Discussiones Mathematicae
Graph Theory 36 (2016) 797–803
doi:10.7151/dmgt.1886

INTEGRAL CAYLEY SUM GRAPHS AND GROUPS

Xuanlong Ma[1,2] and Kaishun Wang[2]

[1] *College of Mathematics and Information Science*
Guangxi University, Nanning 530004, China

[2] *Sch. Math. Sci. & Lab. Math. Com. Sys.*
Beijing Normal University
Beijing 100875, China

e-mail: xuanlma@mail.bnu.edu.cn
wangks@bnu.edu.cn

Abstract

For any positive integer k, let $\mathcal{A}_k$ denote the set of finite abelian groups G such that for any subgroup H of G all Cayley sum graphs $\mathrm{CayS}(H, S)$ are integral if $|S| = k$. A finite abelian group G is called Cayley sum integral if for any subgroup H of G all Cayley sum graphs on H are integral. In this paper, the classes $\mathcal{A}_2$ and $\mathcal{A}_3$ are classified. As an application, we determine all finite Cayley sum integral groups.

Keywords: Cayley sum graph, integral graph, Cayley sum integral group.

2010 Mathematics Subject Classification: 05C25, 05C50.

1. Introduction

A graph is *integral* if all its eigenvalues are integers. Harary and Schwenk [13] introduced integral graphs, and proposed the problem of classifying integral graphs. There are some constructions of graphs from groups in the literature; for example, Cayley graphs, which are integral were studied in [1, 2, 3, 11, 14, 15].

Let G be a finite abelian group. A subset S of G is said to be *square-free* if $x + x \notin S$ for each $x \in G$. The *Cayley sum graph* of G with respect to a square-free subset S of G, denoted by $\mathrm{CayS}(G, S)$, is a simple graph with vertex set G and two distinct vertices x and y form an edge if $x + y \in S$. Some results on Cayley sum graphs can be found in [4, 5, 9, 12, 17].

For any positive integer k, let $\mathcal{A}_k$ denote the set of finite abelian groups G such that for any subgroup H of G all Cayley sum graphs $\mathrm{CayS}(H, S)$ are integral if $|S| = k$. A finite abelian group G is called *Cayley sum integral* if for any subgroup H of G all Cayley sum graphs on H are integral.

In the paper we classify the classes $\mathcal{A}_2$ and $\mathcal{A}_3$. As an application, all finite Cayley sum integral groups are determined. Our main results are the following.

Theorem 1. *The class $\mathcal{A}_2$ consists of the groups:*

$$\mathbb{Z}_2^n, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_3^m, \quad n \geq 2, \quad m \geq 1. \tag{1}$$

Theorem 2. *The class $\mathcal{A}_3$ consists of the groups:*

$$\mathbb{Z}_2^n, \quad \mathbb{Z}_6, \quad \mathbb{Z}_8, \quad n \geq 2. \tag{2}$$

Theorem 3. *All finite Cayley sum integral groups are represented by*

$$\mathbb{Z}_2^n, \quad \mathbb{Z}_4, \quad \mathbb{Z}_6, \quad n \geq 1. \tag{3}$$

2. Cayley Sum Graphs

In this section we recall some results on Cayley sum graphs.

For a finite abelian group G of odd order, since $G = \{2x : x \in G\}$, there exists no Cayley sum graph of G. In fact, an abelian group G has square-free elements if and only if $|G|$ is even, where $|G|$ is the order of G. Thus, in this paper we always consider the finite abelian groups of even order. Observe that $\mathcal{A}_1$ is the set of all finite abelian groups of even order.

Suppose that X is a set. Let $\Omega = \{X_1, X_2, \ldots, X_n\}$ be a family of subsets of X, and f be a complex valued function on X. We denote the sets of integers and complex numbers by $\mathbb{Z}$ and $\mathbb{C}$, respectively. A subset M of X is called *f-integral* if

$$f(M) = \sum_{m \in M} f(m) \in \mathbb{Z}.$$

The *Boolean algebra* generated by Ω in X is the smallest system of subsets of X that contains Ω, and is obtained by arbitrary finite intersections, unions, and complements of the sets. Let G be a finite abelian group. Denote by $\mathbb{B}(G)$ the Boolean algebra generated by all subgroups of G. A *character* of G is a homomorphism from G into the multiplicative group of complex numbers $\mathbb{C} \setminus \{0\}$.

In [10] the authors studied the eigenvalues of a Cayley sum graph.

Proposition 4 [10, Theorem 2.1]**.** *The multiset of eigenvalues of* $\mathrm{CayS}(G, S)$ *is*

$$\{\chi(S) : \chi \text{ is a real character}\} \cup \{\pm|\chi(S)| : \chi \text{ is not a real character}\}.$$

For an elementary abelian 2-group $\mathbb{Z}_2^n$, let $S = \{s_1, \ldots, s_t\}$ be a subset of $\mathbb{Z}_2^n \setminus \{e\}$, where e is the identity element. Then $S = (\langle s_1 \rangle \setminus \{e\}) \cup \cdots \cup (\langle s_t \rangle \setminus \{e\}) \in \mathbb{B}(\mathbb{Z}_2^n)$. It has been shown in [14] that for any character χ of a finite abelian group G, every set in $\mathbb{B}(G)$ is χ-integral. Thus by Proposition 4, we have the next.

Proposition 5. *Let G be a finite abelian group and S a square-free subset of G. If $S \in \mathbb{B}(G)$, then* $\mathrm{CayS}(G, S)$ *is integral. In particular,* $\mathrm{CayS}(\mathbb{Z}_2^n, S)$ *is integral if and only if S does not contain the identity element of $\mathbb{Z}_2^n$.*

Lemma 6 (cf. [6, p. 9])**.** *An n-cycle C_n is integral only for $n = 3, 4,$ or 6.*

Lemma 7 [8, Proposition 2.3]**.** *Let G be an abelian group and S a square-free subset of G. Then* $\mathrm{CayS}(G, S)$ *is connected if and only if $\langle S\rangle = G$ and $|\langle S'\rangle| \geq |G|/2$, where $S' = \{a - b : a, b \in S\}$.*

3. PROOFS OF THE MAIN RESULTS

Denote by $\pi_e(G)$ the set of all orders of elements of a group G. For a graph Γ and a positive integer n, $n\Gamma$ denotes the graph union of n copies of Γ.

Lemma 8. $\mathbb{Z}_2 \times \mathbb{Z}_3^n \in \mathcal{A}_2$ *for each integer $n \geq 1$.*

Proof. Write $G = \mathbb{Z}_2 \times \mathbb{Z}_3^n$. Then G has a unique involution, and $\pi_e(G) = \{1, 2, 3, 6\}$. Let $S := \{a, b\}$ be a square-free subset of size 2 of G.

Case 1. S has an involution. Without loss of generality, let $O(a) = 2$, where $O(a)$ is the order of a. Then $O(b) = 6$ and $a = 3b$. Take any element x in G; one gets that

$$x \sim b - x \sim a - b + x \sim 2b - a - x \sim -2b + x \sim a - x \sim x \tag{4}$$

is a cycle of length 6 in $\mathrm{CayS}(G, S)$. Since $\mathrm{CayS}(G, S)$ is 2-regular, (4) is a connected component of $\mathrm{CayS}(G, S)$. It follows that $\mathrm{CayS}(G, S) \cong 3^{n-1}C_6$. Consequently $\mathrm{CayS}(G, S)$ is integral.

Case 2. S has no involutions. In this case, $O(a) = O(b) = 6$ and $3a = 3b$. For any $x \in G$,

$$x \sim a - x \sim b - a + x \sim 2a - b - x \sim 2b - 2a + x \sim b - x \sim x$$

is a 6-cycle. Similarly to Case 1, we conclude that $\mathrm{CayS}(G, S)$ is integral.

Note that for any subgroup H of G with a square-free subset of size 2, we see that H is isomorphic to a group $\mathbb{Z}_2 \times \mathbb{Z}_3^m$ for some $m \geq 1$. Thus, we have $G \in \mathcal{A}_2$. ■

Lemma 9. *If $G \in \mathcal{A}_2$, then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$.*

Proof. Let g be an element of even order in G. If g is not an involution, then the cycle $C_{O(g)}$ is integral, and so $O(g) = 4$ or 6 by Lemma 6. Suppose, towards a contradiction, that G has an element b with $O(b) \notin \{1, 2, 3, 4, 6\}$. Then $O(b)$ is odd and $O(b) \geq 5$. For an involution a of G, one has $O(a + b) = 2O(b)$, a contradiction. ■

***Proof of Theorem* 1.** Note that there exists precisely one Cayley sum graph $\mathrm{CayS}(\mathbb{Z}_4, \{1,3\})$ of valency 2 on $\mathbb{Z}_4$, which is integral. Note that each subgroup of $\mathbb{Z}_2^n$ is elementary abelian. Then by Lemma 8 and Proposition 5, all groups in (1) belong to $\mathcal{A}_2$.

Suppose that $G \in \mathcal{A}_2$. Since an abelian group is a direct product of some cyclic groups of prime power order, according to Lemma 9, G is isomorphic to one of the following groups:

$$\mathbb{Z}_2^n, \ \ \mathbb{Z}_4^n, \ \ \mathbb{Z}_2^n \times \mathbb{Z}_3^m, \ \ \mathbb{Z}_2^n \times \mathbb{Z}_4^m, \ \ m \geq 1, \ \ n \geq 1.$$

Case 1. $G \cong \mathbb{Z}_4^n$. Suppose that $n \geq 2$. Then G has a subgroup isomorphic to $\mathbb{Z}_4^2$. It follows that $\mathbb{Z}_4^2 \in \mathcal{A}_2$. On the other hand, $\mathrm{CayS}(\mathbb{Z}_4^2, \{(1,0),(0,1)\}) \cong 2C_8$, contrary to Lemma 6. Therefore, in this case we conclude $G \cong \mathbb{Z}_4$.

Case 2. $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_3^m$. Suppose that $n \geq 2$. Note that $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,0), (0,1)\}) \cong C_{12}$. Similarly to Case 1, we get a contradiction.

Case 3. $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_4^m$. Note that G has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_4, \{(1,0),(0,1)\}) \cong C_8$. Similarly to Case 1, we get a contradiction. ∎

Proposition 10. *Let G be an abelian group. Then there is a connected cubic integral Cayley sum graph on G if and only if G is one the following groups:*

$$\mathbb{Z}_2^2, \ \mathbb{Z}_6, \ \mathbb{Z}_8, \ \mathbb{Z}_2^3, \ \mathbb{Z}_2 \times \mathbb{Z}_4, \ \mathbb{Z}_{12}, \ \mathbb{Z}_2 \times \mathbb{Z}_6.$$

In particular, there are exactly five connected cubic integral Cayley sum graphs.

Proof. Let $\mathrm{CayS}(G, S)$ be a connected cubic integral graph. By Schwenk's result [16], independently by Bussemaker and Cvetković [7], there are exactly thirteen cubic connected integral graphs. By checking the list of these thirteen graphs, it follows that

$$|G| \in \{4, 6, 8, 10, 12, 20, 24, 30\}.$$

For each group G of the mentioned orders, finding all 3-element subsets S of G such that all $\mathrm{CayS}(G, S)$ are pairwise non-isomorphic connected integral graphs, we get Table 1.

Note that

$$\begin{aligned} \mathrm{CayS}(\mathbb{Z}_8, \{1,3,5\}) &\cong \mathrm{CayS}(\mathbb{Z}_2^3, \{(1,0,0),(0,1,0),(0,0,1)\}) \\ &\cong \mathrm{CayS}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0),(0,1),(2,1)\}) \end{aligned}$$

and

$$\mathrm{CayS}(\mathbb{Z}_{12}, \{1,3,5\}) \cong \mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2),(1,4),(0,3)\}).$$

We get the desired result. ∎

Table 1. All cubic connected integral Cayley sum graphs.

G	S
$\mathbb{Z}_2^2$	$\{(1,0),(0,1),(1,1)\}$
$\mathbb{Z}_6$	$\{1,3,5\}$
$\mathbb{Z}_8$	$\{1,3,5\}$
$\mathbb{Z}_2^3$	$\{(1,0,0),(0,1,0),(0,0,1)\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\{(1,0),(0,1),(2,1)\}$
$\mathbb{Z}_{12}$	$\{1,3,5\}$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\{(1,0),(1,1),(0,5)\}$, $\{(1,2),(1,4),(0,3)\}$

Lemma 11. *If a group G belongs to $\mathcal{A}_3$, then $\pi_e(G) \subseteq \{1,2,3,4,6,8\}$.*

Proof. Assume that a is a non-identity element of G. We consider two cases.

Case 1. $O(a)$ is odd. Then there exists an element b in G such that $O(b) = 2O(a)$. Note that $O(b) \geq 6$. According to Lemma 7, $\mathrm{CayS}(\langle b\rangle, \{b, 3b, 5b\})$ is a cubic connected graph. It follows from Proposition 10 that $O(a) = 3$, as desired.

Case 2. $O(a)$ is even. Suppose, to derive a contradiction, that $O(a) \notin \{2,4,6,8\}$. By Lemma 7, one gets that $\mathrm{CayS}(\langle a\rangle, \{a, 3a, 5a\})$ is cubic connected. By Proposition 10, one has $O(a) = 12$. It is straightforward to check that $\mathrm{CayS}(\langle a\rangle, \{a, 5a, 11a\})$ is not integral, a contradiction. ∎

***Proof of Theorem* 2.** Firstly, it is easy to check that $\mathbb{Z}_6, \mathbb{Z}_8 \in \mathcal{A}_3$. Thus, by Proposition 5 all groups in (2) belong to $\mathcal{A}_3$.

Suppose that $G \in \mathcal{A}_3$. Then $\pi_e(G) \subseteq \{1,2,3,4,6,8\}$ by Lemma 11.

Case 1. G has an element of order 3. For elements $x, y \in G$, if $O(y)$ and $O(x)$ are relatively prime, then $O(x+y) = O(x)O(y)$. It follows that $\pi_e(G) = \{1,2,3,6\}$. Therefore, $G \cong \mathbb{Z}_6$, or G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_3 \times \mathbb{Z}_6$. If $\mathbb{Z}_3 \times \mathbb{Z}_6$ is a subgroup of G, then by Lemma 7 $\mathrm{CayS}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(0,1),(1,1),(2,3)\})$ is a connected cubic integral graph, contrary to Proposition 10. If $\mathbb{Z}_2 \times \mathbb{Z}_6$ is a subgroup of G, then $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,5),(1,4),(0,1)\})$ is not integral, also a contradiction.

Case 2. G has no elements of order 3. In this case $\pi_e(G) \subseteq \{1,2,4,8\}$. Suppose that G has an element of order 8. Then $G \cong \mathbb{Z}_8$, or G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_8$. Note that $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_8, \{(0,1),(1,1),(0,3)\})$ is not integral. Similarly to Case 1, we get the desired result.

Suppose now that G has no elements of order 8. Then $G \cong \mathbb{Z}_2^n$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$ is a subgroup of G, where $n \geq 2$. Note that $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_8, \{(0,1),(1,1),(1,0)\})$ is not integral. Similarly to Case 1, we end the proof. ∎

***Proof of Theorem* 3.** Clearly, both $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are Cayley sum integral. Thus by Proposition 5, every group in (3) is Cayley sum integral.

Now let G be a finite Cayley sum integral group. Suppose that G has a unique square-free element. Since any element with maximal even order is square-free, every non-identity element is an involution. Then G is an elementary abelian 2-group. This implies that G is isomorphic to $\mathbb{Z}_2$.

Suppose that G has precisely two square-free elements. Then G belongs to $\mathcal{A}_2$. By Theorem 1, one has $G \cong \mathbb{Z}_4$.

Now suppose that the number of square-free elements of G is greater than 2. Then G belongs to $\mathcal{A}_2 \cap \mathcal{A}_3$. In view of Theorems 1 and 2, G is $\mathbb{Z}_2^n$ or $\mathbb{Z}_6$, where $n \geq 2$. ■

Acknowledgement

The authors are grateful to the referees for some helpful suggestions and comments.

This research is supported by National Natural Science Foundation of China (11271047, 11371204), and the Fundamental Research Funds for the Central University of China.

References

[1] A. Abdollahi and M. Jazaeri, *Groups all of whose undirected Cayley graphs are integral*, European J. Combin. **38** (2014) 102–109.
doi:10.1016/j.ejc.2013.11.007

[2] A. Abdollahi and E. Vatandoost, *Which Cayley graphs are integral?*, Electron. J. Combin. **16** (2009) #R122.

[3] A. Ahmady, J.P. Bell and B. Mohar, *Integral Cayley graphs and groups*, SIAM J. Discrete Math. **28** (2014) 685–701.
doi:10.1137/130925487

[4] N. Alon, *Large sets in finite fields are sumsets*, J. Number Theory **126** (2007) 110–118.
doi:10.1016/j.jnt.2006.11.007

[5] M. Amooshahi and B. Taeri, *On Cayley sum graphs of non-Abelian groups*, Graphs Combin. **32** (2016) 17–29.
doi:10.1007/s00373-015-1535-4

[6] A.E. Brouwer and W.H. Haemers, Spectra of Graphs (Springer-Verlag, New York, 2012).
doi:10.1007/978-1-4614-1939-6

[7] F.C. Bussemaker and D.M. Cvetković, *There are exactly* 13 *connected, cubic, integral graphs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **544–576** (1976) 43–48.

[8] B. Cheyne, V. Gupta and C. Wheeler, *Hamilton cycles in addition graphs*, Rose-Hulman Undergrad. Math J. **4** (2003) 1–17.

[9] F.R.K. Chung, *Diameters and eigenvalues*, J. Amer. Math. Soc. **2** (1989) 187–196. doi:10.1090/S0894-0347-1989-0965008-X

[10] M. DeVos, L. Goddyn, B. Mohar and R. Šámal, *Cayley sum graphs and eigenvalues of* $(3,6)$*-fullerenes*, J. Combin. Theory Ser. B **99** (2009) 358–369. doi:10.1016/j.jctb.2008.08.005

[11] I. Estélyi and I. Kovács, *On groups all of whose undirected Cayley graphs of bounded valency are integral*, Electron. J. Combin. **21** (2014) #P4.45.

[12] B.J. Green, *Counting sets with small sumset, and the clique number of random Cayley graphs*, Combinatorica **25** (2005) 307–326. doi:s00493-005-0018-2

[13] F. Harary and A.J. Schwenk, *Which graphs have integral spectra*? in: R.A. Bari and F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Math. **406** (Springer, 1974) 45–51. doi:10.1007/bfb0066434

[14] W. Klotz and T. Sander, *Integral Cayley graphs over abelian groups*, Electron. J. Combin. **17** (2010) #R81.

[15] X. Ma and K. Wang, *On finite groups all of whose cubic Cayley graphs are integral*, J. Algebra Appl. **15** (2016) 1650105, 10 pp. doi:10.1142/S021949881650105X

[16] A.J. Schwenk, *Exactly thirteen connected cubic graphs have integral spectra*, in: Y. Alavi and D.R. Lick (Eds.), Theory and Applications of Graphs, Lecture Notes in Math. **642** (Springer, 1978) 516–533. doi:10.1007/BFb0070407

[17] W. So, *Integral circulant graphs*, Discrete Math. **306** (2006) 153–158. doi:10.1016/j.disc.2005.11.006

Received 6 January 2015
Revised 12 October 2015
Accepted 27 November 2015