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THE VERTEX-RAINBOW INDEX OF A GRAPH

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Abstract

The k-rainbow index $\mathrm{rx}_k(G)$ of a connected graph G was introduced by Chartrand, Okamoto and Zhang in 2010. As a natural counterpart of the k-rainbow index, we introduce the concept of k-vertex-rainbow index $\mathrm{rvx}_k(G)$ in this paper. In this paper, sharp upper and lower bounds of $\mathrm{rvx}_k(G)$ are given for a connected graph G of order n, that is, $0 \leq \mathrm{rvx}_k(G) \leq n-2$. We obtain Nordhaus-Gaddum results for 3-vertex-rainbow index of a graph G of order n, and show that $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) = 4$ for $n = 4$ and $2 \leq \mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \leq n-1$ for $n \geq 5$. Let $t(n,k,\ell)$ denote the minimal size of a connected graph G of order n with $\mathrm{rvx}_k(G) \leq \ell$, where $2 \leq \ell \leq n-2$ and $2 \leq k \leq n$. Upper and lower bounds on $t(n,k,\ell)$ are also obtained.

Keywords: vertex-coloring, connectivity, vertex-rainbow S-tree, vertex-rainbow index, Nordhaus-Gaddum type.

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1. Introduction

The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [6]. Readers can see [6, 7, 9] for details. Consider an edge-coloring (not necessarily proper) of a graph $G = (V, E)$. We say that a path of G is *rainbow*, if no two edges on the path have the same color. An edge-colored graph G is *rainbow connected* if every two vertices are connected by a rainbow path. The minimum number of colors required to rainbow color a graph G is called *the rainbow connection*

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number, denoted by $\mathrm{rc}(G)$. In [15], Krivelevich and Yuster proposed a similar concept, the concept of vertex-rainbow connection. A vertex-colored graph G is *vertex-rainbow connected* if every two vertices are connected by a path whose internal vertices have distinct colors, and such a path is called a *vertex-rainbow path*. The *vertex-rainbow connection number* of a connected graph G, denoted by $\mathrm{rvc}(G)$, is the smallest number of colors that are needed in order to make G vertex-rainbow connected. For more results on the rainbow connection and vertex-rainbow connection, we refer to the survey paper [20] of Li, Shi and Sun and a new book [21] of Li and Sun. All graphs considered in this paper are finite, undirected and simple. We follow the notation and terminology of Bondy and Murty [2], unless otherwise stated.

For a nontrivial graph G and a set $S \subseteq V(G)$ of at least two vertices, a *Steiner tree connecting* S (also called an *S-Steiner tree* or simply *S-tree*) is a subtree T of G such that $S \subseteq V(T)$. Let there be given a vertex coloring c of G that may or may not be proper. An S-tree is a *vertex-rainbow S-tree* if no two vertices in $V(T) \setminus S$ are assigned the same color. For a fixed integer k satisfying $2 \le k \le |V(G)|$, the coloring c is called a *k-vertex-rainbow coloring* if every k-subset S of $V(G)$ has a vertex-rainbow S-tree. If such c exists, then G is *vertex-rainbow k-tree-connected*. The minimum number of colors that are needed in a k-rainbow coloring of G is called the *k-rainbow index* of G, denoted by $\mathrm{rx}_k(G)$. When $k = 2$, $\mathrm{rx}_2(G)$ is the rainbow connection number $\mathrm{rc}(G)$ of G. For more details on k-rainbow index, we refer to [3, 4, 8, 12, 17, 18].

Chartrand, Okamoto and Zhang [9] obtained the following result.

Theorem 1.1 [9]. *For every integer $n \ge 6$, $\mathrm{rx}_3(K_n) = 3$.*

As a natural counterpart of the k-rainbow index, we introduce the concept of k-vertex-rainbow index $\mathrm{rvx}_k(G)$ in this paper. For $S \subseteq V(G)$ and $|S| \ge 2$, an S-Steiner tree T is said to be a *vertex-rainbow S-tree* or a *vertex-rainbow tree connecting S* if the vertices of $V(T) \setminus S$ have distinct colors. For a fixed integer k with $2 \le k \le n$, a vertex-coloring c of G is called a *k-vertex-rainbow coloring* if for every k-subset S of $V(G)$ there exists a vertex-rainbow S-tree. In this case, G is called *vertex-rainbow k-tree-connected*. The minimum number of colors that are needed in a k-vertex-rainbow coloring of G is called the *k-vertex-rainbow index* of G, denoted by $\mathrm{rvx}_k(G)$. When $k = 2$, $\mathrm{rvx}_2(G)$ is nothing new but the vertex-rainbow connection number $\mathrm{rvc}(G)$ of G. It follows, for every nontrivial connected graph G of order n, that

$$\mathrm{rvx}_2(G) \le \mathrm{rvx}_3(G) \le \cdots \le \mathrm{rvx}_n(G).$$

Let G be the graph in Figure 1(a). We give a vertex-coloring c of the graph G shown in Figure 1(b). If $S = \{v_1, v_2, v_3\}$ (see Figure 1(c)), then the tree T induced by the edges in $\{v_1u_1, v_2u_1, u_1u_4, u_4v_3\}$ is a vertex-rainbow S-tree. If

$S = \{u_1, u_2, v_3\}$, then the tree T induced by the edges in $\{u_1u_2, u_2u_4, u_4v_3\}$ is a vertex-rainbow S-tree. One can easily check that there is a vertex-rainbow S-tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, the vertex-coloring c of G is a 3-vertex-rainbow coloring. Thus G is vertex-rainbow 3-tree-connected.

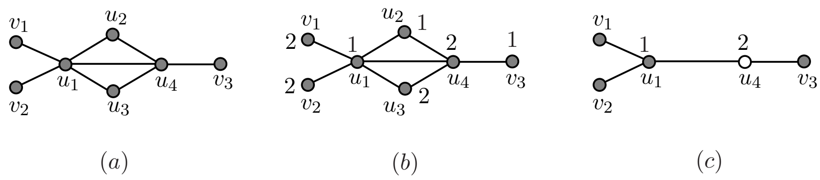

Figure 1. Graphs for the basic definitions.

In some cases $\mathrm{rvx}_k(G)$ may be much smaller than $\mathrm{rx}_k(G)$. For example, $\mathrm{rvx}_k(K_{1,n-1}) = 1$ while $\mathrm{rx}_k(K_{1,n-1}) = n-1$, where $2 \le k \le n$. On the other hand, in some other cases, $\mathrm{rx}_k(G)$ may be much smaller than $\mathrm{rvx}_k(G)$. For $k = 3$, we take n vertex-disjoint cliques of order 4 and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph G has n cut-vertices and hence $\mathrm{rvx}_3(G) \ge n$. In fact, $\mathrm{rvx}_3(G) = n$ by coloring only the cut-vertices with distinct colors. On the other hand, from Theorem 1.1, it is not difficult to see that $\mathrm{rx}_3(G) \le 9$. Just color the edges of the K_n with, say, color $1, 2, 3$ and color the edges of each clique with the colors $4, 5, \ldots, 9$. One can see that the rainbow index and vertex-rainbow index are generalizations of rainbow connection number and vertex-rainbow connection number, respectively.

Steiner tree is used in computer communication networks (see [14]) and optical wireless communication networks (see [13]). As natural combinatorial concepts, the rainbow index and the vertex-rainbow index can also find applications in networking. Suppose we want to route messages in a cellular network in such a way that each link or node on the route between more than two vertices is assigned with a distinct channel. The minimum number of channels that we have to use is exactly the rainbow index and vertex-rainbow index of the underlying graph.

The Steiner distance $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is an S-Steiner tree. The Steiner k-diameter $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G. Then, it is easy to see the following results.

Proposition 1.2. *Let G be a nontrivial connected graph of order n. Then* $\mathrm{rvx}_k(G) = 0$ *if and only if* $\mathrm{sdiam}_k(G) = k - 1$.

Proposition 1.3. *Let G be a nontrivial connected graph of order n $(n \ge 5)$, and let k be an integer with $2 \le k \le n$. Then*

$$0 \le \mathrm{rvx}_k(G) \le n - 2.$$

Proof. We only need to show $\mathrm{rvx}_k(G) \leq n-2$. Since G is connected, there exists a spanning tree of G, say T. We give the internal vertices of the tree T different colors. Since T has at least two leaves, we must use at most $n-2$ colors to color all the internal vertices of the tree T. Color the leaves of the tree T with the used colors arbitrarily. Note that such a vertex-coloring makes T vertex-rainbow k-tree-connected. Then $\mathrm{rvx}_k(T) \leq n-2$ and hence $\mathrm{rvx}_k(G) \leq \mathrm{rvx}_k(T) \leq n-2$, as desired. ■

Observation 1.4. *Let $K_{s,t}$, $K_{n_1,n_2,\ldots,n_r}$, W_n and P_n denote the complete bipartite graph, complete multipartite graph, wheel and path, respectively. Then*
(1) *for integers s and t with $s \geq 2, t \geq 1$, $\mathrm{rvx}_k(K_{s,t}) = 1$ for $2 \leq k \leq \max\{s,t\}$,*
(2) *for $r \geq 2$, $\mathrm{rvx}_k(K_{n_1,n_2,\ldots,n_r}) = 1$ for $2 \leq k \leq \max\{n_i \mid 1 \leq i \leq r\}$,*
(3) *for $n \geq 5$, $\mathrm{rvx}_k(W_n) = 1$ for $2 \leq k \leq n-3$,*
(4) *for $n \geq 4$, $\mathrm{rvx}_k(P_n) = n-2$ for $2 \leq k \leq n-2$; $\mathrm{rvx}_{n-1}(P_n) = 1$; $\mathrm{rvx}_n(P_n) = 0$.*

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n and $\mathcal{G}(n,m)$ the subclass of $\mathcal{G}(n)$ containing graphs with n vertices and m edges. Given a graph parameter $f(G)$ and a positive integer n, the *Nordhaus-Gaddum **(N-G)** Problem* is to determine sharp bounds for: (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [1] by Aouchiche and Hansen.

Chen, Li and Lian [10] gave sharp lower and upper bounds of $\mathrm{rx}_k(G)+\mathrm{rx}_k(\overline{G})$ for $k = 2$. In [11], Chen, Li and Liu obtained sharp lower and upper bounds of $\mathrm{rvx}_k(G) + \mathrm{rvx}_k(\overline{G})$ for $k = 2$. In Section 2, we investigate the case $k = 3$ and prove the following lower and upper bounds on $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G})$.

Theorem 1.5. *Let G be a graph of order n such that G and $\overline{G}$ are connected graphs. If $n = 4$, then $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) = 4$. If $n \geq 5$, then we have*

$$2 \leq \mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \leq n-1.$$

Moreover, the bounds are sharp.

Let $s(n,k,\ell)$ denote the minimal size of a connected graph G of order n with $\mathrm{rx}_k(G) \leq \ell$, where $2 \leq \ell \leq n-1$ and $2 \leq k \leq n$. Schiermeyer [24] focused on the case $k = 2$ and gave exact values and upper bounds for $s(n,2,\ell)$. Later, Li, Li, Sun and Zhao [16] improved Schiermeyer's lower bound of $s(n,2,2)$ and get a lower bound of $s(n,2,\ell)$ for $3 \leq \ell \leq \lceil \frac{n}{2} \rceil$.

In Section 3, we study the vertex case. Let $t(n,k,\ell)$ denote the minimal size of a connected graph G of order n with $\mathrm{rvx}_k(G) \leq \ell$, where $2 \leq \ell \leq n-2$ and $2 \leq k \leq n$. We obtain the following result in Section 3.

Theorem 1.6. *Let k, n, ℓ be three integers with $2 \leq \ell \leq n-3$ and $2 \leq k \leq n$. If n and ℓ have different parity, then*

$$n-1 \leq t(n,k,\ell) \leq n-1+\frac{n-\ell-1}{2}.$$

If n and ℓ have the same parity, then

$$n-1 \leq t(n,k,\ell) \leq n-1+\frac{n-\ell}{2}.$$

2. Nordhaus-Gaddum Results

To begin with, we have the following result.

Proposition 2.1. *Let G be a connected graph of order n. Then the following are equivalent.*

(1) $\mathrm{rvx}_3(G) = 0$;
(2) $\mathrm{sdiam}_3(G) = 2$;
(3) $n-2 \leq \delta(G) \leq n-1$.

Proof. From Proposition 1.2, $\mathrm{rvx}_3(G) = 0$ if and only if $\mathrm{sdiam}_3(G) = 2$. So we only need to show the equivalence of (1) and (3). Suppose $n-2 \leq \delta(G) \leq n-1$. Clearly, G is a graph obtained from the complete graph of order n by deleting some independent edges. For any $S = \{u, v, w\} \subseteq V(G)$, at least two elements in $\{uv, vw, uw\}$ belong to $E(G)$. Without loss of generality, let $uv, vw \in E(G)$. Then the tree T induced by the edges in $\{uv, vw\}$ is an S-Steiner tree and hence $d_G(S) \leq 2$. From the arbitrariness of S, we have $\mathrm{sdiam}_3(G) \leq 2$ and hence $\mathrm{sdiam}_3(G) = 2$. Therefore, $\mathrm{rvx}_3(G) = 0$.

Conversely, we assume $\mathrm{rvx}_3(G) = 0$. If $\delta(G) \leq n-3$, then there exists a vertex $u \in V(G)$ such that $d_G(u) \leq n-3$. Therefore, there are two vertices, say v, w, such that $uv, uw \notin E(G)$. Choose $S = \{u, v, w\}$. Clearly, any rainbow S-tree must contain at least a vertex in $V(G) \setminus S$, which implies that $\mathrm{rvx}_3(G) \geq 1$, a contradiction. So $n-2 \leq \delta(G) \leq n-1$. ■

After the above preparation, we can derive a lower bound of $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G})$.

Lemma 2.2. *Let G be a graph of order n such that G and $\overline{G}$ are connected. For $n \geq 5$, we have $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \geq 2$. Moreover, the bound is sharp.*

Proof. From Proposition 1.3, we have $\mathrm{rvx}_3(G) \geq 0$ and $\mathrm{rvx}_3(\overline{G}) \geq 0$. If $\mathrm{rvx}_3(G) = 0$, then we have $n-2 \leq \delta(G) \leq n-1$ by Proposition 2.1 and hence $\overline{G}$ is disconnected, a contradiction. Similarly, we can get another contradiction for $\mathrm{rvx}_3(\overline{G}) = 0$. Therefore, $\mathrm{rvx}_3(G) \geq 1$ and $\mathrm{rvx}_3(\overline{G}) \geq 1$. So $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \geq 2$. ■

To show the sharpness of the above lower bound, we consider the following example.

Example 1. Let H be a graph of order $n-4$, and let $P = a, b, c, d$ be a path. Let G be the graph obtained from H and the path by adding edges between the vertex a and all vertices of H and adding edges between the vertex d and all vertices of H; see Figure 2(a). We now show that $\text{rvx}_3(G) = \text{rvx}_3(\overline{G}) = 1$. Choose $S = \{a, b, d\}$. Then any S-Steiner tree must occupy at least one vertex in $V(G) \setminus S$. Note that the vertices of $V(G) \setminus S$ in the tree must receive different colors. Therefore, $\text{rvx}_3(G) \geq 1$.

Conversely, we show that $\text{rvx}_3(G) \leq 1$. We give each vertex in G the same color and show that there exists a vertex-rainbow S-tree for any $S \subseteq V(G)$ with $|S| = 3$. Without loss of generality, let $S = \{x, y, z\}$. Suppose first that $|S \cap V(H)| = 3$. Then the tree T induced by the edges in $\{xa, ya, za\}$ is a vertex-rainbow S-tree. Suppose $|S \cap V(H)| = 2$. Without loss of generality, let $x, y \in S \cap V(H)$. If $a \in S$, then the tree T induced by the edges in $\{xa, ya\}$ is a vertex-rainbow S-tree. If $b \in S$, then the tree T induced by the edges in $\{xa, ya, ab\}$ is a vertex-rainbow S-tree. The remaining cases $c \in S$ and $d \in S$ are symmetric.

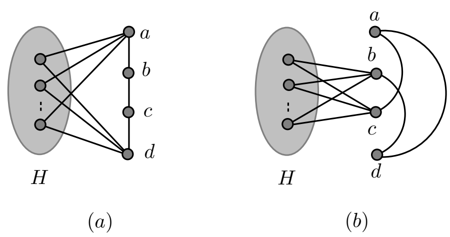

Figure 2. Graphs for Example 1.

Suppose $|S \cap V(H)| = 1$. Without loss of generality, let $x \in S \cap V(H)$. If $a, b \in S$, then the tree T induced by the edges in $\{xa, ab\}$ is a vertex-rainbow S-tree. If $b, c \in S$, then the tree T induced by the edges in $\{xd, cd, bc\}$ is a vertex-rainbow S-tree. If $a, c \in S$, then the tree T induced by the edges in $\{xa, ab, bc\}$ is a vertex-rainbow S-tree. Note that the remaining cases are symmetric. Suppose $|S \cap V(G')| = 0$. If $a, b, c \in S$, then the tree T induced by the edges in $\{ab, bc\}$ is a vertex-rainbow S-tree. If $a, b, d \in S$, then the tree T induced by the edges in $\{ab, bc, cd\}$ is a vertex-rainbow S-tree. We conclude that $\text{rvx}_3(G) \leq 1$. Similarly, one can also check that $\text{rvx}_3(\overline{G}) = 1$ (see Figure 2(b)). So $\text{rvx}_3(G) + \text{rvx}_3(\overline{G}) = 2$.

We are now in a position to give an upper bound of $\text{rvx}_3(G) + \text{rvx}_3(\overline{G})$. For $n = 4$, we have $G = \overline{G} = P_4$, since we only consider connected graphs. Observe that $\text{rvx}_3(G) = \text{rvx}_3(\overline{G}) = \text{rvx}_3(P_4) = 2$.

Observation 2.3. *Let G be a graph of order n $(n = 4)$ such that both G and $\overline{G}$ are connected. Then* $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) = n$.

For $n \geq 5$, we have the following upper bound of $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G})$.

Lemma 2.4. *Let G be a graph of order n $(n = 5)$ such that both G and $\overline{G}$ are connected. Then* $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \leq n - 1$.

Proof. If G is a path of order 5, then $\mathrm{rvx}_3(G) = 3$ by Observation 1.4. Observe that $\mathrm{sdiam}_3(\overline{G}) = 3$. Then $\mathrm{rvx}_3(\overline{G}) \leq 1$ and hence $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \leq 4$, as desired.

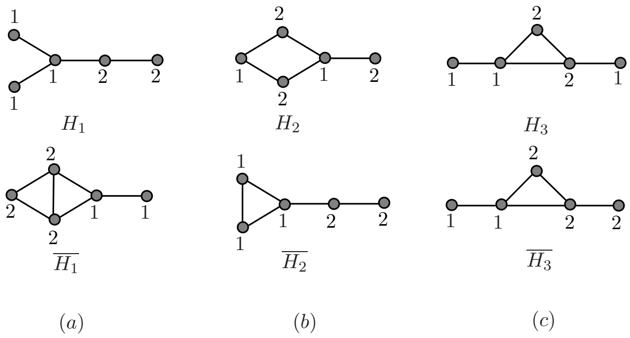

Figure 3. Graphs for Lemma 2.4.

If G is a tree but not a path, then we have $G = H_1$, since $\overline{G}$ is connected (see Figure 3(a)). Clearly, $\mathrm{rvx}_3(G) \leq 2$. Furthermore, $\overline{G}$ consists of a K_2 and a K_3 and two edges between them (see Figure 3(a)). So we assign color 1 to the vertices of K_2 and color 2 to the vertices of K_3, and this vertex-coloring makes the graph G vertex-rainbow 3-tree-connected, that is, $\mathrm{rvx}_3(\overline{G}) \leq 2$. Therefore, $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \leq 4$, as desired.

Suppose that both G and $\overline{G}$ are not trees. Then $e(G) \geq 5$ and $e(\overline{G}) \geq 5$. Since $e(G) + e(\overline{G}) = e(K_5) = 10$, it follows that $e(G) = e(\overline{G}) = 5$. If G contains a cycle of length 5, then $G = \overline{G} = C_5$ and hence $\mathrm{rvx}_3(G) = \mathrm{rvx}_3(\overline{G}) = 2$. If G contains a cycle of length 4, then $G = H_2$ (see Figure 3(b)). Clearly, $\mathrm{rvx}_3(G) = \mathrm{rvx}_3(\overline{G}) = 2$. If G contains a cycle of length 3, then $G = \overline{G} = H_3$ (see Figure 3(c)). One can check that $\mathrm{rvx}_3(G) = \mathrm{rvx}_3(\overline{G}) = 2$. Therefore, $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) = 4$, as desired. ■

Lemma 2.5. *Let G be a nontrivial connected graph of order n, and* $\mathrm{rvx}_3(G) = \ell$. *Let G' be the graph obtained from G by adding a new vertex v to G and making v adjacent to q vertices of G. If $q \geq n - \ell$, then* $\mathrm{rvx}_3(G') \leq \ell$.

Proof. Let $c : V(G) \to \{1, 2, \ldots, \ell\}$ be a vertex-coloring of G such that G is vertex-rainbow 3-tree-connected. Let $X = \{x_1, x_2, \ldots, x_q\}$ be the neighborhood of v in G. Set $V(G) \setminus X = \{y_1, y_2, \ldots, y_{n-q}\}$. We can assume that there exist two vertices y_{j_1}, y_{j_2} such that there is no vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$; otherwise, the result holds obviously.

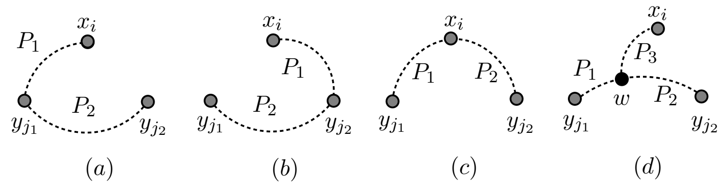

Figure 4. Four type of the Steiner tree T_i.

We define a minimal S-Steiner tree T as a tree connecting S whose subtree obtained by deleting any edge of T does not connect S. Because G is vertex-rainbow 3-tree-connected, there is a minimal vertex-rainbow tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ for each x_i ($i \in \{1, 2, \ldots, q\}$). Then the tree T_i is of one of four types shown in Figure 4. For the type shown in (c), the Steiner tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ is a path induced by the edges in $E(P_1) \cup E(P_2)$ and hence the internal vertices of the path T_i must receive different colors. Therefore, the tree induced by the edges in $E(P_1) \cup E(P_2) \cup \{vx_i\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, a contradiction. Clearly, (b) is symmetric to (a). So we only need to consider remaining two cases shown in Figure 4(a), (d). Obviously, $T_i \cap T_j$ may not be empty. Then we have the following claim.

Claim 1. *No other vertex in* $\{x_1, x_2, \ldots, x_q\}$ *different from* x_i *belongs to* T_i *for each* $1 \leq i \leq q$.

Proof. Assume, to the contrary, that there exists a vertex $x_i' \in \{x_1, x_2, \ldots, x_q\}$ such that $x_i' \neq x_i$ and $x_i' \in V(T_i)$. For the type shown in Figure 4(a), the Steiner tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ is a path induced by the edges in $E(P_1) \cup E(P_2)$ and hence the internal vertices of the path T_i receive different colors. If $x_i' \in V(P_1)$, then the tree induced by the edges in $E(P_1') \cup E(P_2) \cup \{vx_i'\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, where P_1' is the path between the vertex x_i' and the vertex y_{j_1} in P_1, a contradiction. If $x_i' \in V(P_2)$, then the tree induced by the edges in $E(P_2) \cup \{vx_i\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, a contradiction. For the type shown in Figure 4(d), the Steiner tree T_i connecting $\{x_i, y_{j_1}, y_{j_2}\}$ is a tree induced by the edges in $E(P_1) \cup E(P_2) \cup E(P_3)$ and hence the internal vertices of the tree T_i receive different colors. Without loss of generality, let $x_i' \in V(P_1)$. Then the tree induced by the edges in $E(P_1') \cup E(P_2) \cup E(P_3)$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, where P_1' is the path between the vertex x_i' and the vertex w in P_1, a contradiction. □

From Claim 1, since there is no vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, it follows that there exists a vertex y_{k_i} such that $c(x_i) = c(y_{k_i})$ for each tree T_i, which implies that the colors that are assigned to X are among the colors that are assigned to $V(G) \setminus X$. So $\mathrm{rvx}_3(G) = \ell \leq n - q$. Combining this with the hypothesis $q \geq n - \ell$, we have $\mathrm{rvx}_3(G) = n - q$, that is, all vertices in $V(G) \setminus X$ have distinct colors. Now we construct a new graph G', which is induced by the edges in $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_q)$.

Claim 2. *For every y_t not in G', there exists a vertex $y_s \in G'$ such that $y_t y_s \in E(G)$.*

Proof. Assume, to the contrary, that $N(y_t) \subseteq \{x_1, x_2, \ldots, x_q\}$. Since G is vertex-rainbow 3-tree-connected, there is a vertex-rainbow tree T connecting $\{y_t, y_{j_1}, y_{j_2}\}$. Let x_r be the vertex in the tree T such that $x_r \in N_T(y_t)$. Then the tree induced by the edges in $(E(T) \setminus \{y_t x_r\}) \cup \{v x_r\}$ is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$, a contradiction. □

From Claim 2, $G[y_1, y_2, \ldots, y_{n-q}]$ is connected. Clearly, $G[y_1, y_2, \ldots, y_{n-q}]$ has a spanning tree T. Because the tree T has at least two pendant vertices, there must exist a pendant vertex whose color is different from x_1, and we assign its color to x_1. One can easily check that G is still vertex-rainbow 3-tree-connected, and there is a vertex-rainbow tree connecting $\{v, y_{j_1}, y_{j_2}\}$. If there still exist two vertices y_{j_3}, y_{j_4} such that there is no vertex-rainbow tree connecting $\{v, y_{j_3}, y_{j_4}\}$, then we do the same operation until there is a vertex-rainbow tree connecting $\{v, y_{j_r}, y_{j_s}\}$ for each pair $y_{j_r}, y_{j_s} \in \{1, 2, \ldots, n - q\}$. Thus G' is vertex-rainbow 3-tree-connected. So $\mathrm{rvx}_3(G') \leq \ell$. ■

***Proof of Theorem* 1.5.** We prove this theorem by induction on n. By Lemma 2.4, the result is evident for $n = 5$. We assume that $\mathrm{rvx}_3(G) + \mathrm{rvx}_3(\overline{G}) \leq n - 1$ holds for graphs on n vertices. Observe that the union of a connected graph G and its complement $\overline{G}$ is a complete graph of order n, that is, $G \cup \overline{G} = K_n$. We add a new vertex v to G and add q edges between v and $V(G)$. Denote by G' the resulting graph. Clearly, $\overline{G'}$ is a graph of order $n+1$ obtained from $\overline{G}$ by adding a new vertex v to $\overline{G}$ and adding $n - q$ edges between v and $V(\overline{G})$.

Claim 3. $\mathrm{rvx}_3(G') \leq \mathrm{rvx}_3(G) + 1$ *and* $\mathrm{rvx}_3(\overline{G'}) \leq \mathrm{rvx}_3(\overline{G}) + 1$.

Proof. Let c be a $\mathrm{rvx}_3(G)$-vertex-coloring of G such that G is vertex-rainbow 3-tree-connected. Pick up a vertex $u \in N_G(v)$ and give it a new color. It suffices to show that for any $S \subseteq V(G')$ with $|S| = 3$, there exists a vertex-rainbow S-tree. If $S \subseteq V(G)$, then there exists a vertex-rainbow S-tree, since G is vertex-rainbow 3-tree-connected. Suppose $S \nsubseteq V(G)$. Then $v \in S$. Without loss of generality, let $S = \{v, x, y\}$. Since G is vertex-rainbow 3-tree-connected, there exists a vertex-rainbow tree T' connecting $\{u, x, y\}$. Then the tree T induced by the edges in $E(T') \cup \{uv\}$ is a vertex-rainbow S-tree. Therefore, $\mathrm{rvx}_3(G') \leq \mathrm{rvx}_3(G) + 1$. Similarly, $\mathrm{rvx}_3(\overline{G'}) \leq \mathrm{rvx}_3(\overline{G}) + 1$. □

From Claim 3, we have $\text{rvx}_3(G')+\text{rvx}_3(\overline{G'}) \leq \text{rvx}_3(G)+1+\text{rvx}_3(\overline{G})+1 \leq n+1$. Clearly, $\text{rvx}_3(G')+\text{rvx}_3(\overline{G'}) \leq n$ except possibly when $\text{rvx}_3(G') = \text{rvx}_3(G)+1$ and $\text{rvx}_3(\overline{G'}) = \text{rvx}_3(\overline{G})+1$. In this case, by Lemma 2.5, we have $q \leq n-\text{rvx}_3(G)-1$ and $n-q \leq n-\text{rvx}_3(\overline{G})-1$. Thus, $\text{rvx}_3(G)+\text{rvx}_3(\overline{G}) \leq (n-1-q)+(q-1) = n-2$ and hence $\text{rvx}_3(G') + \text{rvx}_3(\overline{G'}) \leq n$, as desired. This completes the induction. ■

To show sharpness of the above bound, we consider the following example.

Example 2. Let G be a path of order n. Then $\text{rvx}_3(G) = n-2$. Observe that $\text{sdiam}_3(\overline{G}) = 3$. Then $\text{rvx}_3(\overline{G}) = 1$, and so we have $\text{rvx}_3(G) + \text{rvx}_3(\overline{G}) = (n-2) + 1 = n-1$.

3. The Minimal Size of Graphs with Given Vertex-Rainbow Index

Recall that $t(n, k, \ell)$ is the minimal size of a connected graph G of order n with $\text{rvx}_k(G) \leq \ell$, where $2 \leq \ell \leq n-2$ and $2 \leq k \leq n$. Let G be a path of order n. Then $\text{rvx}_k(G) \leq n-2$ and hence $t(n, k, n-2) \leq n-1$. Since we only consider connected graphs, it follows that $t(n, k, n-2) \geq n-1$. Therefore, the following result is immediate.

Observation 3.1. *Let k be an integer with $2 \leq k \leq n$. Then*

$$t(n, k, n-2) = n-1.$$

A *rose graph R_p with p petals* (or *p-rose graph*) is a graph obtained by taking p cycles with just a vertex in common. The common vertex is called the *center* of R_p. If the length of each cycle is exactly q, then this rose graph with p petals is called a *(p, q)-rose graph*, denoted by $R_{p,q}$. Now we are able to prove Theorem 1.6.

***Proof of Theorem* 1.6.** From Observation 3.1, we have $t(n, k, \ell) \geq n-1$. It suffices to show the upper bound holds. Suppose that n and ℓ have different parity. Then $n-\ell-1$ is even. Let G be the graph obtained from an $(\frac{n-\ell-1}{2}, 3)$-rose graph $R_{\frac{n-\ell-1}{2},3}$ and a path $P_{\ell+1}$ by identifying the center of the rose graph and one endpoint of the path. Let w_0 be the center of $R_{\frac{n-\ell-1}{2},3}$, and let $C_i = w_0v_iu_iw_0$ $(1 \leq i \leq \frac{n-\ell-1}{2})$ be the cycle of $R_{\frac{n-\ell-1}{2},3}$. Let $P_{\ell+1} = w_0w_1 \cdots w_\ell$ be the path of order $\ell+1$. To show that $\text{rvx}_k(G) \leq \ell$, we define a vertex-coloring $c : V(G) \to \{0, 1, 2, \ldots, \ell-1\}$ of G by

$$c(v) = \begin{cases} i, & \text{if } v = w_i \ (0 \leq i \leq \ell-1); \\ 1, & \text{if } v = u_i \text{ or } v = v_i \ \left(1 \leq i \leq \frac{n-\ell-1}{2}\right); \\ 1, & \text{if } v = w_\ell. \end{cases}$$

One can easily see that there exists a vertex-rainbow S-tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $\text{rvx}_k(G) \leq \ell$ and $t(n,k,\ell) \leq n - 1 + \frac{n-\ell-1}{2}$.

Suppose that n and ℓ have the same parity. Then $n - \ell$ is even. Let G be the graph obtained from an $(\frac{n-\ell}{2}, 3)$-rose graph $R_{\frac{n-\ell}{2},3}$ and a path P_ℓ by identifying the center of the rose graph and one endpoint of the path. Let w_0 be the center of $R_{\frac{n-\ell}{2},3}$, and let $C_i = w_0 v_i u_i w_0$ $(1 \leq i \leq \frac{n-\ell}{2})$ be the cycle of $R_{\frac{n-\ell}{2},3}$. Let $P_\ell = w_0 w_1 \cdots w_{\ell-1}$ be the path of order ℓ. To show that $\text{rvx}_k(G) \leq \ell$, we define a vertex-coloring $c : V(G) \to \{0, 1, 2, \ldots, \ell - 1\}$ of G by

$$c(v) = \begin{cases} i, & \text{if } v = w_i \ (0 \leq i \leq \ell - 1); \\ 1, & \text{if } v = u_i \text{ or } v = v_i \ \left(1 \leq i \leq \frac{n-\ell}{2}\right). \end{cases}$$

One can easily see that there exists a vertex-rainbow S-tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $\text{rvx}_k(G) \leq \ell$ and $t(n,k,\ell) \leq n - 1 + \frac{n-\ell}{2}$. ■

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