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SUM LIST EDGE COLORINGS OF GRAPHS

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Abstract

Let $G = (V, E)$ be a simple graph and for every edge $e \in E$ let $L(e)$ be a set (list) of available colors. The graph G is called *L-edge colorable* if there is a proper edge coloring c of G with $c(e) \in L(e)$ for all $e \in E$. A function $f : E \to \mathbb{N}$ is called an *edge choice function* of G and G is said to be *f-edge choosable* if G is L-edge colorable for every list assignment L with $|L(e)| = f(e)$ for all $e \in E$. Set $\text{size}(f) = \sum_{e \in E} f(e)$ and define the *sum choice index* $\chi'_{sc}(G)$ as the minimum of size(f) over all edge choice functions f of G.

There exists a greedy coloring of the edges of G which leads to the upper bound $\chi'_{sc}(G) \leq \frac{1}{2} \sum_{v \in V} d(v)^2$. A graph is called *sec-greedy* if its sum choice index equals this upper bound.

We present some general results on the sum choice index of graphs including a lower bound and we determine this index for several classes of graphs. Moreover, we present classes of *sec*-greedy graphs as well as all such graphs of order at most 5.

Keywords: sum list edge coloring, sum choice index, sum list coloring, sum choice number, choice function, line graph.

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1. Introduction

Let $G = (V, E)$ be a simple graph and L a *list assignment* of G, that is, a function that assigns a set $L(v)$ of available colors to every vertex $v \in V$. The graph G is called *L-colorable* if there is a coloring c of the vertices of G with $c(v) \in L(v)$ for all $v \in V$ and $c(v) \neq c(w)$ for all $vw \in E$.

A function $f : V \to \mathbb{N}$ is called a *choice function* of G if G is L-colorable for every list assignment L with $|L(v)| = f(v)$ for all $v \in V$. In this case G is called *f-choosable*.

If $f(v) = k$ for all $v \in V$, then G is called *k-choosable*. The *list chromatic number* or *choice number* $\mathrm{ch}(G)$ of G is the minimum k such that G is k-choosable.

If we set $\mathrm{size}(f) = \sum_{v \in V} f(v)$, then the *sum choice number* $\chi_{sc}(G)$ is defined as minimum of $\mathrm{size}(f)$ over all choice functions f of G.

It is easy to see by a greedy coloring that $\chi_{sc}(G) \leq |V| + |E|$ (see [5, 1]). A graph G is called *sc-greedy* if its sum choice number equals this *greedy-bound* $\mathrm{GB}(G) = |V| + |E|$.

Sum list colorings were introduced by Isaak in 2002 [4]. It is known that paths, cycles, trees, complete graphs, and all graphs with at most four vertices are *sc*-greedy. In [10] the sum choice number of all graphs with five vertices is determined. It turned out that exactly the complete bipartite graph $K_{2,3}$, the complete graph K_4 with a subdivided edge, and the wheel W_4 of order 5 are not *sc*-greedy. It was shown in [3] that a broken wheel BW_ℓ (a wheel W_ℓ without an edge of the cycle) is *sc*-greedy if and only if $\ell \leq 9$, and in [8] that a wheel W_ℓ is *sc*-greedy if and only if $\ell \leq 3$ or $\ell = 5$. In [6] it was shown that $\chi_{sc}(G) \geq 2|V| - 1$ for a connected graph G and all connected graphs whose sum choice number attains the lower bound $2|V| - 1$ or $2|V|$ were determined. Moreover, a characterization of all *sc*-greedy complete multipartite graphs is given in [6].

In this paper we transfer the concept of choice functions and sum choice number to edge colorings of graphs.

A graph $G = (V, E)$ is called *L-edge colorable* if there is a proper coloring c of the edges of G with $c(e) \in L(e)$ for all $e \in E$ where $L(e)$ is a set (list) of available colors of e.

A function $f : E \to \mathbb{N}$ is called an *edge choice function* of G if G is L-edge colorable for every list assignment L with $|L(e)| = f(e)$ for all $e \in E$. The graph G is called *f-edge choosable*. If we define $\mathrm{size}(f) = \sum_{e \in E} f(e)$, then the *sum choice index* $\chi'_{sc}(G)$ is defined as the minimum of $\mathrm{size}(f)$ over all edge choice functions f of G.

An *empty graph* has no edges, a *non-empty graph* contains at least one edge. The *line graph* $L(G)$ of a non-empty graph G is the graph with vertex set E in which two vertices are adjacent if and only if the corresponding edges are adjacent in G. Since an edge coloring of G corresponds to a vertex coloring of $L(G)$ it holds that $\chi'_{sc}(G) = \chi_{sc}(L(G))$, that is, known results for the sum choice number can be used to determine the sum choice index of graphs (see also [5]).

If G is empty, then $\chi'_{sc}(G) = 0 = \frac{1}{2}\sum_{v\in V(G)} d(v)^2$. If G is non-empty, then $\chi'_{sc}(G) = \chi_{sc}(L(G)) \leq |V(L(G))| + |E(L(G))| = |E(G)| + \frac{1}{2}\sum_{uv\in E(G)} d_{L(G)}(uv) = |E(G)| + \frac{1}{2}\sum_{uv\in E(G)}(d(u) - 1 + d(v) - 1) = |E(G)| + \frac{1}{2}\sum_{v\in V(G)} d(v)(d(v) - 1) = |E(G)| - |E(G)| + \frac{1}{2}\sum_{v\in V(G)} d(v)^2 = \frac{1}{2}\sum_{v\in V(G)} d(v)^2$ (see [2], p. 72).

Therefore, we obtain an upper bound for $\chi'_{sc}(G)$ and denote this bound *edge greedy bound*

$$\mathrm{GB}'(G) = \frac{1}{2}\sum_{v\in V(G)} d(v)^2$$

of G. A graph is called *sec-greedy* (*sum edge choice greedy*) if the sum choice index equals this upper bound, that is, $\chi'_{sc}(G) = \mathrm{GB}'(G)$.

Remark 1. A non-empty graph G is *sec*-greedy if and only if $L(G)$ is *sc*-greedy.

In Section 2 we present some general results on the sum choice index whereas in Sections 3 and 4 we determine the sum choice index of some graph classes. Moreover, we present classes of *sec*-greedy graphs as well as all such graphs of order at most 5.

2. General Results

Several general results for the sum choice number can be transferred to the sum choice index of a graph.

Proposition 2. *If $H \subseteq G$, then $\chi'_{sc}(H) \leq \chi'_{sc}(G)$.*

Proof. If H is empty, then $0 = \chi'_{sc}(H) \leq \chi'_{sc}(G)$. If H is non-empty and f is an edge choice function of G with $\mathrm{size}(f) = \chi'_{sc}(G)$, then $f' = f|_{E(H)}$ (f restricted to $E(H)$) is an edge choice function of H with $\mathrm{size}(f') \leq \mathrm{size}(f)$ which implies $\chi'_{sc}(H) \leq \chi'_{sc}(G)$. ∎

Due to this result the sum list edge k-colorability ($\chi'_{sc}(G) \leq k$) is a hereditary property.

Proposition 3. *For the disjoint union of two graphs it holds that $\chi'_{sc}(G_1 \cup G_2) = \chi'_{sc}(G_1) + \chi'_{sc}(G_2)$.*

Proof. Let G_1 and G_2 be two non-empty graphs (the result is obvious if G_1 or G_2 is empty since the sum choice index of empty graphs is 0).

If f is an edge choice function of $G_1 \cup G_2$, then $f_i = f|_{E(G_i)}$ is an edge choice function of G_i, $i = 1, 2$, with $\mathrm{size}(f) = \mathrm{size}(f_1) + \mathrm{size}(f_2)$. Since $\mathrm{size}(f_i) \geq \chi'_{sc}(G_i)$ it follows that $\chi'_{sc}(G_1 \cup G_2) \geq \chi'_{sc}(G_1) + \chi'_{sc}(G_2)$.

On the other hand, if f_i is an edge choice function of G_i with $\mathrm{size}(f_i) = \chi'_{sc}(G_i)$, $i = 1, 2$, then these two functions can be used to generate an edge choice function f of $G_1 \cup G_2$ with $\mathrm{size}(f) = \mathrm{size}(f_1) + \mathrm{size}(f_2)$. Thus $\chi'_{sc}(G_1 \cup G_2) \leq \chi'_{sc}(G_1) + \chi'_{sc}(G_2)$. ■

This result implies that one only needs to consider connected graphs since the sum choice index of a graph is the sum of the sum choice indexes of its components.

Proposition 4. *Let G be a connected graph with a bridge e, E_1 and E_2 be the edge sets of the two components of $G - e$, and $G_i = G[E_i \cup \{e\}]$, $i = 1, 2$. Then it holds that $\chi'_{sc}(G) = \chi'_{sc}(G_1) + \chi'_{sc}(G_2) - 1$.*

Proof. This follows directly from $\chi_{sc}(H_1 \cup H_2) = \chi_{sc}(H_1) + \chi_{sc}(H_2) - 1$ for the non-disjoint union of two graphs H_1 and H_2 with one common vertex (see [1]). ■

If G, G_1, and G_2 are defined as in Proposition 4, then it follows that if G_1 and G_2 are *sec*-greedy, then also G itself is *sec*-greedy.

Proposition 5. *If G has a subgraph H that is not sec-greedy, then also G is not sec-greedy.*

Proof. The graph H is non-empty and $L(H)$ is an induced subgraph of $L(G)$. Since H is not *sec*-greedy, $L(H)$ is not *sc*-greedy which implies that $L(G)$ is not *sc*-greedy, that is, G is not *sec*-greedy. ■

Note that the corresponding result for *sc*-greedy graphs requires that the subgraph is an induced subgraph.

The following result gives a lower bound for the sum choice index of G.

Theorem 6. $\chi'_{sc}(G) \geq \frac{1}{2}\,\mathrm{GB}'(G) + \frac{1}{2}|E(G)|$.

Proof. Consider a non-empty graph $G = (V, E)$ and an edge choice function f with $\mathrm{size}(f) = \chi'_{sc}(G)$. The edges incident with a vertex $v \in V$ are pairwise adjacent which means that these edges induce in the line graph $L(G)$ of G a complete graph $K_{d(v)}$ which needs a size of at least $1 + 2 + \cdots + d(v) = \binom{d(v)+1}{2}$. Since each edge is incident to two vertices, it follows that

$$\begin{aligned}\chi'_{sc}(G) &= \mathrm{size}(f) = \sum_{uv\in E} f(uv) = \tfrac{1}{2}\sum_{v\in V}\sum_{u\in N(v)} f(uv)\\ &\geq \tfrac{1}{2}\sum_{v\in V}\binom{d(v)+1}{2} = \tfrac{1}{4}\sum_{v\in V} d(v)(d(v)+1) = \tfrac{1}{4}\sum_{v\in V} d(v)^2 + \tfrac{1}{4}\sum_{v\in V} d(v)\\ &= \tfrac{1}{2}\,\mathrm{GB}'(G) + \tfrac{1}{2}|E|.\end{aligned}$$

■

This bound is tight for K_1 with $\mathrm{GB}'(K_1) = |E(K_1)| = 0$ and K_2 with $\mathrm{GB}'(K_2) = |E(K_2)| = 1$ as well as for unions of these graphs but it is not tight for all other graphs. Consider a graph G and an edge choice function f for which equality holds, that is, $\mathrm{size}(f) = \frac{1}{2}\sum_{v\in V}\binom{d(v)+1}{2}$. This implies that each vertex v is incident to edges with list lengths $1, 2, \ldots, d(v)$ since $\sum_{u\in N(v)} f(uv) = \binom{d(v)+1}{2} = 1 + 2 + \cdots + d(v)$.

If v is a vertex with $d(v) \geq 2$, then there are edges uv and vw with $f(uv) = 1$ and $f(vw) = 2$. If $d(w) \geq 2$, then there is another edge wz with $f(wz) = 1$ which leads to a contradiction since we can set the lists to $L(uv) = \{1\}$, $L(wz) = \{2\}$ and $L(vw) = \{1, 2\}$, and these lists do not allow a proper list edge coloring. Therefore $d(w) = 1$ which implies that the bound will not be achieved since pending edges like vw must have a list length of 1. It follows that $\Delta(G) \leq 1$.

The bound of Theorem 6 is nevertheless interesting since it depends on the upper bound $\mathrm{GB}'(G)$.

3. Sum Choice Index of Some Graph Classes

In this section we use known results for the sum choice number to determine the sum choice index of some graph classes. We will show among others that cycles, cycles with pending edges at one vertex, cycles of order at least 5 with a chord, stars, and trees are *sec*-greedy.

Proposition 7. *For cycles C_n it holds that $\chi'_{sc}(C_n) = \mathrm{GB}'(C_n) = 2n$.*

Proof. Since $L(C_n) \cong C_n$ is *sc*-greedy, a cycle C_n is *sec*-greedy by Remark 1. This implies $\chi'_{sc}(C_n) = \chi_{sc}(C_n) = n + n = 2n$. ■

Proposition 8. *For stars S_n it holds that $\chi'_{sc}(S_n) = \mathrm{GB}'(S_n) = \binom{n}{2}$.*

Proof. Obviously $\chi'_{sc}(S_1) = 0 = \mathrm{GB}'(S_1) = \binom{1}{2}$. Let $n \geq 2$. Since $S_n \cong K_{1,n-1}$ and $L(S_n) \cong K_{n-1}$ is *sc*-greedy, it follows from Remark 1 that S_n is *sec*-greedy: $\chi'_{sc}(S_n) = \chi_{sc}(K_{n-1}) = n - 1 + \binom{n-1}{2} = \binom{n}{2}$. ■

This result implies that the difference between the sum choice index $\chi'_{sc}(G)$ and the lower bound of Theorem 6 can become arbitrarily large. Since $\chi'_{sc}(S_n) = \mathrm{GB}'(S_n) = \binom{n}{2}$ and $|E(S_n)| = n-1$ we obtain for that difference $\frac{1}{2}\binom{n}{2} - \frac{1}{2}(n-1) = \frac{1}{4}(n-1)(n-2) \approx \frac{1}{4}n^2$.

The previous result can be generalized to arbitrary trees (see Isaak [5]).

Theorem 9. *For a tree T it holds that $\chi'_{sc}(T) = \mathrm{GB}'(T)$, that is, trees are sec-greedy.*

Proof. This is obviously true for $T \cong K_1$. Consider a non-empty tree T. The line graph $L(T)$ is a connected graph whose blocks are complete graphs, induced by all edges incident with a vertex, and are therefore *sc*-greedy. It follows that also $L(T)$ is *sc*-greedy and therefore T is *sec*-greedy by Remark 1:

$$\chi'_{sc}(T) = \chi_{sc}(L(T)) = \mathrm{GB}(L(T)) = \mathrm{GB}'(T). \qquad \blacksquare$$

We will now consider some classes of graphs which contain cycles.

Proposition 10. *If G is a cycle C_n with p pending edges attached to a single vertex, then $\chi'_{sc}(G) = \mathrm{GB}'(G) = 2n - 3 + \binom{p+3}{2}$.*

Proof. If $p = 1$ then $L(G)$ consists of a cycle C_n and additionally a vertex e which is connected to two consecutive vertices of the cycle. These two vertices are therefore connected by three internally disjoint paths of length $1, n-1$ (on the cycle), and 2 (using e), that is, $L(G)$ is a so-called theta-graph $\theta_{1,2,n-1}$ which is *sc*-greedy (see [3]) and therefore G is *sec*-greedy.

In general, $L(G)$ consists of a cycle C_n and additionally p pairwise adjacent vertices which are also connected to two consecutive vertices of the cycle. Therefore, $L(G)$ consists of a complete graph K_{p+2} with an ear with $n-2$ vertices connecting these two vertices. In [7] it was proved that these graphs are *sc*-greedy, hence G is *sec*-greedy. Note that $\mathrm{GB}'(G) = \frac{1}{2}((n-1)2^2 + (p+2)^2 + p) = 2n + \frac{1}{2}(p^2 + 5p + 6 - 6) = 2n - 3 + \frac{1}{2}(p+3)(p+2)$. ■

This leads to the following result.

Proposition 11. *If G is a tree with cycles attached to some leaves, then $\chi'_{sc}(G) = \mathrm{GB}'(G)$.*

Proof. The line graph $L(G)$ is a connected graph with blocks that are complete graphs as in the proof of Theorem 9 or theta graphs $\theta_{1,2,n-1}$ as in the proof of Proposition 10, that is, all blocks are *sc*-greedy, and therefore $L(G)$ is *sc*-greedy which implies that G is *sec*-greedy. ■

In the next theorem we consider cycles with exactly one chord.

Theorem 12. *If G is a cycle C_n, $n \geq 5$, with exactly one chord, then $\chi'_{sc}(G) = \mathrm{GB}'(G)$.*

Proof. Let G be a cycle C_n with vertices $v_1, v_2, \ldots, v_n$, $n \geq 5$, and, without loss of generality, a chord $e = v_1v_x$, $x \in \{3, \ldots, 1 + \lfloor n/2 \rfloor\}$.

The line graph $L(G)$ consists of a cycle C_n with an additional vertex $e = v_1v_x$ connected to v_nv_1, v_1v_2, $v_{x-1}v_x$, and v_xv_{x+1}, that is, an hourglass graph H (see bold subgraph of Figure 1) with two additional paths $P_1 = (v_1v_2, \ldots, v_{x-1}v_x)$ with $x - 3 \geq 0$ and $P_2 = (v_xv_{x+1}, \ldots, v_nv_1)$ with $n - x - 1 \geq 1$ new vertices (see

Figure 1, left part). If $x = 3$ then P_1 has only one edge and $L(G)$ is a broken wheel BW_4 with center e and a path P_2 with $n-4$ new vertices connecting the two vertices of degree 2 of the broken wheel (see Figure 1, right part).

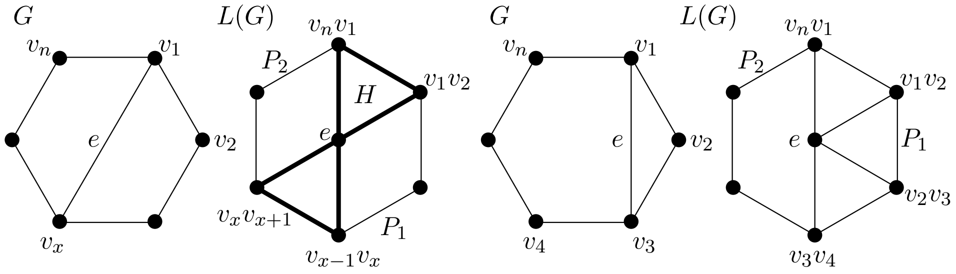

Figure 1. Graphs $G \cong C_n + e$ and line graphs $L(G)$.

Note that $\mathrm{GB}'(G) = \mathrm{GB}(L(G)) = 2n+5$. Assume that there is a choice function f of $L(G)$ with $\mathrm{size}(f) = 2n+4$.

The graph $L(G) - v$ for a vertex v is either a cycle, the graph H with a path P_i and possibly one or two pending paths, or a theta graph $\theta_{1,2,x-1}$ or $\theta_{1,2,n-x+1}$ with possibly a pending path. All these graphs are *sc*-greedy (the proof for H with an additional path is analogous to this one, the other graphs are well-known, see [3]). If there is a vertex v with $f(v) = 1$ or $f(v) > d(v)$, then $\mathrm{size}(f) \geq 1 + d(v) + \chi_{sc}(L(G)-v) = 1 + d(v) + \mathrm{GB}(L(G)-v) = \mathrm{GB}(L(G))$ (see [3]), a contradiction to $\mathrm{size}(f) = 2n+4$. Hence $2 \leq f(v) \leq d(v)$ holds for any vertex v which implies that $f(v) = 2$ for the $n-4$ vertices v that do not belong to the subgraph H of $L(G)$.

Consider a path $P = (w_1, \ldots, w_p)$ with $p \geq 3$. We can set lists of length 2 to the inner vertices of P such that the coloring of w_1 by α and of w_p by $\beta \neq \alpha$ cannot be completed: If p is odd, then set $L(w_i) = \{\alpha, \beta\}$ for $i = 2, \ldots, p-1$. If p is even, then set $L(w_i) = \{\alpha, \beta\}$ for $i = 2, \ldots, p-3$, $L(w_{p-2}) = \{\alpha, \gamma\}$, and $L(w_{p-1}) = \{\beta, \gamma\}$. In such a case we say that the color pair (α, β) is forbidden for P.

Consider the restriction $f' = f|_{V(H)}$ with $\mathrm{size}(f') = 2n+4-2(n-4) = 12$. We discuss all possible choice functions according to the list length of e (see Figure 2).

Case 1. If $f(e) = 4$ then $f(v) = 2$ for the other vertices of H. In this case a list assignment with $L(e) = \{1,2,3,4\}$, $L(v_nv_1) = L(v_1v_2) = \{1,2\}$, and $L(v_{x-1}v_x) = L(v_xv_{x+1}) = \{3,4\}$ does not allow a proper list coloring of H, a contradiction (see Figure 2).

Case 2. If $f(e) = 3$ then there is another vertex of H with list length 3, and the remaining vertices of H have list length 2. Assume, without loss of generality,

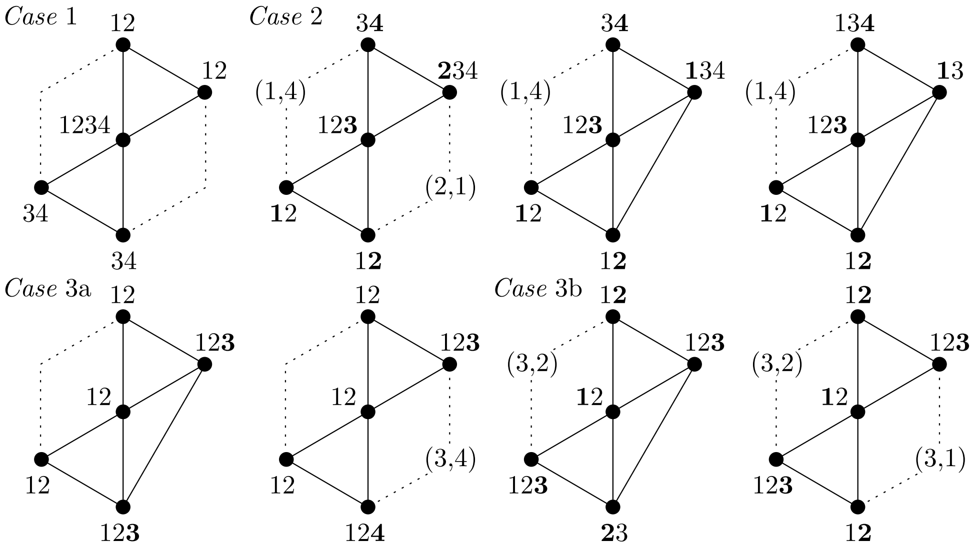

Figure 2. All possible choice functions of $L(G)$ with special list assignments (forced colors are in bold).

that $f(v_{x-1}v_x) = f(v_xv_{x+1}) = 2$ and set $L(v_{x-1}v_x) = L(v_xv_{x+1}) = \{1,2\}$ and $L(e) = \{1,2,3\}$.

If $x \geq 4$ then assume without loss of generality that $f(v_1v_2) = 3$. Set the lists $L(v_nv_1) = \{3,4\}$ and $L(v_1v_2) = \{2,3,4\}$ and the lists of the vertices of P_1 in such a way that the color pair $(2,1)$ is forbidden. If $x = 3$ and $f(v_1v_2) = 3$, then set $L(v_nv_1) = \{3,4\}$ and $L(v_1v_2) = \{1,3,4\}$. If $x = 3$ and $f(v_nv_1) = 3$, then set $L(v_nv_1) = \{1,3,4\}$ and $L(v_1v_2) = \{1,3\}$.

For each of these list assignments, vertex e is forced to be colored by 3, v_nv_1 by 4, v_1v_2 by 1 if $x = 3$ and by 2 if $x \geq 4$, $v_{x-1}v_x$ by 2, and v_xv_{x+1} by 1. Set the remaining lists of G in such a way that the color pair $(1,4)$ is forbidden on the second path P_2, which implies that there is no proper list coloring, a contradiction (see Figure 2).

Case 3. If $f(e) = 2$ then there must exist two vertices of H with list length 3.

Case 3a. These two vertices are connected by the same path P_i, say, without loss of generality, $f(v_1v_2) = f(v_{x-1}v_x) = 3$. Note that if $x = 3$, then these two vertices must be v_1v_2 and v_2v_3 since otherwise there would be a subgraph K_3 of $L(G)$ with all lists of length 2.

Set $L(v_1v_2) = \{1,2,3\}$, $L(v_{x-1}v_x) = \{1,2,3\}$ if $x = 3$ and $L(v_{x-1}v_x) = \{1,2,4\}$ if $x \geq 4$, and $L(v) = \{1,2\}$ for the other vertices v of H. If $x = 3$ then this forces v_1v_2 and $v_2v_3 = v_{x-1}v_x$ to be colored by 3, a contradiction. If $x \geq 4$ then v_1v_2 must be colored by 3 and $v_{x-1}v_x$ by 4. By setting the lists of the vertices of P_1 such that the color pair $(3,4)$ is forbidden we cannot find a proper list coloring of H, a contradiction (see Figure 2).

Case 3b. The two vertices of list lenght 3 are not connected by a path P_i. Without loss of generality, let $f(v_1v_2) = f(v_xv_{x+1}) = 3$. Set $L(v_1v_2) = L(v_xv_{x+1}) = \{1,2,3\}$, $L(v_{x-1}v_x) = \{2,3\}$ if $x = 3$ and $L(v_{x-1}v_x) = \{1,2\}$ if $x \geq 4$, and $L(v) = \{1,2\}$ for the other two vertices of H. This forces v_1v_2 to be colored by 3. If $x \geq 4$, then set the lists of P_1 such that the color pair $(3,1)$ is forbidden which implies that $v_{x-1}v_x$ must be colored by 2 as in the case $x = 3$. It follows that e must be colored by 1, v_nv_1 by 2, and v_xv_{x+1} by 3. Set the lists of the vertices of P_2 in such a way that the color pair $(3,2)$ is forbidden on P_2 which leads again to a list assignment without a proper list coloring, a contradiction (see Figure 2).

Therefore, $\chi_{sc}(L(G)) = \mathrm{GB}(L(G)) = 2n+5$, that is, G is *sec*-greedy. ■

Note that C_4 with a chord is not *sec*-greedy — see next section. Moreover, in this section some other general classes of graphs will be also considered according to their *sec*-greediness.

4. Sum Choice Index of Small Graphs

In this section we consider sum list edge colorings of graphs with small order. We determine the sum choice index of all graphs with at most four vertices. Moreover, we determine all *sec*-greedy graphs with 5 vertices as well as all *sec*-greedy complete multipartite graphs.

Obviously, if a graph G is empty, then $\chi'_{sc}(G) = \mathrm{GB}'(G) = 0$. If G has between one and four edges, then $L(G)$ has at most four vertices and is therefore *sc*-greedy which implies that G itself is *sec*-greedy. This holds for all graphs with at most four vertices except for $K_{1,1,2}$ and for K_4 (see Figure 3 for the graphs and the corresponding line graphs).

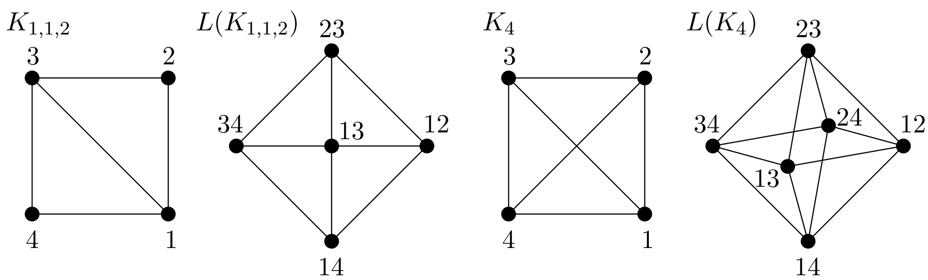

Figure 3. Graphs $K_{1,1,2}$ and K_4 with corresponding line graphs.

It holds that $L(K_{1,1,2}) \cong W_4$ and $\chi_{sc}(W_4) = 12$ (see [10]) which implies that $\chi'_{sc}(K_{1,1,2}) = \chi_{sc}(W_4) = 12$, that is, $K_{1,1,2}$ is not *sec*-greedy. Since $K_{1,1,2}$ is a subgraph of K_4, also K_4 is not *sec*-greedy. The line graph of the complete graph K_4 is the octahedron graph, $L(K_4) \cong K_{2,2,2}$, whose sum choice number is determined in the following lemma.

Lemma 13. $\chi_{sc}(K_{2,2,2}) = 17$.

Proof. Since the wheel W_4 is an induced subgraph of $K_{2,2,2}$ and it is not *sc*-greedy, it follows that $K_{2,2,2}$ is also not *sc*-greedy, that is, $\chi_{sc}(K_{2,2,2}) \leq 17$.

Assume that there is a choice function f of $K_{2,2,2}$ with $\text{size}(f) = 16$. If there is a vertex v with $f(v) = 1$, then $\text{size}(f) \geq 1 + d(v) + \chi_{sc}(W_4) = 17$ (see [3]). Therefore, we may assume that $2 \leq f(v) \leq d(v) = 4$ holds for every vertex v. There are only three possible non-decreasing list length sequences, $(2,2,2,2,4,4)$, $(2,2,2,3,3,4)$, and $(2,2,3,3,3,3)$ and, since K_3 is not 2-choosable, without loss of generality only five possible functions f. Figure 4 shows in each case an assignment L with $|L(v)| = f(v)$ for all $v \in V(K_{2,2,2})$ without proper L-coloring, a contradiction to the assumption. Therefore, $\chi_{sc}(K_{2,2,2}) = 17$. ∎

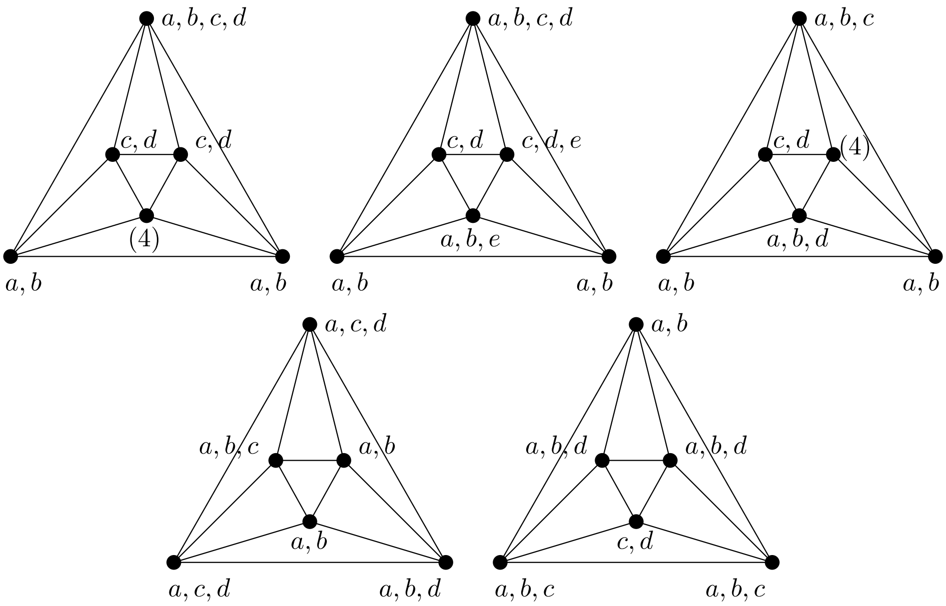

Figure 4. $K_{2,2,2}$ with bad list assignments ((ℓ): arbitrary list of lenght ℓ).

Therefore, $\chi'_{sc}(K_4) = \chi_{sc}(L(K_4)) = \chi_{sc}(K_{2,2,2}) = 17$. Summarizing the previous results we obtain the following statement.

Theorem 14. *If G is a graph with at most 4 vertices, then G is sec-greedy if and only if $|E(G)| \leq 4$.*

Another consequence is that all graphs which contain $K_{1,1,2}$ as a subgraph are not *sec*-greedy. This includes all broken wheels BW_ℓ and all wheels W_ℓ, $\ell \geq 3$, all complete graphs K_n, $n \geq 4$ (while on the other hand K_1, K_2, K_3 are *sec*-greedy).

More generally, if a graph G has 4 vertices that induce a subgraph with 5 or 6 edges, then G is not *sec*-greedy. If k is the maximum number of such subgraphs of G with pairwise disjoint vertex sets, then $\chi'_{sc}(G) \leq \mathrm{GB}'(G) - k$. For example, $\chi'_{sc}(K_n) \leq \mathrm{GB}'(K_n) - \lfloor n/4 \rfloor = \frac{1}{2}n(n-1)^2 - \lfloor n/4 \rfloor$. This bound is tight for $n \leq 4$.

It would be an interesting task to determine $\chi'_{sc}(K_n)$ in general.

In the following we consider graphs with exactly 5 vertices. If the graph contains isolated vertices, then the previous result implies that only $K_{1,1,2} \cup K_1$ with $\chi'_{sc}(K_{1,1,2} \cup K_1) = \chi'_{sc}(K_{1,1,2}) = 12$ and $K_4 \cup K_1$ with $\chi'_{sc}(K_4 \cup K_1) = \chi'_{sc}(K_4) = 17$ are not *sec*-greedy; all other graphs with isolated vertices are *sec*-greedy. Moreover, not-connected graphs without isolated vertices (that is, with exactly two nontrivial components) are *sec*-greedy, as well as all graphs with at most 4 edges. Therefore, we only need to consider connected graphs with at least 5 edges. There are 18 such graphs G_i, $i = 1, \ldots, 18$ (see Figure 5).

It holds that $L(G_1) \cong G_7 \cong \theta_{1,2,3}$, $L(G_2) \cong G_{12} \cong BW_4$, $L(G_3) \cong G_{16}$, $L(G_4) \cong C_5$, $L(G_5) \cong G_9$ are *sc*-greedy since they are not isomorphic to one of the three non-*sc*-greedy graphs with 5 vertices. Hence, $G_1, G_2, \ldots, G_5$ are *sec*-greedy.

It holds that $L(G_6)$ and $L(G_7)$ are *sc*-greedy (see [7]) which implies that G_6 and G_7 are *sec*-greedy with $\chi'_{sc}(G_6) = 15$ and $\chi'_{sc}(G_7) = 16$.

Since $G_8 \cong K_{2,3}$, $L(G_8) \cong K_2 \square K_3$ with $\chi'_{sc}(G_8) = \chi_{sc}(K_2 \square K_3) = 14$ (see [4]) which implies that G_8 is not *sec*-greedy.

The graphs $G_9, G_{10}, \ldots, G_{18}$ contain $K_{1,1,2}$ as a subgraph and are therefore not *sec*-greedy. It holds that $\chi'_{sc}(G_9) = \chi_{sc}(L(G_9)) = 15$ and $\chi'_{sc}(G_{10}) = \chi_{sc}(L(G_{10})) = 16$ (see [7]).

Therefore there is only one new minimal non-*sec*-greedy graph, $K_{2,3}$, and we obtain the following characterizations.

Theorem 15. *A graph G with 5 vertices is sec-greedy if and only if $K_{1,1,2} \not\subseteq G$ and $G \not\cong K_{2,3}$.*

Since $K_{1,1,2}$ and $K_{2,3}$ are not *sec*-greedy we immediately obtain a characterization of all complete multipartite graphs.

Theorem 16. *The complete multipartite graph $K_{r_1,r_2,\ldots,r_t}$ with $r_1 \leq r_2 \leq \cdots \leq r_t$, $t \geq 2$, is sec-greedy if and only if $t = 2$, $r_1 = 1$, or if $t = 2$, $r_1 = r_2 = 2$, or if $t = 3$, $r_1 = r_2 = r_3 = 1$.*

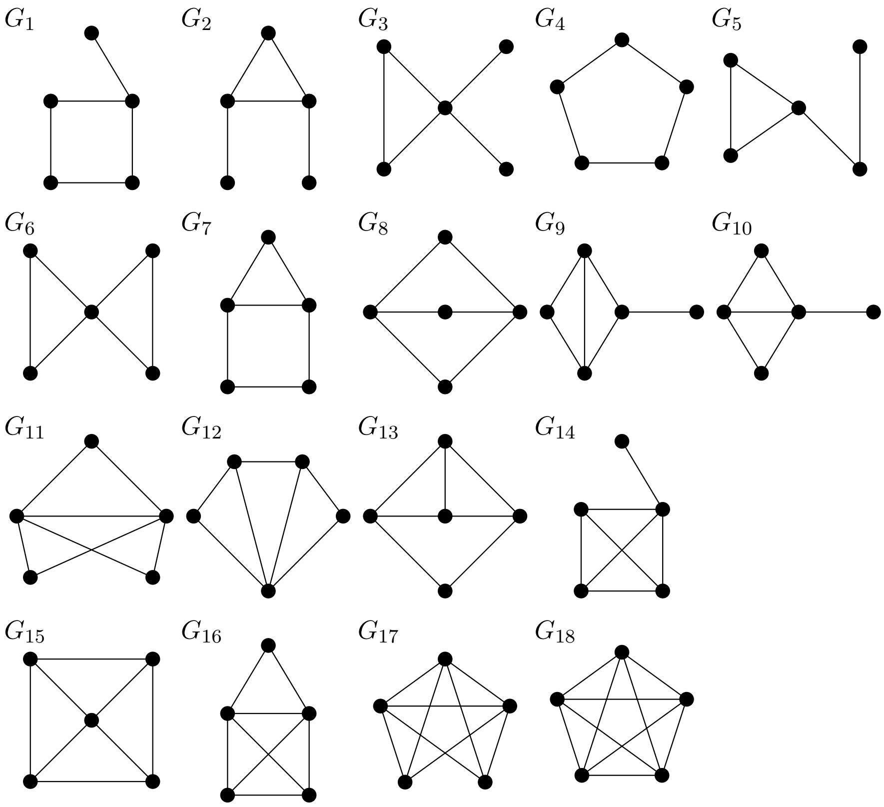

Figure 5. Connected graphs with 5 vertices and at least 5 edges.

Proof. The stars K_{1,r_2} and the cycles $K_{2,2} \cong C_4$ and $K_{1,1,1} \cong C_3$ are *sec*-greedy. All other complete multipartite graphs have $K_{2,3}$ or $K_{1,1,2}$ as a subgraph and are therefore not *sec*-greedy. ■

5. Upper Bound for the Number of Edges

Let $q(n)$ be the maximum number of edges of a *sec*-greedy graph with n vertices. It holds that $q(1) = 0$, $q(2) = 1$, $q(3) = 3$, $q(4) = 4$, and $q(5) = 6$ (see Theorem 14 and the previous section).

Let G be a *sec*-greedy graph with $n \geq 4$ vertices and q edges. There are $\binom{n}{4}$ sets of four vertices in G, and each set must induce a subgraph with at most $q(4) = 4$ edges. Since each edge is contained in $\binom{n-2}{2}$ such sets we obtain

$$q \leq q(n) \leq \frac{q(4)\binom{n}{4}}{\binom{n-2}{2}} = \frac{\frac{4}{24}n(n-1)(n-2)(n-3)}{\frac{1}{2}(n-2)(n-3)} = \tfrac{1}{3}n(n-1) = \tfrac{2}{3}\binom{n}{2}.$$

Therefore, if a graph G has $n \geq 4$ vertices and $q > \frac{2}{3}\binom{n}{2}$ edges, then G is not *sec*-greedy.

This bound is tight at least for $n = 4$ and $n = 5$ since $G \cong C_4$ is *sec*-greedy with $n = 4$ vertices and $q = q(4) = 4 = \frac{2}{3}\binom{4}{2}$ edges, and $G \cong G_7$ (see previous section) is *sec*-greedy with $n = 5$ vertices and $q = q(5) = 6 \leq \frac{2}{3}\binom{5}{2} < 7$ edges.

On the other hand, the bound is not tight for $n = 6$. Assume that there is a *sec*-greedy graph with 6 vertices and $\frac{2}{3}\binom{6}{2} = 10$ edges. By the pigeonhole principle, there is a vertex v with $d(v) \leq 3$. It follows that $G - v$ is *sec*-greedy with 5 vertices and at least 7 edges, a contradiction to $q(5) = 6$.

One way to improve the bound is to generalize it. By the same argument as above we obtain the following result.

Theorem 17. *If $n \geq k \geq 4$ then*

$$q(n) \leq \frac{q(k)\binom{n}{k}}{\binom{n-2}{k-2}} = \frac{q(k)}{k(k-1)}n(n-1) = \frac{q(k)}{\binom{k}{2}}\binom{n}{2}.$$

For $n \geq k = 4$ we obtain $q(n) \leq \frac{2}{3}\binom{n}{2}$ as above. For $n \geq k = 5$ we obtain $q(n) \leq \frac{3}{5}\binom{n}{2}$ which improves the bound. This implies $q(6) \leq 9$ and $q(7) \leq 12$. Using the bound $q(6) \leq 9$ does not improve the result, but using $q(7) \leq 12$ yields $q(n) \leq \frac{4}{7}\binom{n}{2}$ for $n \geq k = 7$.

6. Remark

The list coloring conjecture states that $\chi(G) = \mathrm{ch}(G)$ if G is a line graph. Can a corresponding conjecture be formulated for sum list colorings?

The *chromatic sum* $\Sigma(G)$ of G is the minimum sum of colors in a proper coloring of G by positive integers (see [9]). As it is for chromatic number and list chromatic number, also the chromatic sum arises from the sum choice number if all lists of vertices v are restricted to be initial lists of the form $\{1, 2, \ldots, f(v)\}$, $f(v) \in \mathbb{N}$ (see [5]). By definitions, $\Sigma(G) \leq \chi_{sc}(G)$ for all graphs G.

The difference between $\chi_{sc}(G)$ and $\Sigma(G)$ may become arbitrarily large. To see this consider an even cycle C_n which is a line graph and set $f(v) = 1$ for all vertices of a maximum independent set and $f(v) = 2$ for all other vertices. This forces that all vertices of the first set must be colored with color 1 and all other vertices must be colored with color 2. We obtain $\Sigma(C_n) = n/2 + n$ while $\chi_{sc}(C_n) = 2n$.

Therefore, a conjecture for sum list colorings which corresponds to the list coloring conjecture must be modified. It would be an interesting task to formulate such a conjecture.

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