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INTEGRAL CAYLEY SUM GRAPHS AND GROUPS

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Abstract

For any positive integer k, let $\mathcal{A}_k$ denote the set of finite abelian groups G such that for any subgroup H of G all Cayley sum graphs $\mathrm{CayS}(H,S)$ are integral if $|S|=k$. A finite abelian group G is called Cayley sum integral if for any subgroup H of G all Cayley sum graphs on H are integral. In this paper, the classes $\mathcal{A}_2$ and $\mathcal{A}_3$ are classified. As an application, we determine all finite Cayley sum integral groups.

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1. Introduction

A graph is *integral* if all its eigenvalues are integers. Harary and Schwenk [13] introduced integral graphs, and proposed the problem of classifying integral graphs. There are some constructions of graphs from groups in the literature; for example, Cayley graphs, which are integral were studied in [1, 2, 3, 11, 14, 15].

Let G be a finite abelian group. A subset S of G is said to be *square-free* if $x+x \notin S$ for each $x \in G$. The *Cayley sum graph* of G with respect to a square-free subset S of G, denoted by $\mathrm{CayS}(G,S)$, is a simple graph with vertex set G and two distinct vertices x and y form an edge if $x+y \in S$. Some results on Cayley sum graphs can be found in [4, 5, 9, 12, 17].

For any positive integer k, let $\mathcal{A}_k$ denote the set of finite abelian groups G such that for any subgroup H of G all Cayley sum graphs $\mathrm{CayS}(H,S)$ are integral if $|S|=k$. A finite abelian group G is called *Cayley sum integral* if for any subgroup H of G all Cayley sum graphs on H are integral.

In the paper we classify the classes $\mathcal{A}_2$ and $\mathcal{A}_3$. As an application, all finite Cayley sum integral groups are determined. Our main results are the following.

Theorem 1. *The class $\mathcal{A}_2$ consists of the groups:*

$$\mathbb{Z}_2^n, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_3^m, \quad n \geq 2, \quad m \geq 1. \tag{1}$$

Theorem 2. *The class $\mathcal{A}_3$ consists of the groups:*

$$\mathbb{Z}_2^n, \quad \mathbb{Z}_6, \quad \mathbb{Z}_8, \quad n \geq 2. \tag{2}$$

Theorem 3. *All finite Cayley sum integral groups are represented by*

$$\mathbb{Z}_2^n, \quad \mathbb{Z}_4, \quad \mathbb{Z}_6, \quad n \geq 1. \tag{3}$$

2. Cayley Sum Graphs

In this section we recall some results on Cayley sum graphs.

For a finite abelian group G of odd order, since $G = \{2x : x \in G\}$, there exists no Cayley sum graph of G. In fact, an abelian group G has square-free elements if and only if $|G|$ is even, where $|G|$ is the order of G. Thus, in this paper we always consider the finite abelian groups of even order. Observe that $\mathcal{A}_1$ is the set of all finite abelian groups of even order.

Suppose that X is a set. Let $\Omega = \{X_1, X_2, \ldots, X_n\}$ be a family of subsets of X, and f be a complex valued function on X. We denote the sets of integers and complex numbers by $\mathbb{Z}$ and $\mathbb{C}$, respectively. A subset M of X is called *f-integral* if

$$f(M) = \sum_{m \in M} f(m) \in \mathbb{Z}.$$

The *Boolean algebra* generated by Ω in X is the smallest system of subsets of X that contains Ω, and is obtained by arbitrary finite intersections, unions, and complements of the sets. Let G be a finite abelian group. Denote by $\mathbb{B}(G)$ the Boolean algebra generated by all subgroups of G. A *character* of G is a homomorphism from G into the multiplicative group of complex numbers $\mathbb{C} \setminus \{0\}$.

In [10] the authors studied the eigenvalues of a Cayley sum graph.

Proposition 4 [10, Theorem 2.1]**.** *The multiset of eigenvalues of* $\mathrm{CayS}(G, S)$ *is*

$$\{\chi(S) : \chi \text{ is a real character}\} \cup \{\pm|\chi(S)| : \chi \text{ is not a real character}\}.$$

For an elementary abelian 2-group $\mathbb{Z}_2^n$, let $S = \{s_1, \ldots, s_t\}$ be a subset of $\mathbb{Z}_2^n \setminus \{e\}$, where e is the identity element. Then $S = (\langle s_1 \rangle \setminus \{e\}) \cup \cdots \cup (\langle s_t \rangle \setminus \{e\}) \in \mathbb{B}(\mathbb{Z}_2^n)$. It has been shown in [14] that for any character χ of a finite abelian group G, every set in $\mathbb{B}(G)$ is χ-integral. Thus by Proposition 4, we have the next.

Proposition 5. *Let G be a finite abelian group and S a square-free subset of G. If $S \in \mathbb{B}(G)$, then* $\mathrm{CayS}(G, S)$ *is integral. In particular,* $\mathrm{CayS}(\mathbb{Z}_2^n, S)$ *is integral if and only if S does not contain the identity element of $\mathbb{Z}_2^n$.*

Lemma 6 (cf. [6, p. 9])**.** *An n-cycle C_n is integral only for $n = 3, 4,$ or 6.*

Lemma 7 [8, Proposition 2.3]**.** *Let G be an abelian group and S a square-free subset of G. Then* $\mathrm{CayS}(G, S)$ *is connected if and only if $\langle S\rangle = G$ and $|\langle S'\rangle| \geq |G|/2$, where $S' = \{a - b : a, b \in S\}$.*

3. PROOFS OF THE MAIN RESULTS

Denote by $\pi_e(G)$ the set of all orders of elements of a group G. For a graph Γ and a positive integer n, $n\Gamma$ denotes the graph union of n copies of Γ.

Lemma 8. $\mathbb{Z}_2 \times \mathbb{Z}_3^n \in \mathcal{A}_2$ *for each integer $n \geq 1$.*

Proof. Write $G = \mathbb{Z}_2 \times \mathbb{Z}_3^n$. Then G has a unique involution, and $\pi_e(G) = \{1, 2, 3, 6\}$. Let $S := \{a, b\}$ be a square-free subset of size 2 of G.

Case 1. S has an involution. Without loss of generality, let $O(a) = 2$, where $O(a)$ is the order of a. Then $O(b) = 6$ and $a = 3b$. Take any element x in G; one gets that

$$x \sim b - x \sim a - b + x \sim 2b - a - x \sim -2b + x \sim a - x \sim x \tag{4}$$

is a cycle of length 6 in $\mathrm{CayS}(G, S)$. Since $\mathrm{CayS}(G, S)$ is 2-regular, (4) is a connected component of $\mathrm{CayS}(G, S)$. It follows that $\mathrm{CayS}(G, S) \cong 3^{n-1}C_6$. Consequently $\mathrm{CayS}(G, S)$ is integral.

Case 2. S has no involutions. In this case, $O(a) = O(b) = 6$ and $3a = 3b$. For any $x \in G$,

$$x \sim a - x \sim b - a + x \sim 2a - b - x \sim 2b - 2a + x \sim b - x \sim x$$

is a 6-cycle. Similarly to Case 1, we conclude that $\mathrm{CayS}(G, S)$ is integral.

Note that for any subgroup H of G with a square-free subset of size 2, we see that H is isomorphic to a group $\mathbb{Z}_2 \times \mathbb{Z}_3^m$ for some $m \geq 1$. Thus, we have $G \in \mathcal{A}_2$. ■

Lemma 9. *If $G \in \mathcal{A}_2$, then $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$.*

Proof. Let g be an element of even order in G. If g is not an involution, then the cycle $C_{O(g)}$ is integral, and so $O(g) = 4$ or 6 by Lemma 6. Suppose, towards a contradiction, that G has an element b with $O(b) \notin \{1, 2, 3, 4, 6\}$. Then $O(b)$ is odd and $O(b) \geq 5$. For an involution a of G, one has $O(a + b) = 2O(b)$, a contradiction. ■

***Proof of Theorem* 1.** Note that there exists precisely one Cayley sum graph $\mathrm{CayS}(\mathbb{Z}_4, \{1,3\})$ of valency 2 on $\mathbb{Z}_4$, which is integral. Note that each subgroup of $\mathbb{Z}_2^n$ is elementary abelian. Then by Lemma 8 and Proposition 5, all groups in (1) belong to $\mathcal{A}_2$.

Suppose that $G \in \mathcal{A}_2$. Since an abelian group is a direct product of some cyclic groups of prime power order, according to Lemma 9, G is isomorphic to one of the following groups:

$$\mathbb{Z}_2^n,\ \ \mathbb{Z}_4^n,\ \ \mathbb{Z}_2^n \times \mathbb{Z}_3^m,\ \ \mathbb{Z}_2^n \times \mathbb{Z}_4^m,\ \ m \geq 1,\ \ n \geq 1.$$

Case 1. $G \cong \mathbb{Z}_4^n$. Suppose that $n \geq 2$. Then G has a subgroup isomorphic to $\mathbb{Z}_4^2$. It follows that $\mathbb{Z}_4^2 \in \mathcal{A}_2$. On the other hand, $\mathrm{CayS}(\mathbb{Z}_4^2, \{(1,0),(0,1)\}) \cong 2C_8$, contrary to Lemma 6. Therefore, in this case we conclude $G \cong \mathbb{Z}_4$.

Case 2. $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_3^m$. Suppose that $n \geq 2$. Note that $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,0), (0,1)\}) \cong C_{12}$. Similarly to Case 1, we get a contradiction.

Case 3. $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_4^m$. Note that G has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_4, \{(1,0),(0,1)\}) \cong C_8$. Similarly to Case 1, we get a contradiction. ■

Proposition 10. *Let G be an abelian group. Then there is a connected cubic integral Cayley sum graph on G if and only if G is one the following groups:*

$$\mathbb{Z}_2^2,\ \mathbb{Z}_6,\ \mathbb{Z}_8,\ \mathbb{Z}_2^3,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_{12},\ \mathbb{Z}_2 \times \mathbb{Z}_6.$$

In particular, there are exactly five connected cubic integral Cayley sum graphs.

Proof. Let $\mathrm{CayS}(G, S)$ be a connected cubic integral graph. By Schwenk's result [16], independently by Bussemaker and Cvetković [7], there are exactly thirteen cubic connected integral graphs. By checking the list of these thirteen graphs, it follows that

$$|G| \in \{4, 6, 8, 10, 12, 20, 24, 30\}.$$

For each group G of the mentioned orders, finding all 3-element subsets S of G such that all $\mathrm{CayS}(G, S)$ are pairwise non-isomorphic connected integral graphs, we get Table 1.

Note that

$$\begin{aligned}\mathrm{CayS}(\mathbb{Z}_8, \{1,3,5\}) &\cong \mathrm{CayS}(\mathbb{Z}_2^3, \{(1,0,0),(0,1,0),(0,0,1)\})\\ &\cong \mathrm{CayS}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0),(0,1),(2,1)\})\end{aligned}$$

and

$$\mathrm{CayS}(\mathbb{Z}_{12}, \{1,3,5\}) \cong \mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2),(1,4),(0,3)\}).$$

We get the desired result. ■

Table 1. All cubic connected integral Cayley sum graphs.

G	S
$\mathbb{Z}_2^2$	$\{(1,0),(0,1),(1,1)\}$
$\mathbb{Z}_6$	$\{1,3,5\}$
$\mathbb{Z}_8$	$\{1,3,5\}$
$\mathbb{Z}_2^3$	$\{(1,0,0),(0,1,0),(0,0,1)\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\{(1,0),(0,1),(2,1)\}$
$\mathbb{Z}_{12}$	$\{1,3,5\}$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\{(1,0),(1,1),(0,5)\}$, $\{(1,2),(1,4),(0,3)\}$

Lemma 11. *If a group G belongs to $\mathcal{A}_3$, then $\pi_e(G) \subseteq \{1,2,3,4,6,8\}$.*

Proof. Assume that a is a non-identity element of G. We consider two cases.

Case 1. $O(a)$ is odd. Then there exists an element b in G such that $O(b) = 2O(a)$. Note that $O(b) \geq 6$. According to Lemma 7, $\mathrm{CayS}(\langle b\rangle, \{b, 3b, 5b\})$ is a cubic connected graph. It follows from Proposition 10 that $O(a) = 3$, as desired.

Case 2. $O(a)$ is even. Suppose, to derive a contradiction, that $O(a) \notin \{2,4,6,8\}$. By Lemma 7, one gets that $\mathrm{CayS}(\langle a\rangle, \{a, 3a, 5a\})$ is cubic connected. By Proposition 10, one has $O(a) = 12$. It is straightforward to check that $\mathrm{CayS}(\langle a\rangle, \{a, 5a, 11a\})$ is not integral, a contradiction. ∎

Proof of Theorem 2. Firstly, it is easy to check that $\mathbb{Z}_6, \mathbb{Z}_8 \in \mathcal{A}_3$. Thus, by Proposition 5 all groups in (2) belong to $\mathcal{A}_3$.

Suppose that $G \in \mathcal{A}_3$. Then $\pi_e(G) \subseteq \{1,2,3,4,6,8\}$ by Lemma 11.

Case 1. G has an element of order 3. For elements $x, y \in G$, if $O(y)$ and $O(x)$ are relatively prime, then $O(x+y) = O(x)O(y)$. It follows that $\pi_e(G) = \{1,2,3,6\}$. Therefore, $G \cong \mathbb{Z}_6$, or G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_3 \times \mathbb{Z}_6$. If $\mathbb{Z}_3 \times \mathbb{Z}_6$ is a subgroup of G, then by Lemma 7 $\mathrm{CayS}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(0,1),(1,1),(2,3)\})$ is a connected cubic integral graph, contrary to Proposition 10. If $\mathbb{Z}_2 \times \mathbb{Z}_6$ is a subgroup of G, then $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,5),(1,4),(0,1)\})$ is not integral, also a contradiction.

Case 2. G has no elements of order 3. In this case $\pi_e(G) \subseteq \{1,2,4,8\}$. Suppose that G has an element of order 8. Then $G \cong \mathbb{Z}_8$, or G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_8$. Note that $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_8, \{(0,1),(1,1),(0,3)\})$ is not integral. Similarly to Case 1, we get the desired result.

Suppose now that G has no elements of order 8. Then $G \cong \mathbb{Z}_2^n$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$ is a subgroup of G, where $n \geq 2$. Note that $\mathrm{CayS}(\mathbb{Z}_2 \times \mathbb{Z}_8, \{(0,1),(1,1),(1,0)\})$ is not integral. Similarly to Case 1, we end the proof. ∎

Proof of Theorem 3. Clearly, both $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are Cayley sum integral. Thus by Proposition 5, every group in (3) is Cayley sum integral.

Now let G be a finite Cayley sum integral group. Suppose that G has a unique square-free element. Since any element with maximal even order is square-free, every non-identity element is an involution. Then G is an elementary abelian 2-group. This implies that G is isomorphic to $\mathbb{Z}_2$.

Suppose that G has precisely two square-free elements. Then G belongs to $\mathcal{A}_2$. By Theorem 1, one has $G \cong \mathbb{Z}_4$.

Now suppose that the number of square-free elements of G is greater than 2. Then G belongs to $\mathcal{A}_2 \cap \mathcal{A}_3$. In view of Theorems 1 and 2, G is $\mathbb{Z}_2^n$ or $\mathbb{Z}_6$, where $n \geq 2$. ∎

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