Discussiones Mathematicae
Graph Theory 36 (2016) 773–778
doi:10.7151/dmgt.1885

Note

SPANNING TREES WHOSE STEMS HAVE A BOUNDED NUMBER OF BRANCH VERTICES

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Abstract

Let T be a tree, a vertex of degree one and a vertex of degree at least three is called a leaf and a branch vertex, respectively. The set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the stem of T and denoted by $Stem(T)$. In this paper, we give two sufficient conditions for a connected graph to have a spanning tree whose stem has a bounded number of branch vertices, and these conditions are best possible.

Keywords: spanning tree, stem, branch vertex.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For two vertices u and v of G, the distance between u and v in G is denoted by $d_G(u, v)$. For an integer $l \geq 2$, let $\alpha^l(G)$ denote the number defined by

$$\alpha^l(G) = \max\{|S| : S \subset V(G), d_G(x, y) \geq l \text{ for all distinct } x, y \in S\}.$$

For an integer $k \geq 2$, we define

$$\sigma_k^l(G) = \min\left\{\sum_{x \in S} \deg_G(x) : S \subset V(G),\ |S| = k,\ d_G(x, y) \geq l \text{ for all distinct } x, y \in S\right\}.$$

For convenience, we define $\sigma_k^l(G) = \infty$ if $\alpha^l(G) < k$. Note that $\alpha^2(G)$ is often written $\alpha(G)$, which is the *independence number* of G, and $\sigma_k^2(G)$ is often written $\sigma_k(G)$, which is the minimum degree sum of k independent vertices.

For a tree T, a vertex of degree at least three is called a *branch vertex*, and a tree having at most one branch vertex is called a *spider*. Many researchers have investigated the independence number conditions and the degree sum conditions for the existence of a spanning tree with bounded number of branch vertices [1, 2, 3, 4, 7, 8]. A vertex of T, which has degree one, is often called a *leaf* of T, and the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the *stem* of T and is denote by $Stem(T)$. A spanning tree with specified stem was first considered in [5], and the following theorem was obtained.

Theorem 1 (Kano, Tsugaki and Yan [5])**.** *Let $k \geq 2$ be an integer, and G be a connected graph. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has maximum degree at most k.*

The following theorems give two sufficient conditions for a connected graph to have a spanning tree whose stem has a few number of leaves.

Theorem 2 (Tsugaki and Zhang [9])**.** *Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree whose stem has at most k leaves.*

Theorem 3 (Kano and Yan [6])**.** *Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has at most k leaves.*

In this paper, we give two sufficient conditions for a connected graph to have a spanning tree whose stem has a bounded number of branch vertices, and these conditions are best possible.

Theorem 4. *Let G be a connected graph and k be a non-negative integer. If one of the following conditions holds, then G has a spanning tree whose stem has at most k branch vertices.*

(i) $\alpha^4(G) \leq k + 2$.

(ii) $\sigma_{k+3}^4 \geq |G| - 2k - 3$.

Before proving Theorem 4, we first show that the conditions of Theorem 4 are best possible. Let $m, k \geq 1$ be integers, and let $D_0, D_1, \ldots, D_{k+2}$ be disjoint copies of K_m. Let $P = z_1 z_2, \ldots, z_{k+1}$ be a path. Let $v_0, v_1, \ldots, v_{k+2}$ be vertices not contained in $D_0 \cup D_1 \cdots \cup D_{k+2}$. Join z_i, v_i to all the vertices of D_i $(1 \leq i \leq k+1)$ by edges, and join z_1, v_0 (z_{k+1}, v_{k+2}) to all vertices of D_0 (D_{k+1}) by edges, respectively. Let G denote the resulting graph. Then G satisfies $\alpha^4(G) = k + 3$ and $\sigma_{k+3}^4(G) = |G| - 2k - 4$. Since for any spanning tree T of G, $z_1, z_2, \ldots, z_{k+1}$ have to be the branch vertices of $Stem(T)$, G has no spanning tree whose stem has at most k branch vertices.

2. Proof of Theorem 4

In order to prove Theorem 4, we need the following lemma.

Lemma 5. *Let T be a tree, and let X be the set of vertices of degree at least 3. Then the number of leaves in T is counted as follows:*

$$|Leaf(T)| = \sum_{x \in X} (\deg_T(x) - 2) + 2.$$

***Proof of Theorem* 4.** Assume that G satisfies the conditions in Theorem 4 and does not have a spanning tree whose stem has at most k branch vertices. We choose a tree T whose stem has k branch vertices in G so that

(T1) $|T|$ is as large as possible.
(T2) $|Leaf(Stem(T))|$ is as small as possible subject to (T1).
(T3) $|Stem(T)|$ is as small as possible subject to (T1) and (T2).

For the remaining of the proof v is a vertex of G not in T. By the choice (T1), we have the following claim.

Claim 1. For every $v \in V(G) - V(T)$, $N_G(v) \subseteq Leaf(T) \cup (V(G) - V(T))$.

$Stem(T)$ has k branch vertices. Denote the number of leaves of $Stem(T)$ by l. By Lemma 5, $|Leaf(Stem(T))| = l \geq k + 2$. Let $x_1, x_2, \ldots, x_l$ be the leaves of $Stem(T)$. Since T is not a spanning tree of G, there exist two vertices $v \in V(G) - V(T)$ and $u \in Leaf(T)$ which are adjacent in G.

By the choice (T2), we have the following claim.

Claim 2. $Leaf(Stem(T))$ is an independent set of G.

Proof. Assume that there exists two vertices x_i and x_j of $Leaf(Stem(T))$ adjacent in G. Then add $x_i x_j$ to T. The resulting subgraph of G includes the unique cycle, which contains an edge e_1 of $Stem(T)$ incident with a branch vertex. By removing the edge e_1, we obtain a tree T^* whose stem has at most k branch vertices, $|T^*| = |T|$ and $|Leaf(Stem(T^*))| \leq |Leaf(Stem(T))| - 1$. If $Stem(T^*)$ has $k-1$ branch vertices, then add uv to T^*; we obtain a tree whose stem has at most k branch vertex and the order of the tree is greater than $|T|$, which contradicts the condition (T1). Otherwise, T^* contradicts the condition (T2). Hence $Leaf(Stem(T))$ is an independent set of G. □

Claim 3. For every x_i $(1 \leq i \leq l)$, there exists a vertex $y_i \in Leaf(T)$ adjacent to x_i and $N_G(y_i) \subset Leaf(T) \cup \{x_i\}$.

Proof. It is easy to see that for every leaf y of T adjacent to a leaf of $Stem(T)$ in T, y is not adjacent to any vertex of $V(G) - V(T)$ since otherwise we can add an edge joining y to a vertex of $V(G) - V(T)$ to T.

Suppose that for some $1 \leq i \leq l$, each leave y_{i_j} of T adjacent to x_i is also adjacent to a vertex $z_{i_j} \in (Stem(T) - \{x_i\})$. Then for every leaf y_{i_j} adjacent to x_i in T, remove the edge $y_{i_j}x_i$ from T and add the edge $y_{i_j}z_{i_j}$. Denote the resulting tree of G by T_1. Then T_1 is a tree whose stem has at most k branch vertices. If x_i is adjacent with a branch of $Stem(T)$, then $Leaf(Stem(T_1)) = Leaf(Stem(T)) - \{x_i\}$, which contradicts the condition (T2). If x_i is not adjacent with a branch of $Stem(T)$, then $Stem(T_1) = Stem(T) - \{x_i\}$, which contradicts the condition (T3). Therefore, the claim holds. □

Claim 4. For any two distinct vertices $y, z \in \{v, y_1, y_2, \ldots, y_l\}$, $d_G(y, z) \geq 4$.

Proof. First, we show that $d_G(v, y_i) \geq 4$ for every $1 \leq i \leq l$. Let P_i be a shortest path connecting v and y_i in G. Then there exists a vertex $s \in V(P_i)$ with $s \in V(Stem(T)) - \{x_i\}$. Otherwise, all vertices of P_i between v and y_i are contained in $Leaf(T) \cup (V(G) - V(T)) \cup \{x_i\}$. Then add P_i to T (if P_i passes through x_i, we just add the segment of P_i between v and x_i) and remove the edges of T joining $V(P_i \cap Leaf(T))$ to $V(Stem(T))$ except the edge y_ix_i. Then resulting tree of G is a tree whose stem has at most k branch vertices and the order of the resulting tree is greater than $|T|$, which contradicts the condition (T1).

Hence, by Claim 3, $d_G(v, s) \geq 2$ and $d_G(s, y_i) \geq 2$. Therefore $d_G(v, y_i) = d_G(v, s) + d_G(s, y_i) \geq 4$.

Next, we show that $d_G(y_i, y_j) \geq 4$ for all $1 \leq i < j \leq l$. Let P_{ij} be the shortest path connecting y_i and y_j in G. Then there exists a vertex $t \in V(P_{ij})$ with $t \in V(Stem(T)) - \{x_i, x_j\}$. Otherwise, all vertices of P_{ij} between y_i and y_j are contained in $Leaf(T) \cup (V(G) - V(T)) \cup \{x_i, x_j\}$. If P_{ij} passes through x_i (or x_j), then $y_ix_i \in E(P_{ij})$ (or $y_jx_j \in E(P_{ij})$), respectively.

Then add P_{ij} to T and remove the edges of T joining $V(P_{ij} \cap Leaf(T))$ to $V(Stem(T))$ except the edges y_ix_i and y_jx_j. Then the resulting subgraph of G includes the unique cycle, which contains an edge e_2 of $Stem(T)$ incident with a branch vertex. By removing the edge e_2, we obtain a tree T_2 whose stem has at most k branch vertices. If P_{ij} contains a vertex of $V(G) - V(T)$, then the order of T_2 is greater than $|T|$, which contradicts the condition (T1). Otherwise, $|T_2| = |T|$ and $|Leaf(Stem(T_2))| = |Leaf(Stem(T))| - 1$. This contradicts the condition (T2). Hence P_{ij} passes through a vertex s in $Stem(T) - \{x_i, x_j\}$.

Hence, by Claims 1 and 3, $d_G(y_i, s) \geq 2$ and $d_G(s, y_j) \geq 2$. Therefore $d_G(y_i, y_j) = d_G(y_i, s) + d_G(s, y_j) \geq 4$ for $1 \leq i < j \leq k$. □

By Claim 4, we have $\alpha^4(G) \geq l+1 \geq k+3$, which contradicts the condition (i). Next, by Claim 4, we can obtain Claim 5.

Claim 5. (i) $N_G(v) \cap N_G(y_i) = \emptyset$ for $1 \leq i \leq l$; and (ii) $N_G(y_i) \cap N_G(y_j) = \emptyset$ for $1 \leq i \neq j \leq l$.

Claim 6. There exists one vertex $w \in Stem(T)$ with $\deg_{Stem(T)}(w) = 2$.

Proof. Otherwise, all vertices of $Stem(T)$ are leaves or branch vertices of $Stem(T)$. If u is adjacent to a leaf or branch vertex of $Stem(T)$, then we add v to T by adding edge uv; we can get a tree $T + uv$ whose stem has k branch vertices and $|T + uv| = |T| + 1$, which contradicts (T1). □

By Claim 6, we have $|Stem(T)| \geq l + k + 1$.

Denote $Y = \{y_1, y_2, \ldots, y_l\}$. By Claims 1–5, we have

$$\begin{aligned} N_G(v) &\subseteq (V(G) - V(T) - \{v\}) \cup (N_G(v) \cap (Leaf(T) - Y)), \\ \bigcup_{i=1}^{k+2} N_G(y_i) &\subseteq (Leaf(T) - Y - N_G(v)) \cup \{x_1, \ldots, x_{k+2}\}. \end{aligned}$$

Hence by letting $m = |N_G(v) \cap (Leaf(T) - Y)|$, we have

$$\begin{aligned} \deg_G(v) + \sum_{i=1}^{k+2} \deg_G(y_i) &\leq |G| - |T| - 1 + m + |Leaf(T)| - m - l + k + 2 \\ &= |G| - |Stem(T)| - l + k + 1 \\ &\leq |G| - 2l \leq |G| - 2k - 4. \end{aligned}$$

Which contradicts the condition (ii) of theorem.

The theorem follows since we either reach a contradiction to condition (i) or a contradiction to condition (ii). ■

Acknowledgement

The author would like to thank Professor Mikio Kano for his valuable comments. The author is graceful to the referees for careful reading and useful comments. The author was supported by the Natural Science Foundation of China (61273179), the Natural Science Foundation of Hubei (2014CFB248), the Doctoral Fund of Yangtze University (80107001), the Yangtze Youth Fund (70107021), Open Research Fund Program of Institute of Applied Mathematics Yangtze University(KF1601

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Received 1 June 2015
Revised 29 October 2015
Accepted 19 November 2015