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TRIANGLE DECOMPOSITIONS OF PLANAR GRAPHS

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Abstract

A multigraph G is triangle decomposable if its edge set can be partitioned into subsets, each of which induces a triangle of G, and rationally triangle decomposable if its triangles can be assigned rational weights such that for each edge e of G, the sum of the weights of the triangles that contain e equals 1.

We present a necessary and sufficient condition for a planar multigraph to be triangle decomposable. We also show that if a simple planar graph is rationally triangle decomposable, then it has such a decomposition using only weights $0, 1$ and $\frac{1}{2}$. This result provides a characterization of rationally triangle decomposable simple planar graphs. Finally, if G is a multigraph with K_4 as underlying graph, we give necessary and sufficient conditions on the multiplicities of its edges for G to be triangle and rationally triangle decomposable.

Keywords: planar graphs, triangle decompositions, rational triangle decompositions.

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1. Introduction

We consider multigraphs, in which multiple edges between vertices are allowed, but loops are not, and reserve the term *graph* for a simple graph. For a graph H, a multigraph G is *H-decomposable* if its edge set can be partitioned into subsets, each of which induces a subgraph isomorphic to H. Such a partition is called an *H-decomposition* of G. A K_3-decomposition is also called a *triangle decomposition*, and a K_3-decomposable multigraph is also said to be *triangle decomposable.* Given a multigraph G, a *rational* (or *fractional*) *K_3-decomposition* of G is an assignment of nonnegative rational numbers, called *weights*, to the copies of K_3 in G such that for each edge e of G, the sum of the weights of the triangles that contain e equals 1. If G admits a rational K_3-decomposition, we say that G is *rationally triangle decomposable* or *rationally K_3-decomposable.*

Our work was motivated by Garaschuk's study of rational triangle decompositions of dense graphs in [7]. Instead of dense graphs we consider planar graphs and characterize rationally K_3-decomposable planar graphs. To achieve our goal, we need results on K_3-decomposable planar multigraphs; thus we also present a necessary and sufficient condition for a planar multigraph to be triangle decomposable.

Triangle decompositions of graphs have a long history, beginning with the following problem raised by Woolhouse in 1844 in *The Lady's and Gentleman's Diary* [18, as cited by Biggs in [3]]:

> "Determine the number of combinations that can be made of n symbols, p symbols in each; with this limitation, that no combination of q symbols which may appear in any one of them shall be repeated in any other."

A version of this problem (in which each pair of symbols appears *exactly* once) was solved for $p = 3$ and $q = 2$ by Kirkman [13, as cited in [3]] in 1847. Structures satisfying these constraints became known as Steiner triple systems in honour of Jakob Steiner [15, as cited in [3]], who independently posed the question of their existence.

Simple necessary conditions for a connected multigraph G to be triangle decomposable are that G be Eulerian and $|E(G)| \equiv 0 \pmod 3$. A multigraph that satisfies these conditions is called *K_3-divisible.* Kirkman showed that being K_3-divisible is also sufficient for a complete graph to possess a triangle decomposition. A natural question, therefore, concerns the density of non-complete triangle decomposable graphs. Some work on this topic concerns a conjecture due to Nash-Williams [14]. A graph G of order n and minimum degree $\delta(G)$ is $(1-\varepsilon)$-*dense* if $\delta(G) \geq (1-\varepsilon)(n-1)$. Nash-Williams conjectured that any sufficiently large K_3-divisible $\frac{3}{4}$-dense graph is K_3-decomposable. Keevash [11] obtained an asymptotic result, a special case of which applies to this conjecture, with a value of ε much smaller than $\frac{1}{4}$.

The Nash-Williams conjecture also stands for rational K_3-decomposability (except that no divisibility conditions are required). Garaschuk [7] showed that sufficiently large graphs of order n and minimum degree at least $\frac{22}{23}n \approx 0.956n$ are rationally K_3-decomposable. Dross [5] improved the degree condition for rational K_3-decomposability to $(0.9+\varepsilon)n$, for all $\varepsilon > 0$. Combined with a result by Barber, Kühn, Lo and Osthus [1], this implies that every sufficiently large graph of order n and minimum degree at least $(0.9 + \varepsilon)n$ admits a triangle decomposition.

Holyer [10] showed that the problem of deciding whether a given general graph is K_n-decomposable is NP-complete for $n \geq 3$. Conditions for different classes of planar graphs to be decomposable into paths of length 3 are presented in [9]. For decompositions of graphs into other graphs H of size $|E(H)| = 3$, see e.g. [2, 6, 8]. On a somewhat different note, planar graphs decomposable into a forest and a matching are considered in several publications, including [4, 16], while it is shown in [12] that any planar graph is decomposable into three forests, one of which has maximum degree at most four.

In contrast to the asymptotic results on K_n-decompositions of dense graphs, we consider planar multigraphs and, in Section 2, characterize those that are triangle decomposable. We begin with some definitions and the statement of the characterization in Section 2.1, followed by a number of lemmas in Section 2.2 and the proof in Section 2.3. In Section 3 we turn to rational decompositions of planar multigraphs. We show in Section 3.1 that any rationally K_3-decomposable (simple) graph admits such a decomposition using only weights $0, 1$ or $\frac{1}{2}$, a result which leads to a characterization of these graphs. As evidence that the $0, 1, \frac{1}{2}$ result also holds for planar multigraphs, we characterize rationally K_3-decomposable multigraphs that have K_4 as underlying graph in Section 3.2. We close with some ideas for further work in Section 4.

2. Triangle Decompositions of Planar Multigraphs

2.1. Definitions and statement of main result

Since a multigraph is K_3-decomposable if and only if each of its blocks is K_3-decomposable, we consider only 2-connected planar multigraphs. In addition to being K_3-divisible, a K_3-decomposable multigraph also needs to satisfy the condition that each of its edges is contained in a triangle, a condition that holds trivially for (large enough) complete graphs. A K_3-divisible multigraph that satisfies this third necessary condition is called *strongly K_3-divisible.* The planar graph H obtained by joining the two vertices of $K_{2,7}$ of degree seven shows that a strongly K_3-divisible graph need not be K_3-decomposable: the removal of any triangle of H results in a triangle-free graph.

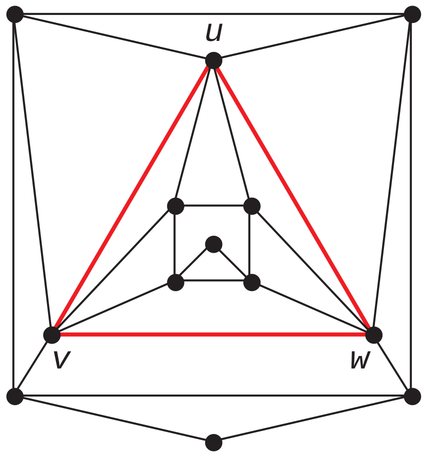

Figure 1. Triangle uvw is a faceless triangle.

We denote a triangle with vertex set $\{u, v, w\}$ by $\tau = uvw$ if we are not interested in the specific edges between its vertices. If specific edges are important, we denote τ by efg, where $e = uv$, $f = vw$, and $g = wu$. A triangle τ of a planar multigraph G is called *faced* if there exists a plane embedding $\widetilde{G}$ of G such that τ is a face of $\widetilde{G}$; otherwise τ is called *faceless*. The triangle uvw of the graph in Figure 1 is a faceless triangle; this can be seen without much effort, but also follows from Lemma 2 below. A *separating triangle* uvw of G is one such that $G - \{u, v, w\}$ is disconnected.

For vertices $u, v \in V(G)$, denote the number of edges joining u and v by $\mu(u, v)$. A *duplicate triangle* is a triangle $u_1u_2u_3$ such that $\mu(u_i, u_j) \geq 2$ for each $i \neq j$, and may be faced or faceless, separating or non-separating. By *deleting the edges of a duplicate triangle* we mean that we delete exactly one edge between each pair of vertices u_i and u_j of a duplicate triangle $u_1u_2u_3$.

A *triangle depletion*, or simply a *depletion*, of G is any spanning subgraph of G obtained by sequentially deleting edges of (any number of) faceless or duplicate triangles. Note that G is a depletion of itself.

The *dual multigraph* G^* of a plane multigraph G is a plane multigraph having a vertex for each face of G. The edges of G^* correspond to the edges of G as follows: if e is an edge of G that has a face F on one side and a face F' on the other side, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices f and f' of G^* that correspond to the faces F and F' of G. Note that under our assumption that G is 2-connected, G^* has no loops, and, using a careful geometric description of the placement of vertices and edges in the dual, as in [17, Remark 7.1.8], we see that $(G^*)^* \cong G$.

The statement of the main result of this section follows.

Theorem 1. *A planar multigraph G is triangle decomposable if and only if some depletion of G has a plane embedding whose dual is a bipartite multigraph in which all vertices of some partite set have degree three.*

2.2. Lemmas

In our first result we present a characterization of faceless triangles of planar multigraphs.

Lemma 2. *A triangle $\tau = v_1v_2v_3$ of a planar multigraph G is faceless if and only if there exist two components H_1 and H_2 of $G - \{v_1, v_2, v_3\}$ such that each v_i is adjacent, in G, to a vertex in each H_j, $i = 1, 2, 3$, $j = 1, 2$.*

Proof. Let $\widetilde{G}$ be a plane embedding of G having τ as a face, but $G - \{v_1, v_2, v_3\}$ has components H_j as described. Let G' be the multigraph obtained by joining a new vertex v to each v_i. By inserting v in the face τ of $\widetilde{G}$, we get a plane embedding of G'. However, by contracting each H_i to a single vertex we now obtain a $K_{3,3}$ minor of G', a contradiction.

Conversely, suppose two such components H_j do not exist. Let $\widetilde{G}$ be a plane embedding of G and suppose τ is not a face of $\widetilde{G}$. Then $\widetilde{G}$ has vertices interior and exterior to τ. By assumption we may assume without loss of generality that each component of $G - \{v_1, v_2, v_3\}$ interior to τ has vertices adjacent, in G, to at most two vertices v_i, $i = 1, 2, 3$. Let H be a component of $G - \{v_1, v_2, v_3\}$ interior to τ such that no vertex of H is adjacent to (say) v_3. Let F be the face of $\widetilde{G}$ exterior to τ that contains v_1v_2 on its boundary. By moving H to F we obtain an embedding of G such that H is exterior to τ. By repeating this procedure we eventually obtain an embedding $\widetilde{G}'$ of G such that τ is a face of $\widetilde{G}'$. ■

Evidently, then, a faceless triangle is a separating triangle.

Lemma 3. *If a planar multigraph G is 2-connected, then so is any depletion of G.*

Proof. Suppose the statement of the lemma does not hold, and let G be a 2-connected planar multigraph with the minimum number of edges such that a depletion of G is not 2-connected. Then there exists a faceless or duplicate triangle $\tau = uvw$ whose edges can be deleted from G to obtain a planar multigraph G' that is not 2-connected. This is impossible if τ is a duplicate triangle, hence τ is a faceless triangle. Since G is 2-connected but G' is not, some vertex of τ is a cut-vertex of G' but not of G. Assume without loss of generality that v is such a vertex.

Let H be a component of $G - \{u, v, w\}$ whose existence is guaranteed by Lemma 2. Then both u and w are adjacent, in $G' - v$, to vertices of H. Therefore

u and w belong to the same component, say A, of $G' - v$. Let B be the union of all other components of $G' - v$. Then no vertex of A is adjacent, in $G' - v$, to a vertex of B. Reinserting the edge uw in A, we see that no vertex of $A + uw$ is adjacent, in $G - v$, to a vertex of B; that is, v is also a cut-vertex of G, a contradiction. ■

We also need the following result.

Proposition 4 [17, Theorem 7.1.13]. *A plane multigraph is Eulerian if and only if its dual is bipartite.*

2.3. Proof of Theorem 1

We restate the characterization of triangle decomposable planar multigraphs for convenience.

Theorem 1. *A planar multigraph G is triangle decomposable if and only if some depletion G_Δ of G has a plane embedding whose dual is a bipartite multigraph in which all vertices of some partite set have degree three.*

Proof. We may assume that G is 2-connected. Suppose G is triangle decomposable. Then G is strongly K_3-divisible. Let $\mathcal{S}$ be the collection of triangles in some triangle decomposition of G and let $\mathcal{S}'$ consist of all faceless triangles, or triangles forming part of duplicate triangles, in $\mathcal{S}$. Since the triangles in $\mathcal{S}'$ are pairwise edge-disjoint, deleting their edges results in a depletion G_Δ of G. Since $\mathcal{S}$ is a triangle decomposition of G, $\mathcal{S} - \mathcal{S}'$ is a triangle decomposition of G_Δ, and every vertex of G_Δ is even.

Among all plane embeddings of G_Δ, let $\widetilde{G}_\Delta$ be one that maximizes the number of triangles in $\mathcal{S} - \mathcal{S}'$ that are faces of the embedding. Suppose $\tau = uvw$ is a triangle in $\mathcal{S} - \mathcal{S}'$ that is not a face of $\widetilde{G}_\Delta$. Since τ is a faced triangle, Lemma 2 implies that we may assume without loss of generality that each component of $G - \{u, v, w\}$ interior to τ has vertices adjacent, in G, to at most two of u, v and w. Since τ is not a duplicate triangle of G_Δ, we may further assume that there is at least one component of $G - \{u, v, w\}$ interior to τ. Let H be such a component; say no vertex of H is adjacent to w. Let F and F' be the faces interior and exterior to τ, respectively, containing the edge uv on their boundaries. Then neither F nor F' is contained in $\mathcal{S}$. By moving H from F to F' we obtain an embedding of G_Δ such that H is exterior to τ. By repeating this procedure we eventually obtain an embedding $\widetilde{G}'_\Delta$ of G such that τ is a face of $\widetilde{G}'_\Delta$ and such that each triangle in $\mathcal{S} - \mathcal{S}'$ that is a face of $\widetilde{G}_\Delta$ is also a face of $\widetilde{G}'_\Delta$. This contradicts the choice of $\widetilde{G}_\Delta$.

Hence all triangles in $\mathcal{S} - \mathcal{S}'$ are faces of $\widetilde{G}_\Delta$. By Lemma 3, G_Δ is 2-connected. Thus each edge of $\widetilde{G}_\Delta$ lies on two faces. Since G_Δ is Eulerian, the dual G^*_Δ of

$\widetilde{G}_\Delta$ is bipartite (Proposition 4). Let (A, B) be a bipartition of G^*_Δ. Let τ, τ' be two triangles in $\mathcal{S} - \mathcal{S}'$, let t, t' be the corresponding vertices of G^*_Δ and assume without loss of generality that $t \in A$. Consider any $t{-}t'$ path $t = t_0, t_1, \ldots, t_k = t'$ in G^*_Δ and say t_i corresponds to a face F_i of $\widetilde{G}_\Delta$, $i = 1, \ldots, k$. Then F_1 is adjacent to τ, hence $F_1 \notin \mathcal{S}$. Since F_2 is adjacent to F_1 and the shared edge on the boundaries of F_1 and F_2 belongs to a triangle in $\mathcal{S} - \mathcal{S}'$, $F_2 \in \mathcal{S} - \mathcal{S}'$. Continuing this argument we see that $F_i \in \mathcal{S} - \mathcal{S}'$ if and only if i is even. Since $F_k = \tau' \in \mathcal{S} - \mathcal{S}'$, k is even. Therefore $t' = t_k \in A$. We conclude that A consists of all vertices of G^*_Δ that correspond to triangles in $\mathcal{S} - \mathcal{S}'$, while all other vertices of G^*_Δ correspond to faces of $\widetilde{G}_\Delta$ that are adjacent to triangles in $\mathcal{S} - \mathcal{S}'$. Hence these vertices belong to B. Therefore $\deg v = 3$ for all $v \in A$.

Conversely, suppose some depletion G_Δ of G has a plane embedding $\widetilde{G}_\Delta$ whose dual G^*_Δ possesses the stated properties. By Proposition 4, G_Δ is Eulerian. Let $\mathcal{S}'$ be the collection of edge-disjoint triangles of G whose deletion resulted in G_Δ. Let (A, B) be a bipartition of G^*_Δ such that all vertices in A have degree three and let $\mathcal{S}$ be the faces of $\widetilde{G}_\Delta$ corresponding to the vertices in A. Since A is an independent set of vertices that cover all edges of G^*_Δ (since G^*_Δ is a multigraph, it has no loops), $\mathcal{S}$ consists of mutually edge-disjoint triangles covering all edges of G_Δ. Therefore $\mathcal{S}$ is a triangle decomposition of G_Δ and $\mathcal{S} \cup \mathcal{S}'$ is a triangle decomposition of G. ■

Triangle decompositions of a graph G and its depletion G_Δ are illustrated in Figure 2. Since G itself is Eulerian, the dual of any embedding of G is bipartite. However, no embedding of G has a dual in which all vertices of one partite set of its bipartition have degree three: the edge vw always lies on two nontriangular faces, and the corresponding vertices (of degree at least four) of the dual are in different partite sets. A K_3-decomposition of G is obtained by first deleting uvw, partitioning the faces into two sets so that one set contains only triangles, which form part of the decomposition, and reinserting uvw to complete the decomposition.

Theorem 1 implies that the necessary conditions for a multigraph to be triangle decomposable are also sufficient for maximal planar graphs, which trivially satisfy two of the conditions (of being strongly K_3-divisible) provided they have order at least three. Corollary 5 can also be obtained directly from duality, and is probably known, although we found no reference to it.

Corollary 5. *A maximal planar graph is triangle decomposable if and only if it is Eulerian.*

Proof. Any plane embedding of a maximal planar graph G of order at least three is a triangulation of the plane. Its dual is cubic, and bipartite because G is Eulerian, and either partite set corresponds to a triangle decomposition of G. ■

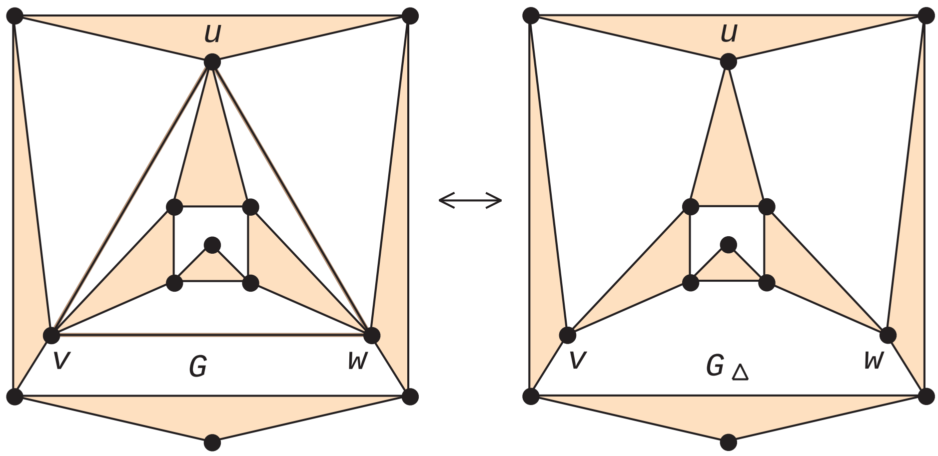

Figure 2. Triangle decompositions of G and $G_\triangle$.

We conclude this section by remarking that the characterization in Theorem 1 does not provide a polynomial-time algorithm to decide the triangle-decomposability of planar multigraphs, since there may exist many different depletions of the same graph.

3. Rational Triangle Decompositions

The main purpose of this section is to characterize rationally triangle decomposable planar graphs, which we do in Corollary 7, after first showing, in Theorem 6, that each such graph admits a rational triangle decomposition using only weights $0, 1$ and $\frac{1}{2}$. In Section 3.2 we characterize K_3-decomposable and rationally K_3-decomposable planar multigraphs that have K_4 as underlying graph in terms of the multiplicities of their edges.

Dense graphs that admit rational K_3-decompositions were studied in [7]. The only condition among the three for a multigraph G to be K_3-decomposable that remains necessary for G to be rationally K_3-decomposable is the condition that each edge of G be contained in a triangle. Clearly, maximal planar graphs of order at least three are rationally K_3-decomposable: assign a weight of $\frac{1}{2}$ to each face triangle in a plane embedding of the graph. In fact, each multigraph whose edges can be partitioned into sets that induce maximal planar subgraphs is rationally K_3-decomposable.

3.1. Rationally triangle decomposable planar graphs

Suppose G is a rationally K_3-decomposable multigraph and consider such a decomposition of G. For a triangle τ of G, we denote the weight of τ by $w(\tau)$, and for any edge e of G, we denote the sum of the weight of the triangles that contain e by $w(e)$. Since G is rationally K_3-decomposable, $w(e) = 1$ for each edge e.

While it is easy to find planar multigraphs that possess rational triangle decompositions with weights $\frac{p}{q}$ and $\frac{q-p}{q}$ for arbitrary integers $q \geq 1$ and $0 \leq p \leq q$, for example Eulerian maximal planar graphs, any rationally K_3-decomposable multigraph we know of also admits a decomposition using only weights 0, 1 and $\frac{1}{2}$, which we call a $0, 1, \frac{1}{2}$-K_3-*decomposition*. We show that this is true for all rationally K_3-decomposable (simple) planar graphs. (This result is not true for nonplanar graphs; for example, K_5 is rationally K_3-decomposable [7] but does not have a $0, 1, \frac{1}{2}$-K_3-decomposition.)

For a triangle $\tau = xyz$ of a plane graph G, let I_τ denote the subgraph of G induced by $\{x, y, z\}$ and all vertices interior to τ. We call I_τ the *interior graph* of τ. A separating triangle of G is an *innermost* (or an *outermost*) *separating triangle* if its interior (or its exterior) contains no separating triangles. Similarly, a separating triangle containing an edge e is an *outermost separating triangle containing* e if no separating triangle in its exterior contains e.

Theorem 6. *If G is a rationally K_3-decomposable planar graph, then G has a $0, 1, \frac{1}{2}$-K_3-decomposition.*

Proof. Suppose there exists a planar graph that is rationally K_3-decomposable but does not have a $0, 1, \frac{1}{2}$-K_3-decomposition. Let H be such a graph with the minimum number of edges. We establish the following properties of H:

1. *H is not maximal planar.* A maximal planar graph has a K_3-decomposition where each face receives weight $\frac{1}{2}$.

2. *Every edge of H is in at least two triangles.* Suppose $e \in E(H)$ is in only one triangle τ. Then in any rational K_3-decomposition of H, τ receives weight 1. Hence $H - \tau$ is rationally K_3-decomposable, and since $H - \tau$ has fewer edges than H, it has a rational K_3-decomposition using only weights 0, 1, and $\frac{1}{2}$. But then H has a $0, 1, \frac{1}{2}$-K_3-decomposition, which is a contradiction.

3. *In any embedding of H, every edge incident with a nontriangular face belongs to a separating triangle.* Let e be an edge incident with a nontriangular face. As e is in at least two triangles, and is incident with exactly two faces, one such triangle τ is not a face. Thus there are vertices interior and exterior to τ, which is therefore a separating triangle.

4. *In any embedding of H, there exists a separating triangle τ, incident with an edge e of a nontriangular face, whose exterior contains no triangles containing e and whose interior graph I_τ is maximal planar.* Let e_1 be an edge of a nontriangular face and let τ_1 be the outermost separating triangle containing e. If I_{τ_1} is maximal planar, we are done. Otherwise, I_{τ_1} contains a nontriangular

face; choose an edge e_2 of this face not also contained in τ_1 and its outermost separating triangle τ_2. Note that τ_2 lies interior to τ_1. As H is finite, this process terminates.

Now, assume that H is embedded in the plane and that τ is a separating triangle incident with an edge e of a nontriangular face f, whose exterior contains no triangles containing e and whose interior graph I_τ is maximal planar. We next prove the following claim regarding an innermost separating triangle of H.

Claim 6.1. *Let T be an innermost separating triangle of H. Then I_T is maximal planar and any rational K_3-decomposition of H gives the same weight to the faces of I_T adjacent to T.*

Proof. Suppose I_T is not maximal planar. Then it contains a nontriangular face. But every edge of this face that does not belong to T is in a separating triangle interior to T, which contradicts the choice of T. Hence I_T is maximal planar.

Let I_T^* be the dual of I_T and let $D = I_T^* - T$. First suppose D is bipartite with bipartition (X, Y). We show that the vertices x, y and z of D corresponding to the three faces of I_T adjacent to T are in the same partite set. Otherwise, assume without loss of generality that $x, y \in X$ and $z \in Y$. Since I_T is maximal planar, x, y and z have degree 2 and every other vertex of D has degree 3. But then the number of edges incident with a vertex in X is congruent to 1 (mod 3) and the number of edges incident with a vertex in Y is congruent to 2 (mod 3), which is impossible. Hence x, y and z belong to the same partite set.

Now, since every edge in D corresponds to an edge of I_T that lies on exactly two triangle faces, and I_T has no separating triangles, every face in the same partite set receives the same weight, and the weights of the two sets sum to 1. Hence, the faces of I_T adjacent to T receive the same weight.

Now suppose D is not bipartite. Then D contains an odd cycle $f_1 f_2 f_3 \cdots f_k f_1$. Suppose $w(f_1) = x$. Then as every edge is incident with exactly two triangles, $w(f_2) = 1 - x$, $w(f_3) = x, \ldots, w(f_k) = x$, and $w(f_1) = 1 - x = x$. Hence $x = \frac{1}{2}$. Filling in the remaining weights from this cycle, every face receives weight $\frac{1}{2}$. Hence, the faces of I_T adjacent to T receive the same weight. □

Continuing the proof of Theorem 6, consider a rational K_3-decomposition of H. Suppose I_τ contains a separating triangle other than τ. Then choose an innermost separating triangle τ' of I_τ and let H' be the graph obtained by deleting the interior of τ'. By Claim 6.1, the interior faces of $I_{\tau'}$ adjacent to τ' receive the same weight, say x. Then the rational K_3-decomposition of H induces a rational K_3-decomposition of H' in which $w_{H'}(\tau') = w_H(\tau') + x$. We continue this process until τ has no separating triangles in its interior. Finally, apply this process to τ itself, obtaining the graph $H^\dagger$. Now e is contained in only one triangle in $H^\dagger$, namely τ, so $w_{H^\dagger}(\tau) = 1$. Then $H^\dagger - \tau$ has a rational K_3-decomposition and since

$H^\dagger - \tau$ has fewer edges than H, it has a $0, 1, \frac{1}{2}$-K_3-decomposition. As a result, we obtain a $0, 1, \frac{1}{2}$-K_3-decomposition of H by extending the decomposition of $H^\dagger - \tau$ and giving each face of I_τ (including τ) weight $\frac{1}{2}$. This decomposition contradicts the assumption that H does not have a $0, 1, \frac{1}{2}$-K_3-decomposition, completing the proof. ■

Let ${}^2G$ denote the multigraph obtained from a simple graph G by replacing each edge by a pair of parallel edges. For an edge e of G, denote the corresponding pair of edges of ${}^2G$ by e_1 and e_2. If τ_1 and τ_2 are edge-disjoint triangles of ${}^2G$ with the same vertex set, denote the corresponding triangle of G by τ. For $u, v \in V({}^2G)$, denote the set of edges joining u and v by $E(u, v)$. The characterization of rationally triangle decomposable planar graphs follows.

Corollary 7. *A simple planar graph G is rationally K_3-decomposable if and only if ${}^2G$ is K_3-decomposable.*

Proof. Suppose G is rationally K_3-decomposable. By Theorem 6, G has a $0, 1, \frac{1}{2}$-K_3-decomposition. Let $\mathcal{T}_{\frac{1}{2}}$ and $\mathcal{T}_1$ denote the sets of triangles of G with weights $\frac{1}{2}$ and 1, respectively. For any triangle $efg \in \mathcal{T}_1$, partition the edges $e_i, f_i, g_i,\ i = 1, 2$, of ${}^2G$ arbitrarily into two triangles τ_1 and τ_2, and let $w(\tau_1) = w(\tau_2) = 1$. Any edge of G that belongs to a triangle in $\mathcal{T}_{\frac{1}{2}}$ belongs to exactly two triangles in $\mathcal{T}_{\frac{1}{2}}$ and to no triangles in $\mathcal{T}_1$. Therefore, for the set of edges of G that belong to triangles in $\mathcal{T}_{\frac{1}{2}}$, the corresponding set of edge pairs of ${}^2G$ can be partitioned into edge-disjoint triangles, each being allocated weight 1, to give a K_3-decomposition of ${}^2G$.

Conversely, assume ${}^2G$ is K_3-decomposable. For vertices x, y, z of ${}^2G$ and edges $e_1, e_2 \in E(x, y),\ f_1, f_2 \in E(y, z)$ and $g_1, g_2 \in E(x, z)$, if $e_i, f_i, g_i,\ i = 1, 2$, can be partitioned into triangles τ_1 and τ_2 such that $w(\tau_1) = w(\tau_2) = 1$, let $w(\tau) = 1$, and if e_i, f_i, g_i can be partitioned into triangles τ_1 and τ_2 such that (say) $w(\tau_1) = 0$ and $w(\tau_2) = 1$, let $w(\tau) = \frac{1}{2}$. Since each edge that belongs to τ_1 also belongs to another triangle of ${}^2G$ with weight 1, this gives a rational K_3-decomposition of G. ■

Corollary 8. *If G is a rationally K_3-decomposable planar graph, then $|E(G)| \equiv 0 \pmod 3$.*

Proof. By Corollary 7, ${}^2G$ has a K_3-decomposition. Hence $|E({}^2G)| \equiv 0 \pmod 3$. Since $|E({}^2G)| = 2|E(G)|$, we also have $|E(G)| \equiv 0 \pmod 3$. ■

3.2. Multigraphs with K_4 as underlying graph

One reason why the proof of Theorem 6 fails for multigraphs is that multiple edges that do not lie on triangular faces are not necessarily contained in separating triangles. Hence statement (4) in the proof of Theorem 6 does not necessarily hold; certainly, if its underlying graph is complete, a multigraph contains no separating triangles at all. While it is perhaps premature to conjecture that every rationally K_3-decomposable planar multigraph has a $0, 1, \frac{1}{2}$-K_3-decomposition, we now present evidence to support such a conjecture by characterizing K_3-decomposable and rationally K_3-decomposable multigraphs that have K_4 as underlying graph. This result may aid our intuitive understanding of the behaviour of rationally K_3-decomposable multigraphs.

It is easy to see that a multigraph G with K_3 as underlying graph is rationally K_3-decomposable if and only if all edges have the same multiplicity, say k. In this case, $|E(G)| = 3k$ and G can be decomposed into k edge-disjoint triangles.

Denote the set of all multigraphs that have K_4 as underlying graph by $\mathcal{K}_4$. For any $G \in \mathcal{K}_4$ and distinct edges e and f, let $w(e, f)$ be the sum of the weight of the triangles that contain both e and f, and for any vertices u, v of G, let $w(uv, e)$ be the sum of the weight of the triangles that contain e and some edge joining u and v. Also, for $u, v \in V(G)$, denote the set of edges joining u and v by $E(u, v)$.

Say $V(G) = \{a, b, c, d\}$. The following notation will be used throughout this subsection (see Figure 3). Let $\mu(a, b) = r$, $\mu(a, c) = s$, $\mu(a, d) = t$, $\mu(b, c) = x$, $\mu(b, d) = y$ and $\mu(c, d) = z$, and let

$$\begin{aligned} E(a, b) &= \{e_1, \ldots, e_r\}, \\ E(a, c) &= \{f_1, \ldots, f_s\}, \\ E(a, d) &= \{g_1, \ldots, g_t\}, \\ E(b, c) &= \{h_1, \ldots, h_x\}, \\ E(b, d) &= \{\ell_1, \ldots, \ell_y\}, \\ E(c, d) &= \{m_1, \ldots, m_z\}. \end{aligned} \tag{1}$$

Theorem 9. *Let $G \in \mathcal{K}_4$, let $u \in V(G)$ and let $V(G) \setminus \{u\} = \{v_1, v_2, v_3\}$. Then G is K_3-decomposable if and only if*

(i) *there exists an integer n such that $0 \leq n \leq \min\{\mu(v_i, v_j) : i, j \in \{1, 2, 3\}, i \neq j\}$ and $\mu(u, v_i) = \mu(v_i, v_j) + \mu(v_i, v_k) - 2n$ for each $i \in \{1, 2, 3\}$, each $j \in \{1, 2, 3\} \setminus \{i\}$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$,*

and rationally K_3-decomposable if and only if

(ii) *there exists an integer n' such that $0 \leq \frac{n'}{2} \leq \min\{\mu(v_i, v_j) : i, j \in \{1, 2, 3\}, i \neq j\}$ and $\mu(u, v_i) = \mu(v_i, v_j) + \mu(v_i, v_k) - n'$ for each $i \in \{1, 2, 3\}$, each $j \in \{1, 2, 3\} \setminus \{i\}$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$.*

Moreover, if G is rationally K_3-decomposable, it has a $0, 1, \frac{1}{2}$-K_3-decomposition.

Proof. To simplify notation, let $V(G) = \{a, b, c, d\}$ and assume without loss of generality that (i) holds with $u = a$. With notation as in (1), if $n > 0$, let

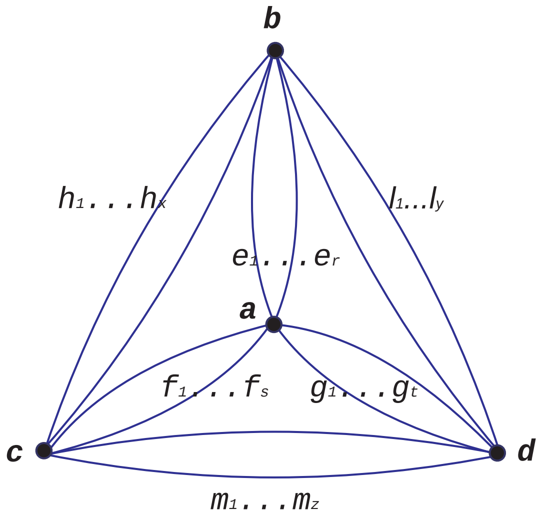

Figure 3. Labels of the vertices and edges of the multigraph G with K_4 as underlying graph.

$$G' = G - \bigcup_{i=0}^{n-1} \{h_{x-i}\ell_{y-i}m_{z-i}\} \tag{2}$$

and let $x' = x - n,\ y' = y - n$ and $z' = z - n$. Then in G',

$$\begin{aligned} E'(a,b) &= \{e_1, \ldots, e_r\}, \\ E'(a,c) &= \{f_1, \ldots, f_s\}, \\ E'(a,d) &= \{g_1, \ldots, g_t\}, \\ E'(b,c) &= \{h_1, \ldots, h_{x'}\}, \\ E'(b,d) &= \{\ell_1, \ldots, \ell_{y'}\}, \\ E'(c,d) &= \{m_1, \ldots, m_{z'}\}, \end{aligned}$$

and (i) holds for G' and a with $n = 0$. Now

$$E(G') = \left(\bigcup_{i=1}^{x'} \{e_i f_i h_i\}\right) \cup \left(\bigcup_{i=1}^{y'} \{e_{x'+i} g_i \ell_i\}\right) \cup \left(\bigcup_{i=1}^{z'} \{f_{x'+i} g_{y'+i} m_i\}\right). \tag{3}$$

As each set of three edges in (2) and (3) induces a triangle, G is K_3-decomposable.

Conversely, suppose G is K_3-decomposable into α triangles induced by $\{b, c, d\}$, β triangles induced by $\{a, c, d\}$, ψ triangles induced by $\{a, b, d\}$ and δ triangles induced by $\{a, b, c\}$. Then

$$\begin{aligned}
\mu(a,b) &= \psi + \delta,\\
\mu(a,c) &= \beta + \delta,\\
\mu(a,d) &= \beta + \psi,\\
\mu(b,c) &= \alpha + \delta, \text{ hence } \delta = \mu(b,c) - \alpha,\\
\mu(b,d) &= \alpha + \psi, \text{ hence } \psi = \mu(b,d) - \alpha,\\
\mu(c,d) &= \alpha + \beta, \text{ hence } \beta = \mu(c,d) - \alpha,
\end{aligned}$$

and thus

$$\begin{aligned}
\mu(a,b) &= \mu(b,c) + \mu(b,d) - 2\alpha,\\
\mu(a,c) &= \mu(b,c) + \mu(c,d) - 2\alpha,\\
\mu(a,d) &= \mu(b,d) + \mu(c,d) - 2\alpha.
\end{aligned}$$

Since $\beta, \psi, \delta \geq 0$, $\alpha \leq \min\{\mu(b,c), \mu(b,d), \mu(c,d)\}$. Therefore (i) holds for $u = a$ and $n = \alpha$. Similarly, (i) holds for b, c and d with $n = \beta, \psi$ and δ, respectively.

Suppose (ii) holds with $u = a$. If n' is even, let $n' = 2n$. Then (i) holds and G is K_3-decomposable. Hence assume n' is odd. Say $n' = 2n + 1$ and let

$$G' = G - \{e_r, f_s, g_t, h_x, \ell_y, m_z\}.$$

Since $n + 1 \leq \min\{x, y, z\}$, $n \leq \min\{x - 1, y - 1, z - 1\}$. The equations $r = x + y - 2n - 1$, $s = x + z - 2n - 1$ and $t = y + z - 2n - 1$ in G imply the equations $r - 1 = (x-1) + (y-1) - 2n$, $s - 1 = (x-1) + (z-1) - 2n$ and $t - 1 = (y-1) + (z-1) - 2n$ in G'. Hence (i) holds for G' with $u = a$, and G' is K_3-decomposable.

As $\{e_r, f_s, g_t, h_x, \ell_y, m_z\}$ induces a K_4, which is rationally K_3-decomposable into four triangles, each of weight $\frac{1}{2}$, G is rationally K_3-decomposable using only weights of 0, 1 and $\frac{1}{2}$.

Conversely, say G is rationally K_3-decomposable and consider such a decomposition of G. For each edge $e_j \in E(a,b)$, any triangle that contains e_j also contains one edge in $E(b,c) \cup E(b,d)$. Since $w(e_j) = 1$,

$$\sum_{i=1}^{x} w(e_j, h_i) + \sum_{i=1}^{y} w(e_j, \ell_i) = 1,$$

hence

$$\sum_{i=1}^{x} w(ab, h_i) + \sum_{i=1}^{y} w(ab, \ell_i) = r. \tag{4}$$

Similarly,

$$\sum_{i=1}^{x} w(ac, h_i) + \sum_{i=1}^{z} w(ac, m_i) = s$$

and

$$\sum_{i=1}^{y} w(ad, \ell_i) + \sum_{i=1}^{z} w(ad, m_i) = t.$$

Let $\mathcal{T}$ be the set of all triangles that do not contain any edges incident with a, that is, triangles of the form $h_i\ell_j m_k$, $i = 1,\dots,x$, $j = 1,\dots,y$, $k = 1,\dots,z$, and let ω be the total weight of the triangles in $\mathcal{T}$. Then $\omega \le \min\{x, y, z\}$. For any edge h_i, any triangle that contains h_i but no edge in $E(a, b)$ belongs to $\mathcal{T}$. Hence $\sum_{i=1}^{x} w(ab, h_i) + \omega = x$. Similarly, $\sum_{i=1}^{y} w(ab, \ell_i) + \omega = y$. Substitution in (4) gives $r = x + y - 2\omega$. Similarly, $s = x + z - 2\omega$ and $t = y + z - 2\omega$. Since r, x, y are integers, 2ω is an integer, say $2\omega = n'$. Then (ii) holds for a. As before, similar arguments show that (ii) also holds for b, c and d.

As shown above, if (ii) holds, then G has a $0, 1, \frac{1}{2}$-K_3-decomposition. This proves the last part of the theorem. ■

By taking $\mu(v_1, v_2) = 0$ in Theorem 9(ii), we get the following corollary.

Corollary 10. *Let G be a multigraph whose underlying graph is $K_4 - e$. Say $V(G) = \{u, v, v_1, v_2\}$, where u and v correspond to the vertices of $K_4 - e$ of degree three. The following conditions are equivalent.*

1. *G is rationally K_3-decomposable.*
2. *G is K_3-decomposable.*
3. *$\mu(u, v) = \mu(v, v_1) + \mu(v, v_2)$, $\mu(u, v_1) = \mu(v, v_1)$ and $\mu(u, v_2) = \mu(v, v_2)$.*

The final corollary now follows similar to Corollary 8.

Corollary 11. *If G is a rationally K_3-decomposable multigraph whose underlying graph is K_3, K_4 or $K_4 - e$, then $|E(G)| \equiv 0 \pmod 3$.*

4. Open Questions

1. Does Theorem 6 hold for rationally K_3-decomposable planar multigraphs?
2. Can we characterize rationally K_3-decomposable planar multigraphs or outerplanar multigraphs?
3. What can we say about graphs embeddable on other surfaces?

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