. ■

Remark 5. If we choose G to be a conference graph on n vertices, then as proved in [19],

$$R(G)=R(\overline{G})=2n.$$

Noticing that both G and $\overline{G}$ are $\dfrac{n-1}{2}$-regular, it follows that

$$R^+(G)+R^+(\overline{G})=2n(n-1)+2n(n-1)=4n(n-1),$$

which indicates that the lower bound obtained in Theorem 4 is asymptotically best.

We now give an upper bound for $R^+(G)+R^+(\overline{G})$.

Theorem 6. *Let G be a connected graph of order n with m edges, maximum degree Δ and minimum degree δ. Then*

$$\begin{aligned} R^+(G) + R^+(\overline{G}) &\le 2(n-1)(n-1+\Delta-\delta) + [n(n-1)-2m](n+3-2\delta)\Delta \\ &\quad + 2(n-1-\delta)(2\Delta+5-n)m. \end{aligned}$$

Proof. By graph theoretic knowledge, it is easily seen that if $d_{ij} \ge 3$, then the length of any path connecting i and j must be less than or equal to $n+1-d_i-d_j$, thus

$$r_{ij} \le n+1-d_i-d_j. \tag{10}$$

Similarly, if $\overline{d}_{ij} \ge 3$, then

$$\overline{r}_{ij} \le n+1-\overline{d}_i-\overline{d}_j = d_i+d_j+3-n.$$

Using the above results, we get

$$\begin{aligned} R^+(G) + R^+(\overline{G}) &= \sum_{i<j} \Big[(d_i+d_j)\, r_{ij} + (2n-d_i-d_j-2)\, \overline{r}_{ij}\Big] \\ &= \sum_{v_iv_j \in E(G)} (d_i+d_j)\, r_{ij} + \sum_{v_iv_j \in E(\overline{G})} (2n-d_i-d_j-2)\, \overline{r}_{ij} + \sum_{d_{ij}=2} (d_i+d_j)\, r_{ij} \\ &+ \sum_{\overline{d}_{ij}=2} (2n-d_i-d_j-2)\overline{r}_{ij} + \sum_{d_{ij}\ge 3} (d_i+d_j)\, r_{ij} + \sum_{\overline{d}_{ij}\ge 3} (2n-d_i-d_j-2)\overline{r}_{ij} \\ &\le 2(n-1)\,\Delta + 2(n-\delta-1)\,(n-1) + 2\sum_{d_{ij}=2} (d_i+d_j) + 2\sum_{\overline{d}_{ij}=2} (2n-d_i-d_j-2) \\ &+ \sum_{d_{ij}\ge 3} (d_i+d_j)(n+1-d_i-d_j) + \sum_{\overline{d}_{ij}\ge 3} (2n-d_i-d_j-2)\,(d_i+d_j+3-n) \\ &\le 2(n-1)\,\Delta + 2(n-\delta-1)\,(n-1) + 2\sum_{d_{ij}=2} (2\Delta) + 2\sum_{\overline{d}_{ij}=2} (2n-2-2\delta) \\ &+ \sum_{d_{ij}\ge 3} [2\Delta(n+1-2\delta)] + \sum_{\overline{d}_{ij}\ge 3} [(2n-2-2\delta)\,(2\Delta+3-n)]. \end{aligned}$$

Then the desired result can be derived by using the following inequalities

$$\sum_{d_{ij}=2} 1 \le \frac{n(n-1)}{2} - m, \quad \sum_{d_{ij}\ge 3} 1 \le \frac{n(n-1)}{2} - m, \quad \sum_{\overline{d}_{ij}=2} 1 \le m, \quad \sum_{\overline{d}_{ij}\ge 3} \le m.$$

■

If $\delta \le \frac{n-1}{2} \le \Delta$, we have $2 \le 2\Delta+3-n$ and $2 \le n+1-2\delta$. Therefore by the proof of Theorem 6, the following corollary can be easily obtained.

Corollary 7. *Let G be a connected graph of order n with m edges, maximum degree Δ and minimum degree δ such that $\delta \leq \frac{n-1}{2} \leq \Delta$. Then*

$$R^{+}(G) + R^{+}(\overline{G}) \leq 2(n-1)(n-1+\Delta-\delta) + [n(n-1)-2m]\Delta(n+1-2\delta) + m(2n-2-2\delta)(2\Delta+3-n).$$

4. Nordhaus-Gaddum-Type Results for the Multiplicative Degree-Kirchhoff Index

We now give lower bound for $R^{*}(G) + R^{*}(\overline{G})$ in terms of n, m, Δ and $M_2^{*}(G)$.

Theorem 8. *Let G be a connected graph of order n with m edges and maximum degree $\Delta < n-1$. Then*

$$R^{*}(G) + R^{*}(\overline{G}) > n(n^2-3n+4) - \frac{1}{(n-\Delta-1)^2}\,[n\,(n-1)-2m] - 2\,M_2^{*}(G).$$

Proof. From (4),

$$\begin{aligned}
R^{*}(G) + R^{*}(\overline{G}) &= \sum_{i<j}\Big[d_i\,d_j\,r_{ij} + (n-d_i-1)(n-d_j-1)\,\overline{r}_{ij}\Big] \\
&\geq \sum_{v_iv_j\in E(G)}\left[\frac{d_i+d_j-2}{d_id_j-1}\,d_i\,d_j + \Big(\frac{1}{n-d_i-1}+\frac{1}{n-d_j-1}\Big)(n-d_i-1)(n-d_j-1)\right] \\
&\quad + \sum_{i<j,\,d_{ij}\geq 2}\left[\Big(\frac{1}{d_i}+\frac{1}{d_j}\Big)d_i\,d_j + \frac{2n-d_i-d_j-4}{(n-d_i-1)(n-d_j-1)-1}\,(n-d_i-1)\,(n-d_j-1)\right] \\
&= \sum_{v_iv_j\in E(G)}\left[2n-4+\frac{d_i+d_j-2}{d_id_j-1}\right] + \sum_{i<j,\,d_{ij}\geq 2}\left[2n-4+\frac{2n-d_i-d_j-4}{(n-d_i-1)\,(n-d_j-1)-1}\right] \\
&> \sum_{i<j}(2n-4) + \sum_{v_iv_j\in E(G)}\Big(\frac{1}{d_i}+\frac{1}{d_j}-\frac{2}{d_i\,d_j}\Big) \\
&\quad + \sum_{i<j,\,d_{ij}\geq 2}\left[\frac{1}{n-d_i-1}+\frac{1}{n-d_j-1}-\frac{2}{(n-d_i-1)\,(n-d_j-1)}\right] \\
&= n(n-1)(n-2) + 2n - 2\sum_{v_iv_j\in E(G)}\frac{1}{d_i\,d_j} - 2\sum_{i<j,\,d_{ij}\geq 2}\frac{1}{(n-d_i-1)\,(n-d_j-1)} \\
&\geq n(n^2-3n+4) - \frac{1}{(n-\Delta-1)^2}\Big[n\,(n-1)-2m\Big] - 2\,M_2^{*}(G) \quad \text{as } \Delta \geq d_i\,.
\end{aligned}$$

This completes the proof. ■

Remark 9. If we choose G to be a conference graph, then

$$R^*(G) + R^*(\overline{G}) = 2n \cdot \frac{(n-1)^2}{4} + 2n \cdot \frac{(n-1)^2}{4} = n(n-1)^2,$$

which indicates that the lower bound obtained in Theorem 8 is asymptotically best.

Similarly to Theorem 6, we obtain an upper bound for $R^*(G) + R^*(\overline{G})$ in terms of n, m, δ and Δ.

Theorem 10. *Let G be a connected graph of order n with m edges, minimum degree δ and maximum degree Δ. Then*

$$\begin{aligned} R^*(G) + R^*(\overline{G}) \le\ & [\Delta^2 + (n-\delta-1)^2](n-1) + \left[\frac{n(n-1)}{2} - m\right](n+3-2\delta)\Delta^2 \\ & + m(n-1-\delta)^2(2\Delta+5-n). \end{aligned}$$

Proof. Bearing in mind that

$$r_{ij} \le n+1-d_i-d_j \le n+1-2\delta \quad \text{for} \quad d_{ij} \ge 3,$$

we have

$$\begin{aligned} & R^*(G) + R^*(\overline{G}) \\ & = \sum_{v_iv_j \in E(G)} d_i d_j\, r_{ij} + \sum_{v_iv_j \in E(\overline{G})} (n-1-d_i)(n-1-d_j)\, \overline{r}_{ij} + \sum_{d_{ij}=2} d_i d_j\, r_{ij} \\ & + \sum_{\overline{d}_{ij}=2} (n-1-d_i)(n-1-d_j)\, \overline{r}_{ij} + \sum_{d_{ij}\ge 3} d_i d_j\, r_{ij} + \sum_{\overline{d}_{ij}\ge 3} (n-1-d_i)(n-1-d_j)\, \overline{r}_{ij} \\ & \le \Delta^2(n-1) + (n-\delta-1)^2(n-1) + \sum_{d_{ij}=2} (2\Delta^2) + \sum_{\overline{d}_{ij}=2} 2(n-1-\delta)^2 \\ & + \sum_{d_{ij}\ge 3} \Delta^2(n+1-2\delta) + \sum_{\overline{d}_{ij}\ge 3} (n-1-\delta)^2(2\Delta+3-n) \\ & \le [\Delta^2 + (n-\delta-1)^2](n-1) + \left[\frac{n(n-1)}{2} - m\right](n+3-2\delta)\Delta^2 \\ & + m(n-1-\delta)^2(2\Delta+5-n), \end{aligned}$$

as required. ■

Similarly as before, we can deduce the following corollary.

Corollary 11. *Let G be a connected graph of order n with m edges, minimum degree δ and maximum degree Δ such that $\delta \le \frac{n-1}{2} \le \Delta$. Then*

$$R^*(G) + R^*(\overline{G}) \leq [\Delta^2 + (n-\delta-1)^2](n-1) + \left[\frac{n(n-1)}{2} - m\right](n+1-2\delta)\Delta^2 + m(n-1-\delta)^2(2\Delta+3-n). \tag{11}$$

5. A Relation Between $R(G)$, $R^+(G)$ and $R^*(G)$

Theorem 12. *Let G be a connected graph of order n with m edges and maximum degree $\Delta > 1$, minimum degree $\delta > 1$. Then*

$$R^*(G) - R^+(G) + R(G) \geq 2m(n-1) - n^2 + (n-1)\,ID(G) - \frac{(\Delta+\delta-2)^2}{(\Delta-1)(\delta-1)}\,m - \left[\frac{n(n-1)}{2} - m\right]\left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right). \tag{12}$$

Proof. We have

$$\sum_{i<j}\left(\frac{1}{d_i} + \frac{1}{d_j}\right) = (n-1)\sum_{i=1}^{n}\frac{1}{d_i} = (n-1)\,ID(G),$$

$$\sum_{i<j,\, d_{ij}\geq 2}\left(\frac{1}{d_i} + \frac{1}{d_j}\right) = (n-1)\,ID(G) - \sum_{v_iv_j\in E(G)}\left(\frac{1}{d_i} + \frac{1}{d_j}\right) = (n-1)\,ID(G) - n. \tag{13}$$

$$\sum_{i<j}(d_i + d_j) = \frac{1}{2}\sum_{i=1}^{n}[(n-1)\,d_i + 2m - d_i] = 2(n-1)m. \tag{14}$$

Moreover,

$$\frac{d_i-1}{d_j-1} + \frac{d_j-1}{d_i-1} + 2 = \left(\sqrt{\frac{d_i-1}{d_j-1}} + \sqrt{\frac{d_j-1}{d_i-1}}\right)^2 = \left(\sqrt{\frac{d_i-1}{d_j-1}} - \sqrt{\frac{d_j-1}{d_i-1}}\right)^2 + 4$$
$$\leq \left(\sqrt{\frac{\Delta-1}{\delta-1}} - \sqrt{\frac{\delta-1}{\Delta-1}}\right)^2 + 4 = \frac{(\Delta+\delta-2)^2}{(\Delta-1)(\delta-1)}\,.$$

Using the above results, we get

$$\sum_{v_iv_j\in E(G)}\frac{(d_i+d_j-2)^2}{d_i\,d_j-1} \leq \sum_{v_iv_j\in E(G)}\frac{(d_i+d_j-2)^2}{(d_i-1)(d_j-1)} = \sum_{v_iv_j\in E(G)}\left[\frac{d_i-1}{d_j-1} + \frac{d_j-1}{d_i-1} + 2\right] \leq \frac{(\Delta+\delta-2)^2}{(\Delta-1)(\delta-1)}\,m. \tag{15}$$

Similarly, as before, we have

$$\sum_{\substack{i<j\\ d_{ij}\geq 2}} \left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right) \leq \left(\frac{n(n-1)}{2}-m\right)\left(\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\right). \tag{16}$$

Now,

$$R^*(G)-R^+(G)=\sum_{i<j}\left[d_id_j-(d_i+d_j)\right]r_{ij}=\sum_{i<j}(d_i-1)(d_j-1)\,r_{ij}-\sum_{i<j} r_{ij}$$

$$\geq \sum_{v_iv_j\in E(G)}(d_i-1)(d_j-1)\frac{d_i+d_j-2}{d_i\,d_j-1}+\sum_{\substack{i<j\\ d_{ij}\geq 2}}(d_i-1)(d_j-1)\cdot\left(\frac{1}{d_i}+\frac{1}{d_j}\right)-R(G)$$

$$=\sum_{v_iv_j\in E(G)}\left[(d_i+d_j-2)-\frac{(d_i+d_j-2)^2}{d_i\,d_j-1}\right]+\sum_{\substack{i<j\\ d_{ij}\geq 2}}(d_i+d_j-2)$$

$$-\sum_{\substack{i<j\\ d_{ij}\geq 2}}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)+\sum_{\substack{i<j\\ d_{ij}\geq 2}}\left(\frac{1}{d_i}+\frac{1}{d_j}\right)-R(G)\,.$$

Using the results (13), (14), (15) and (16) above, we get the required result in (12). ■

Corollary 13. *Let G be a connected graph of order n with m edges and maximum degree $\Delta>1$, minimum degree $\delta>1$. Then*

$$R^*(G)-R^+(G)+R(G)\;\geq\;2m(n-1)-n^2+\frac{(n-1)\,n^2}{2m}-\frac{(\Delta+\delta-2)^2}{(\Delta-1)(\delta-1)}\,m$$

$$-\left[\frac{n(n-1)}{2}-m\right]\left(\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\right). \tag{17}$$

Proof. By the arithmetic-harmonic mean inequality, we have

$$ID(G)\geq\frac{n^2}{2m}.$$

Using the above result in Theorem 12, we get the required result in (17). ■

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References

[1] D. Bonchev, A.T. Balaban, X. Liu and D.J. Klein, *Molecular cyclicity and centricity of polycyclic graphs. I. Cyclicity based on resistance distances or reciprocal distances*, Int. J. Quantum Chem. **50** (1994) 1–20.
doi:10.1002/qua.560500102

[2] H. Chen and F. Zhang, *Resistance distance and the normalized Laplacian spectrum*, Discrete Appl. Math. **155** (2007) 654–661.
doi:10.1016/j.dam.2006.09.008

[3] H. Chen and F. Zhang, *Resistance distance local rules*, J. Math. Chem. **44** (2008) 405–417.
doi:10.1007/s10910-007-9317-8

[4] P. Dankelmann, H.C. Swart and P. van den Berg, *Diameter and inverse degree*, Discrete Math. **308** (2008) 670–673.
doi:10.1016/j.disc.2007.07.053

[5] K.Ch. Das, I. Gutman and B. Zhou, *New upper bounds on Zagreb indices*, J. Math. Chem. **46** (2009) 514–521.
doi:10.1007/s10910-008-9475-3

[6] R.M. Foster, *The average impedance of an electrical network*, in: Contributions to Applied Mechanics (Edwards Bros., Michigan, Ann Arbor, 1949) 333–340.

[7] I. Gutman, L. Feng and G. Yu, *Degree resistance distance of unicyclic graphs*, Trans. Comb. **1** (2012) 27–40.

[8] I. Gutman and B. Mohar, *The Quasi-Wiener and the Kirchhoff indices coincide*, J. Chem. Inf. Comput. Sci. **36** (1996) 982–985.
doi:10.1021/ci960007t

[9] D.J. Klein, *Graph geometry, graph metrics, & Wiener*, MATCH Commun. Math. Comput. Chem. **35** (1997) 7–27.

[10] D.J. Klein, *Centrality measure in graphs*, J. Math. Chem. **47** (2010) 1209–1223.
doi:10.1007/s10910-009-9635-0

[11] D.J. Klein and O. Ivanciuc, *Graph cyclicity, excess conductance, and resistance deficit*, J. Math. Chem. **30** (2001) 271–287.
doi:10.1023/A:1015119609980

[12] D.J. Klein and M. Randić, *Resistance distance*, J. Math. Chem. **12** (1993) 81–95.
doi:10.1007/BF01164627

[13] D.J. Klein and H.-Y. Zhu, *Distances and volumina for graphs*, J. Math. Chem. **23** (1998) 179–195.
doi:10.1023/A:1019108905697

[14] E.A. Nordhaus and J.W. Gaddum, *On complementary graphs*, Amer. Math. Monthly **63** (1956) 175–177.
doi:10.2307/2306658

[15] J.L. Palacios and J.M. Renom, *Another look at the degree-Kirchhoff index*, Int. J. Quantum Chem. **111** (2011) 3453–3455.
doi:10.1002/qua.22725

[16] H. Wiener, *Structural determination of paraffin boiling points*, J. Amer. Chem. Soc. **69** (1947) 17–20.
doi:10.1021/ja01193a005

[17] W. Xiao and I. Gutman, *Resistance distance and Laplacian spectrum*, Theor. Chem. Acc. **110** (2003) 284–289.
doi:10.1007/s00214-003-0460-4

[18] Y. Yang, *Relations between resistance distances of a graph and its complement or its contraction*, Croat. Chem. Acta **87** (2014) 61–68.
doi:10.5562/cca2318

[19] Y. Yang, H. Zhang and D.J. Klein, *New Nardhaus-Gaddum-type results for the Kirchhoff index*, J. Math. Chem. **49** (2011) 1587–1598.
doi:10.1007/s10910-011-9845-0

[20] B. Zhou and N. Trinajstić, *A note on Kirchhoff index*, Chem. Phys. Lett. **455** (2008) 120–123.
doi:10.1016/j.cplett.2008.02.060

[21] B. Zhou and N. Trinajstić, *On resistance-distance and Kirchhoff index*, J. Math. Chem. **46** (2009) 283–289.
doi:10.1007/s10910-008-9459-3

[22] H.-Y. Zhu, D.J. Klein and I. Lukovits, *Extensions of the Wiener number*, J. Chem. Inf. Comput. Sci. **36** (1996) 420–428.
doi:10.1021/ci950116s

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