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PERFECT SET OF EULER TOURS OF $K_{p,p,p}$

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Abstract

Bermond conjectured that if G is Hamilton cycle decomposable, then $L(G)$, the line graph of G, is Hamilton cycle decomposable. In this paper, we construct a perfect set of Euler tours for the complete tripartite graph $K_{p,p,p}$ for any prime p and hence prove Bermond's conjecture for $G = K_{p,p,p}$.

Keywords: compatible Euler tour, line graph, Hamilton cycle decomposition.

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1. Introduction

In this paper, a graph, its vertex set and its edge set are, respectively, denoted by G, $V(G)$ and $E(G)$. The *line graph* of a graph G, denoted by $L(G)$, is defined to be the graph with vertex set $E(G)$, where two vertices of $L(G)$ are adjacent if and only if the corresponding edges induce a 2-path in G. *Decomposition* of G is a partition of G into edge-disjoint subgraphs of G. If $H_1, H_2, \ldots, H_t$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_t)$, then we say that $H_1, H_2, \ldots, H_t$ *decompose* G. Furthermore, G is said to have a *H-decomposition* if $H_i \cong H$ for each $i = 1, 2, \ldots, t$. A *Hamilton cycle* of G is a 2-regular connected spanning subgraph of G. If G can be decomposed into Hamilton cycles, then G is said to have a *Hamilton cycle decomposition* (in short, HC-*decomposition*). A factor of G is a spanning subgraph of G and a k-factor of G is a k-regular spanning subgraph of G. Decomposition of G into k-factors is called a k-factorization of G. A 1-factorization is called *perfect* 1*-factorization* if any two 1-factors induce a Hamilton cycle in G. In a bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, an edge x_iy_j is said to be of *distance* k if $j - i = k$ for $j \geq i$, or $n + j - i = k$ for $j < i$. If every edge x_iy_j of a 1-factor of the bipartite graph is of distance k, then the 1-factor is said to be of distance k from X to Y. Obviously, if a 1-factor is of distance k from X to Y, then it is of distance $n - k$ from Y to X. By *transition* at a vertex v of G we mean traversing the 2-path e_ive_j through the edges e_i and e_j incident with v. If G has colored edges and if the two edges incident with v are of colors r and s (not necessarily distinct), then it is said to be a transition between colors (r, s) or simply (r, s)-transition. In a transition, the order of traversing the edges is immaterial. An *Euler tour* of a graph G is a closed trail that traverses every edge of G exactly once. Two Euler tours are said to be *compatible* when they have no transition in common. A *perfect set of Euler tours* of a regular graph is a set of $\triangle(G) - 1$ Euler tours which are pairwise compatible. It is easy to observe that if G has an Euler tour, then its line graph $L(G)$ has a Hamilton cycle. For definitions and notations not defined here the reader is referred to [2].

Bermond [1] conjectured that if G has a Hamilton cycle decomposition, then its line graph $L(G)$ has a Hamilton cycle decomposition. Muthusamy and Paulraja [5] have proved that if G is a $4r$-regular HC-decomposable graph, then $L(G)$ is HC-decomposable. They also proved that if G is a $(4r + 2)$-regular HC-decomposable graph, then $L(G)$ has a decomposition into Hamilton cycles and a 2-factor. Zhan [8] has also proved independently that the line graph of a $2r$-regular HC-decomposable graph can be decomposed into at least $2r - 2$ Hamilton cycles and a 2-factor. Heinrich and Verrall [4] constructed a perfect set of Euler tours of K_{2k+1} and thereby proved the existence of HC-decomposition in $L(K_{2k+1})$. Pike [7] proved the existence of HC-decomposition of $L(G)$ when G is a perfectly 1-factorable graph of even degree. Pike [6] proved the HC-decomposition of $L(G)$ when G is a 5-regular HC-decomposable graph. Verrall [9] showed the existence of a perfect set of Euler tours of $K_{2k} + I$, where I is a 1-factor of K_{2k}, and thereby proved the HC-decomposition of $L(K_{2k})$. Govindan and Muthusamy [3] have recently proved that for each $k > 5$, there exists a 2-diregular directed HC-decomposable digraph D of order $2k$ such that $L(D)$ is not directed HC-decomposable. In this paper, we give a construction of perfect set of Euler tours for $G = K_{p,p,p}$, where p is a prime number, and hence prove the HC-decomposition of $L(G)$.

2. Main Results

First we consider the complete tripartite graph $K_{2,2,2}$. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$ be the partite sets of $K_{2,2,2}$. Then the 1-factors $E_1 = \{x_1y_1, x_2z_1, y_2z_2\}$, $E_2 = \{x_1y_2, y_1z_1, x_2z_2\}$, $E_3 = \{x_1z_1, x_2y_2, y_1z_2\}$ and $E_4 = \{x_1z_2, x_2y_1, y_2z_1\}$ give a perfect 1-factorization of $K_{2,2,2}$. Thus it is easy to obtain a perfect set of Euler tours of $K_{2,2,2}$ and therefore we consider only $p \geq 3$ in the forthcoming results.

In order to prove our main results, we construct the following $2p$-regular tripartite graph. Let $X_{p,p,2p-2}$ be a $2p$-regular tripartite graph with partite sets $V_1 = \{a_1, a_2, \ldots, a_p\}$, $V_2 = \{b_1, b_2, \ldots, b_p\}$ and $V_3 = \{c_1, c_2, \ldots, c_{2p-2}\}$ and edge set $E(X_{p,p,2p-2}) = \{a_ic_j, b_ic_j, a_ib_i, a_ib_{i+1} |\ 1 \leq i \leq p, 1 \leq j \leq 2p-2\}$. Let $X'_{p,p,2p-2}$ and $X''_{p,p,2p-2}$ be two p-regular spanning subgraphs of $X_{p,p,2p-2}$ with edge sets $E(X'_{p,p,2p-2}) = \{a_ic_j, b_ic_{p+j-1}, a_ib_i | 1 \leq i \leq p, 1 \leq j \leq p-1\}$ and $E(X''_{p,p,2p-2}) = \{a_ic_{p+j-1}, b_ic_j, a_ib_{i+1} | 1 \leq i \leq p, 1 \leq j \leq p-1\}$, respectively. Note that $X'_{p,p,2p-2}$ and $X''_{p,p,2p-2}$ are edge disjoint subgraphs of $X_{p,p,2p-2}$ and hence we write $X_{p,p,2p-2} = X'_{p,p,2p-2} \oplus X''_{p,p,2p-2}$.

Lemma 1. *The tripartite graph $X_{p,p,2p-2}$ has a perfect 1-factorization for any prime p.*

Proof. We deal the proof in three cases. First we show that the spanning subgraphs $X'_{p,p,2p-2}$ and $X''_{p,p,2p-2}$ possess perfect 1-factorizations and then find a perfect 1-factorization in the whole graph.

Case 1. Consider the graph $X'_{p,p,2p-2}$. Note that $F_1 = \{a_ic_{p+1-i}, b_ic_{p-2+i} | 2 \leq i \leq p\} \oplus \{a_1b_1\}$ is a 1-factor of $X'_{p,p,2p-2}$ (see Figure 1). Let σ be the permutation $(c_1)(c_2)\cdots(c_{2p-2})(a_1\ a_2\cdots a_p)(b_1\ b_2\cdots b_p)$. Then $\{\sigma^{i-1}(F_1), 1 \leq i \leq p\}$

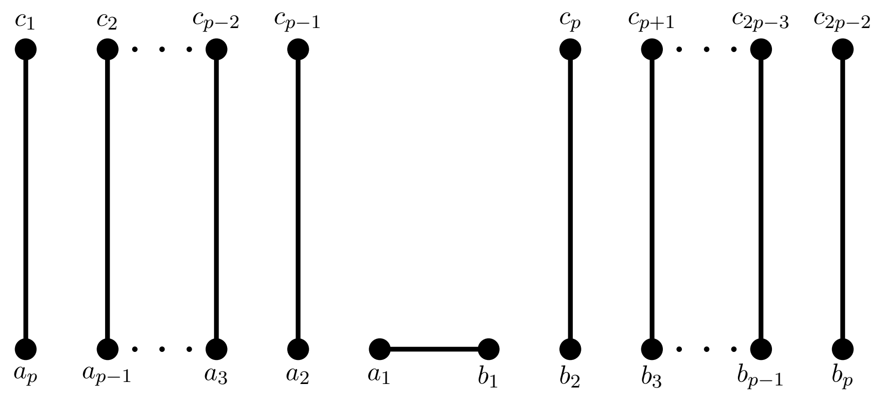

Figure 1. The 1-factor F_1 in $X'_{p,p,2p-2}$.

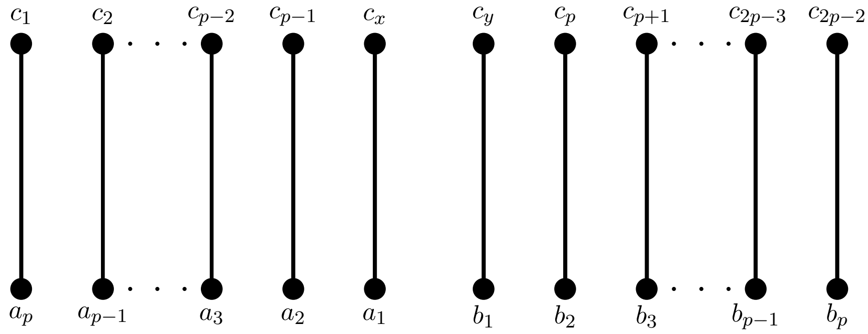

Figure 2. The 1-factor F_1' of G'.

gives a 1-factorization of $X'_{p,p,2p-2}$. It remains to show that the 1-factors defined above form a perfect 1-factorization of $X'_{p,p,2p-2}$. Let G_1 and G_2 denote the complete bipartite graphs with partite sets $\{V_1, V_3'\}$ and $\{V_2, V_3''\}$, respectively, where $V_1 = \{a_1, a_2, \ldots, a_p\}$, $V_2 = \{b_1, b_2, \ldots, b_p\}$, $V_3' = \{c_1, c_2, \ldots, c_{p-1}, c_x\}$ and $V_3'' = \{c_y, c_p, \ldots, c_{2p-2}\}$. Let $F_1' = \{a_i c_{p+1-i}, b_i c_{p-2+i} | 2 \leq i \leq p\} \oplus \{a_1 c_x, b_1 c_y\}$ be a 1-factor of $G' = G_1 \cup G_2$ (see Figure 2).

Let σ_1 be the permutation $(c_1)(c_2)\cdots(c_{p-1})(c_x)(c_y)(c_p)\cdots(c_{2p-2})(a_1 a_2 \cdots a_p)$ $(b_1 b_2 \cdots b_p)$. Since the partite sets of G_1 and G_2 have a prime number of vertices, $\{\sigma_1^{i-1}(F_1'),\ 1 \leq i \leq p\}$, gives a perfect 1-factorization within the subgraphs G_1 and G_2 of G'. Color the edges of $\sigma_1^{i-1}(F_1')$ with color i for $1 \leq i \leq p$. Now remove the vertices c_x and c_y from V_3' and V_3'', respectively. Then it is obvious that, for every $i, 1 \leq i \leq p$, there is an $a_i \in V_1$ and a $b_i \in V_2$ which are not represented by color i; join those vertices by an edge of color i. Now the resulting graph is isomorphic to $X'_{p,p,2p-2}$. Since $\{\sigma_1^{i-1}(F_1'),\ 1 \leq i \leq p\}$ gives a perfect 1-factorization within G_1 and G_2, the operation defined above guarantees that the 1-factorization

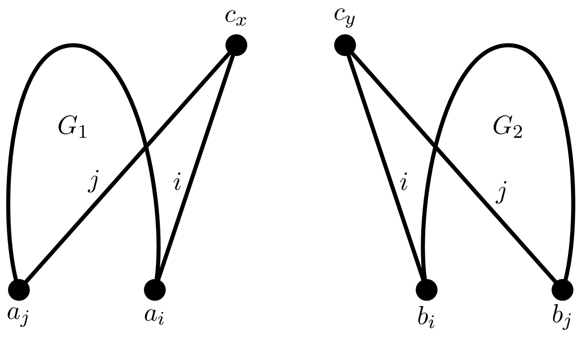

Figure 3. Hamilton cycles in G_1 and G_2.

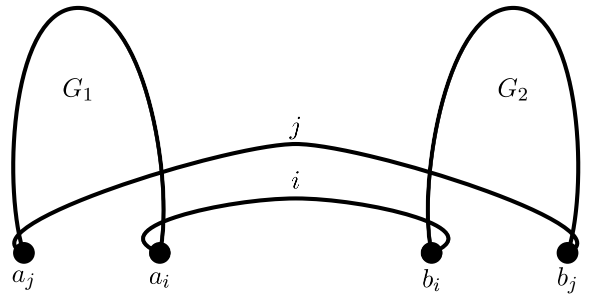

Figure 4. Hamilton cycle in $G' - \{c_x, c_y\}$.

given by $\{\sigma_1^{i-1}(F_1' - \{a_1c_x, b_1c_y\}) \oplus a_ib_i, 1 \leq i \leq p\}$ is a perfect 1-factorization of $G' - \{c_x, c_y\}$ (see Figure 3 and Figure 4, where edges of two arbitrary colors i and j are shown to induce a Hamilton cycle). Further, since the edges given by the 1-factor $\sigma_1^{i-1}(F_1' - \{a_1c_x, b_1c_y\}) \oplus a_ib_i$ for any $i, 1 \leq i \leq p$, are exactly the same as defined by the 1-factor $\sigma^{i-1}(F_1)$, it follows that $\{\sigma^{i-1}(F_1), 1 \leq i \leq p\}$ forms a perfect 1-factorization of the p-regular tripartite graph $X'_{p,p,2p-2}$.

Case 2. Consider the graph $X''_{p,p,2p-2}$. Note that $F_2 = \{a_jc_{2p-j}, b_{j+1}c_{j-1} | 2 \leq j \leq p\} \oplus \{a_1b_2\}$, where the suffixes in b_i are taken addition modulo p, is a 1-factor of $X''_{p,p,2p-2}$ as given in Figure 5. Let σ be the permutation as defined in Case 1. Then $\{\sigma^{j-1}(F_2), 1 \leq j \leq p\}$ gives a 1-factorization of $X''_{p,p,2p-2}$. It remains to show that the 1-factors defined above form a perfect 1-factorization of $X''_{p,p,2p-2}$. Let G_3 and G_4 denote, respectively, the complete bipartite graphs with partite sets $\{V_2, V_3'\}$ and $\{V_1, V_3''\}$. Let $F_2' = \{a_jc_{2p-j}, b_{j+1}c_{j-1} | 2 \leq j \leq p\} \oplus \{a_1c_y, b_2c_x\}$ be a 1-factor of $G'' = G_3 \cup G_4$ (see Figure 6). Then for the same reason as above, $\{\sigma_1^{j-1}(F_2'),\ 1 \leq j \leq p\}$ defines a perfect 1-factorization within the subgraphs G_3 and G_4 of G''. Color the edges of $\sigma_1^{j-1}(F_2')$ with color $p+j$ for $1 \leq j \leq p$. Remove the vertices c_x and c_y, respectively, from V_3' and V_3''. Then note that, for every $p+j, 1 \leq j \leq p$, there is a vertex $a_j \in V_1$ and a vertex $b_{j+1} \in V_2$ which are not represented by color $p+j$; join them by an edge of color $p+j$. Clearly, the resulting graph is isomorphic to $X''_{p,p,2p-2}$. Since the edges given by $\{\sigma_1^{j-1}(F_2'), 1 \leq j \leq p\}$ gives a perfect 1-factorization within G_3 and G_4, it is guaranteed by the above operation that $\{\sigma_1^{j-1}(F_2' - \{a_1c_y, b_2c_x\}) \oplus a_jb_{j+1}\}$ is a perfect 1-factorization of $G'' - \{c_x, c_y\}$. Further, the edges given by the 1-factor $\sigma_1^{j-1}(F_2' - \{a_1c_y, b_2c_x\}) \oplus a_jb_{j+1}$ for any $j, 1 \leq j \leq p$, are exactly the same as defined by the 1-factor $\sigma^{j-1}(F_2)$ and therefore $\{\sigma^{j-1}(F_2), 1 \leq j \leq p\}$ forms a perfect 1-factorization of the p-regular tripartite graph $X''_{p,p,2p-2}$.

Case 3. Since $X_{p,p,2p-2} = X'_{p,p,2p-2} \oplus X''_{p,p,2p-2}$, it remains to show that for any fixed i and j, $\sigma^{i-1}(F_1) \oplus \sigma^{j-1}(F_2)$ induce a Hamilton cycle in $X_{p,p,2p-2}$.

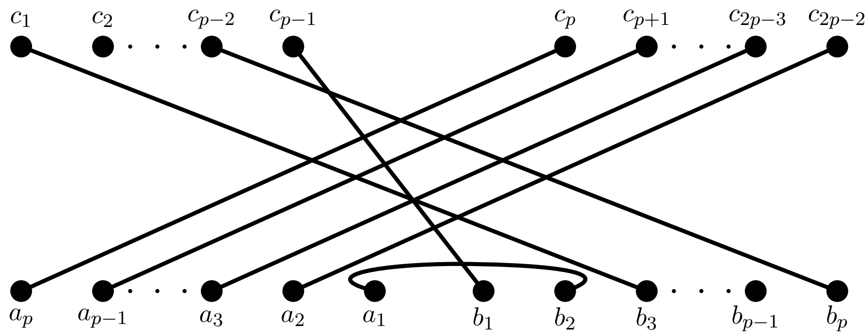

Figure 5. The 1-factor F_2 in $X''_{p,p,2p-2}$.

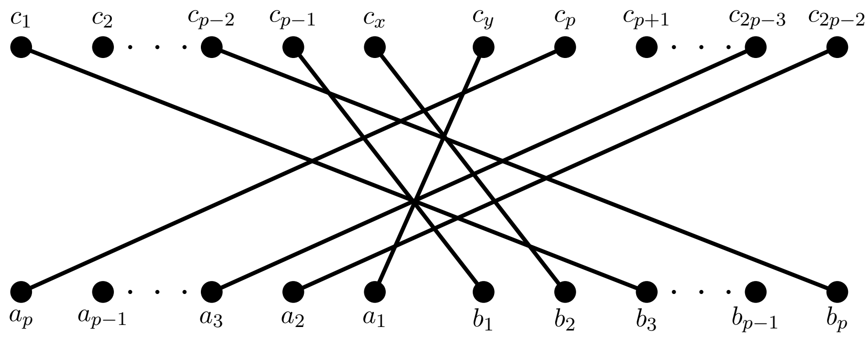

Figure 6. The 1-factor F'_2 of G''.

In other words, it is enough if we prove that the edges of any two colors s_1 and s_2, say, constitute a Hamilton cycle in $X_{p,p,2p-2}$ when $s_1 \in \{1, 2, \ldots, p\}$ and $s_2 \in \{p+1, p+2, \ldots, 2p\}$. Now, we consider the graph $G = G' \oplus G''$ with two 1-factors of colors s_1 and s_2, respectively from G' and G''. We will prove that these two 1-factors induce a Hamilton cycle in G. For the sake of convenience, relabel the vertices of V_2, V'_3, V''_3 as $\{y_m\}$, $\{z_m\}$, $\{t_m\}$, respectively, where $1 \le m \le p$, while the vertices $\{a_1, a_2, \ldots, a_p\}$ are relabeled in the order $\{x_p, x_{p-1}, \ldots, x_1\}$. Now the four complete bipartite graphs induced by the partite sets $\{V_1, V'_3\}$, $\{V_2, V''_3\}$, $\{V_2, V'_3\}$ and $\{V_1, V''_3\}$ are isomorphic to G_1, G_2, G_3 and G_4, respectively. Let us find the distance of the edges x_iz_j, y_it_j, z_iy_j and t_ix_j, respectively, from the complete bipartite graphs G_1, G_2, G_3 and G_4. Note that, by the definition of $\sigma_1^{i-1}(F'_1)$, if the edges of a particular color s_1 have distance k_0 in G_1, then the edges of the same color s_1 in G_2 have distance $p - k_0$ for any $k_0 \in \{0, 1, \ldots, p-1\}$, where $p - k_0 \equiv 0 \,(\bmod\, p)$ (see Figure 7 for the edges of distance $k_0 = 1$ in G_1 and the edges of the same color in G_2 with distance $p-k_0 = p-1$). Similarly, by the definition of $\sigma_1^{j-1}(F'_2)$, if the edges of a particular

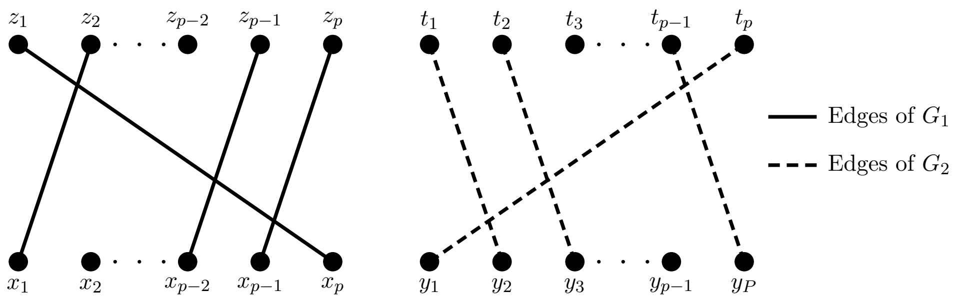

Figure 7. Edges of the same color with distance 1 in G_1 and distance $p-1$ in G_2.

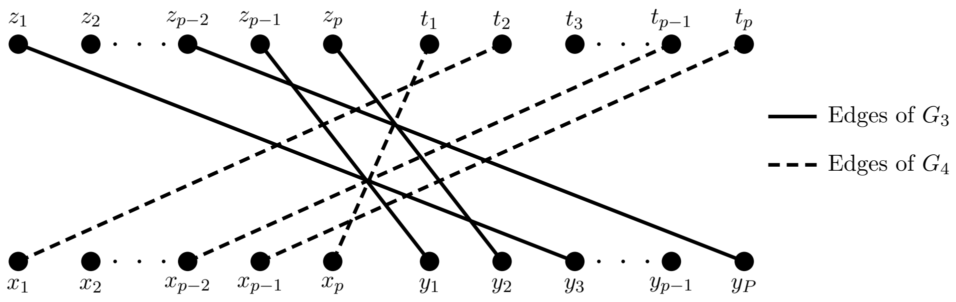

Figure 8. Edges of the same color with distance 2 in G_3 and distance $p-1$ in G_4.

color s_2 have distance l_0 in G_3, then the edges of the same color s_2 in G_4 have distance $p+1-l_0$ for any $l_0 \in \{0,1,\ldots,p-1\}$ (see Figure 8 for the edges of distance $l_0 = 2$ in G_3 and the edges of the same color in G_4 with distance $p+1-2 = p-1$).

Without loss of generality, consider the partite set V_1. Then sum of distances of edges of the 2-colored 4-path $x_m z_{r_1} y_{r_2} t_{r_3} x_n$, $1 \leq m, r_1, r_2, r_3, n \leq p$, between the vertices x_m and x_n is $k_0 + (p-k_0) + l_0 + (p+1-l_0) = 1$. Since $\gcd(p,1) = 1$, it follows that the two 1-factors with colors s_1 and s_2 induce a Hamilton cycle in G. Now the removal of vertices z_p and t_1 breaks the Hamilton cycle into a path or two paths according as the 2-paths with middle vertices z_p and t_1 have a common vertex or not. Suppose that the edges incident with z_p and t_1 have no vertex in common. Then the removal of z_p and t_1 breaks the Hamilton cycle into two paths with their end vertices in V_1 and V_2. Without loss of generality, suppose that the removal of vertex z_p removes the edge that represents color s_1 at $x_i \in V_1$. Then, the removal of t_1 must remove an edge that represents color s_2 at some other vertex $x_j \in V_1$. Consequently, deletion of z_p must also remove an edge of color s_2 at $y_k \in V_2$ and t_1 must remove an edge of color s_1 at another vertex $y_r \in V_2$. But, from the pattern in which the graphs $X'_{p,p,2p-2}$ and $X''_{p,p,2p-2}$

are constructed, there must be an edge of color s_1 between x_i and y_r, say e', in $X'_{p,p,2p-2}$ and an edge of color s_2 between x_j and y_k, say e'', in $X''_{p,p,2p-2}$.

The two path components obtained from the Hamilton cycle after removing the vertices z_p and t_1 can be one of the following type:

(i) one path with end vertices x_i and x_j, other with ends y_k and y_r.

(ii) one path with ends x_i and y_k other with ends x_j and y_r.

(iii) one path with ends x_i and y_r, other with ends x_j and y_k.

Paths of type (ii) cannot exist, since they form two cycles along with the removed 2-paths, contradicting the Hamilton cycle. Paths of type (iii) cannot exist, since a path between x_i and y_r should be of even length having the same color edges at the ends, but no such path exists. Thus paths of type (i) exists. Hence the paths of type (i) along with the edges e' and e'' form a Hamilton cycle (see Figure 9).

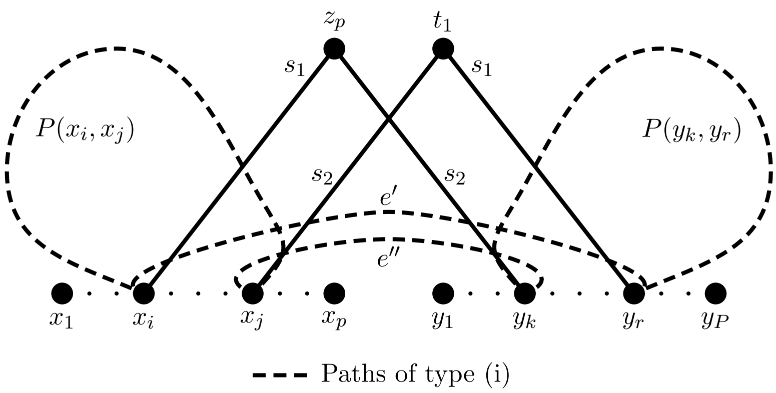

Figure 9. $P(x_i, x_j) \oplus P(y_k, y_r) \oplus \{e', e''\}$ form a Hamilton cycle.

Suppose edges incident with z_p and t_1 have a common vertex, say $x_i \in V_1$. Then the removal of z_p and t_1 from the Hamilton cycle has a single path between y_k and y_r and the isolated vertex x_i. In such case, the path between y_k and y_r along with the edges $e' = x_i y_r$ of color s_1 in $X'_{p,p,2p-2}$ and $e'' = x_i y_k$ of color s_2 in $X''_{p,p,2p-2}$ form a Hamilton cycle in $X_{p,p,2p-2}$. Since the colors s_1 and s_2 are arbitrary, it follows that $F = \{\sigma^{i-1}(F_1), \sigma^{j-1}(F_2), 1 \leq i, j \leq p\}$ gives a perfect 1-factorization of $X_{p,p,2p-2}$. Thus the lemma is proved. ■

Remark. Let us partition the color classes into two sets, namely, $A=\{1, 2, \ldots, p\}$ and $B = \{p + 1, p + 2, \ldots, 2p\}$. Since the edges of any two colors induce a Hamilton cycle in $X_{p,p,2p-2}$ and since there are $2p$ colors, we can construct $2p-1$ different Hamilton cycle decompositions in $X_{p,p,2p-2}$. Now we consider the different Hamilton cycle decompositions $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{2p-1}$ of $X_{p,p,2p-2}$ in two types as follows.

Type-I Decompositions

$$
\begin{array}{rl}
\mathcal{H}_1: & \{(1,p+2),(2,p+3),(3,p+4),\ldots,(p,p+1)\} \\
\mathcal{H}_2: & \{(1,p+3),(2,p+4),(3,p+5),\ldots,(p,p+2)\} \\
 & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
 & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\mathcal{H}_{p-1}: & \{(1,2p),(2,p+1),(3,p+2),\ldots,(p,2p-1)\}.
\end{array}
$$

In the above $p-1$ decompositions, the first element of the pairs denote a color from A while the second element denote a color from B and the addition in the second elements are taken modulo $2p$ with residues $p+1, p+2, \ldots, 2p$. Similarly, we call the following p Hamilton cycle decompositions as Type-II decompositions.

Type-II Decompositions

$$
\begin{array}{rl}
\mathcal{H}_p: & \{(1,p+1)\} \cup \\
 & \left\{(2,p),(3,p-1),\ldots,\left(\dfrac{p+1}{2},\dfrac{p+3}{2}\right)\right\} \cup \\
 & \left\{(p+2,2p),(p+3,2p-1),\ldots,\left(\dfrac{3p+1}{2},\dfrac{3(p+1)}{2}\right)\right\}, \\
\mathcal{H}_{p+1}: & \{(2,p+2)\} \cup \\
 & \left\{(3,1),(4,p),\ldots,\left(\dfrac{p+3}{2},\dfrac{p+5}{2}\right)\right\} \cup \\
 & \left\{(p+3,p+1),(p+4,2p),\ldots,\left(\dfrac{3(p+1)}{2},\dfrac{3p+5}{2}\right)\right\}, \\
 & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
 & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\mathcal{H}_{2p-1}: & \{(p,2p)\} \cup \\
 & \left\{(1,p-1),(2,p-2),\ldots,\left(\dfrac{p-1}{2},\dfrac{p+1}{2}\right)\right\} \cup \\
 & \left\{(p+1,2p-1),(p+2,2p-2),\ldots,\left(\dfrac{3p-1}{2},\dfrac{3p+1}{2}\right)\right\}.
\end{array}
$$

Each Hamilton cycle decomposition in Type-II is given as the union of three sets. The elements of pairs in the second set of each decomposition denote colors from A and therefore the additions are taken modulo p with residues $1, 2, \ldots, p$. Similarly, as the elements of pairs of the third set of each decomposition denote colors from B, they are taken addition modulo $2p$ with residues $p+1, p+2, \ldots, 2p$.

Theorem 2. *For any prime p, $K_{p,p,p}$ has a perfect set of Euler tours.*

Proof. First we consider different sets of Hamilton cycles of the tripartite graph $X_{p,p,2p-2}$. Then we find Euler tours which are not compatible at some vertices of the partite set consisting of $2p-2$ vertices in $X_{p,p,2p-2}$. By reducing the partite set of size $2p-2$ to p by a suitable technique, we get the compatible Euler tours of the required tripartite graph $K_{p,p,p}$.

Now we prove the remaining part in four claims as follows.

Claim 1. *HC-decompositions given by* Type-I *form Euler tours of* $X_{p,p,2p-2}$ *if the transitions are fixed between colors* $(k, p+k)$, $1 \le k \le p$, *at the vertices* $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$.

Proof. It is obvious that the transitions between colors $(k, p+k)$ at any vertex of $X_{p,p,2p-2}$ will connect the Hamilton cycles of each HC-decomposition of Type-I into an Euler tour.

Further, note that transitions between colors $(k, p+k)$ at the given vertices shift the tour from one Hamilton cycle to the other at each such fixed transition vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ in a uniform manner while traversing the Euler tour. Since the Hamilton cycles are formed based on the distance of 1-factors, the vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ will appear in the same cyclic order of succession in all the Hamilton cycles of any HC-decomposition of Type-I. Further observe that, since the gcd of the number of vertices at which transitions being fixed and the number of Hamilton cycles is one, i.e., $\gcd(p-2, p) = 1$, the fixation of such transitions never affect the formation of Euler tour and hence we have the claim (see Figure 10 for fixed transition vertices in $X_{p,p,2p-2}$ for the Hamilton cycle decomposition $\mathcal{H}_1$). □

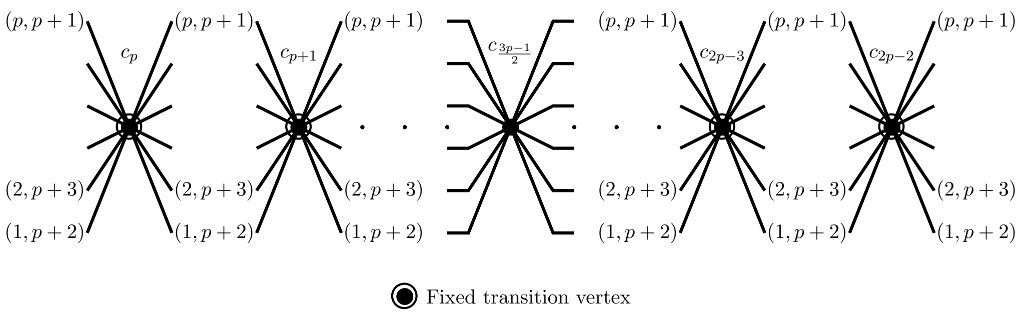

Figure 10. Fixed transition vertices in $X_{p,p,2p-2}$.

Claim 2. *If the transitions at the vertices* $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ *are fixed between colors* $(k, p+k)$, $1 \le k \le p$, *then each HC-decomposition of* Type-II *gives* $(p-1)/2$ *number of* 4*-regular spanning subgraphs and a Hamilton cycle.*

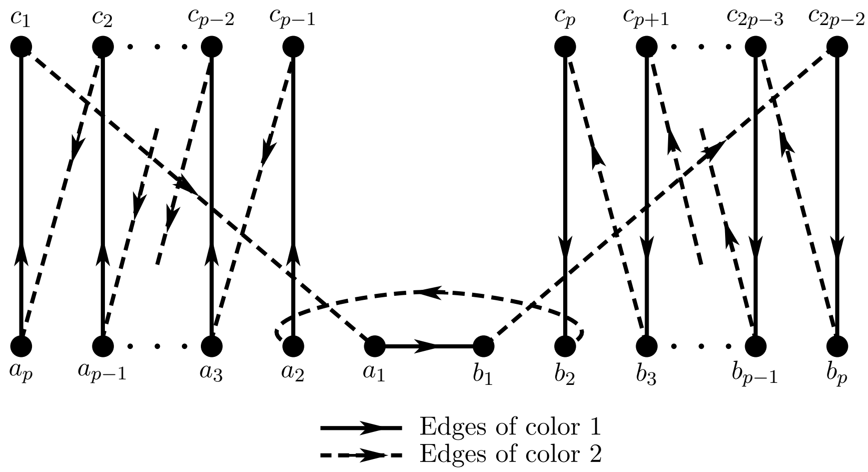

Figure 11. Hamilton cycle of $X_{p,p,2p-2}$ with edge colors $(1, 2)$.

Proof. In each HC-decomposition of Type-II, there is exactly one Hamilton cycle with edge colors $(k, p+k)$ for some $k, 1 \leq k \leq p$. Among the remaining Hamilton cycles, for each Hamilton cycle with edge colors r and s, $1 \leq r$, $s \leq p$, there is another Hamilton cycle with edge colors $p+r$ and $p+s$. Since $\gcd(p-2, 2) = 1$, those two Hamilton cycles together form a 4-regular subgraph for all possible values of r and s by fixing the said transitions at the $p-2$ vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$. □

Claim 3. *At any vertex of V_1 and V_2, one of the two trails of the 4-regular spanning subgraphs starts with the edge of color r and ends with the edge of color $p+s$, while the other trail starts with the edge of color s and ends with the edge of color $p+r$.*

Proof. By Claim 2, it is obvious that any 4-regular subgraph is the union of two Hamilton cycles with colors given by the pairs (r, s) and $(p+r, p+s)$ for some r and s, $1 \leq r, s \leq p$. We prove the claim by giving orientation to edges of an arbitrary 4-regular component. First consider the Hamilton cycle induced by the pair of colors (r, s) in any 4-regular subgraph. Note that it is the union of a spanning path of the bipartite graph with partite sets $\{V_1, V_3' - \{c_x\}\}$, a spanning path of the bipartite graph with vertex sets $\{V_2, V_3'' - \{c_y\}\}$ and two edges of colors r and s, between V_1 and V_2. Now orient the edges of the Hamilton cycle by giving an orientation to the edge of color r between V_1 and V_2 towards V_1. Then at any vertex of V_1 the edge of color s will be an outward edge and at any vertex of V_2 the edge of color s will be an inward edge (see Figure 11).

Now consider the Hamilton cycle induced by the pair of colors $(p+r,$

$p+s$). It is the union of a spanning path of the bipartite graph with partite sets $\{V_1, V_3'' - \{c_y\}\}$, a spanning path of the bipartite graph with partite sets $\{V_2, V_3' - \{c_x\}\}$ and two edges one with color $p+r$ and another with color $p+s$, between V_1 and V_2. Now we orient the edges of the Hamilton cycle as follows. By Claim 2, the transitions are fixed between colors k and $p+k$ at vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ of $V_3'' - \{c_y\}$. Since the edges of color r are inward edges to vertices of $V_3'' - \{c_y\}$, the edges of color $p+r$ between $V_3'' - \{c_y\}$ and V_1 need to be oriented towards V_1 to form a required spanning 4-regular directed trail. Thus, by orienting the edges of color $p+r$ towards V_1 and the edges of color $p+s$ towards $V_3'' - \{c_y\}$, we see that there exists a trail between the edges of color s and $p+r$ at any vertex of V_1. The same can be proved for any vertex of V_2 and hence the claim. $\square$

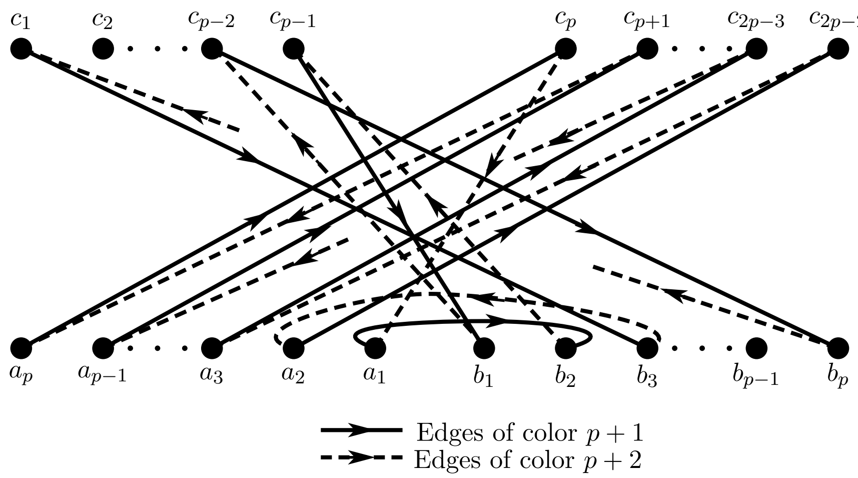

Figure 12. Hamilton cycle of $X_{p,p,2p-2}$ with edge colors $(p+1, p+2)$.

Claim 4. *Each Hamilton cycle decomposition $\mathcal{H}_i, p \leq i \leq 2p-1$, of* Type-II *gives an Euler tour of $X_{p,p,2p-2}$, if*

(i) *transitions at vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ are fixed between colors k and $p+k$, $1 \leq k \leq p$; and*

(ii) *transitions given by the $\mathcal{H}_{i+1}$ of* Type-II *are used at any one vertex of V_1 or V_2.*

Proof. By Claim 2, fixing transitions between colors k and $p+k$, $1 \leq k \leq p$, at vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ converts the Hamilton cycle decomposition into 4-regular spanning subgraphs and a Hamilton cycle. By Claim 3, each 4-regular subgraph is the union of two trails, one from the edge of color r to the edge of color $p+s$ and the another from the edge of color s to the edge of color

$p+r$, at any vertex of V_1 and V_2. Thus it is easy to observe that using transitions of $\mathcal{H}_{i+1}$, where $i+1$ is taken addition modulo $2p$ with residue p, at any vertex $a_x \in V_1$, say, the transitions of each Hamilton cycle decomposition $\mathcal{H}_i$ gives an Euler tour of $X_{p,p,2p-2}$ and hence the claim. □

We have now constructed $2p-1$ Euler tours of $X_{p,p,2p-2}$ and are all compatible except for the common transitions at the $p-2$ vertices $c_p, c_{p+1}, \ldots, c_{\frac{3p-3}{2}}, c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$. Note that $X_{p,p,2p-2}$ does not have edges of distances $2, 3, \ldots, p-1$ between V_1 and V_2. Now replace the 2-path consisting of edges of colors k and $p+k$ with middle vertex c_p, say $a_i c_p b_j$, by a single edge $a_i b_j$ and color half the edge with end a_i (resp. b_j) by the color of edge $a_i c_p$ (resp. $c_p b_j$). Repeat the process for all 2-paths with middle vertex c_p. Now the vertex c_p is isolated and removed from $X_{p,p,2p-2}$. Observe that the newly added double colored edges are all of distance 2 between V_1 and V_2. Continue the process to remove $c_{p+1}, c_{p+2}, \ldots, c_{\frac{3p-3}{2}}$ from $X_{p,p,2p-2}$ and obtain the double colored edges of distances, respectively $4, 6, \ldots, p-1$, between V_1 and V_2. Similarly, obtain the double colored edges of distances $3, 5, \ldots, p-2$ ($p > 3$) between V_1 and V_2 by removing the vertices $c_{\frac{3p+1}{2}}, \ldots, c_{2p-2}$ from $X_{p,p,2p-2}$. Rename the vertex $c_{\frac{3p-1}{2}}$ to be c_p. Now the resulting graph is a complete tripartite graph $K_{p,p,p}$ with a perfect set of $2p-1$ Euler tours. Hence the theorem is proved. ■

Corollary 3. *The line graph $L(K_{p,p,p})$ has a HC-decomposition for any prime p.*

Proof. The proof follows immediately from Theorem 2. ■

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