Discussiones Mathematicae
Graph Theory 36 (2016) 661–668
doi:10.7151/dmgt.1880

SOME VARIATIONS OF PERFECT GRAPHS

Magda Dettlaff, Magdalena Lemańska

Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
ul. Narutowicza 11/12, 80–233 Gdańsk, Poland

e-mail: mdettlaff,magda@pg.gda.pl

Gabriel Semanišin

Institute of Computer Science
P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovakia

e-mail: gabriel.semanisin@upjs.sk

and

Rita Zuazua

Faculty of Science
National Autonomous University of Mexico
Av. Universidad 3000, Mexico City, Mexico

e-mail: ritazuazua@ciencias.unam.mx

Abstract

We consider $(\psi_k-\gamma_{k-1})$-perfect graphs, i.e., graphs G for which $\psi_k(H) = \gamma_{k-1}(H)$ for any induced subgraph H of G, where ψ_k and γ_{k-1} are the k-path vertex cover number and the distance $(k-1)$-domination number, respectively. We study $(\psi_k-\gamma_{k-1})$-perfect paths, cycles and complete graphs for $k \geq 2$. Moreover, we provide a complete characterisation of $(\psi_2-\gamma_1)$-perfect graphs describing the set of its forbidden induced subgraphs and providing the explicit characterisation of the structure of graphs belonging to this family.

Keywords: k-path vertex cover, distance k-domination number, perfect graphs.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

The graphs considered here are finite, undirected and simple, with n vertices. In general we follow the terminology of [5].

Let $G = (V, E)$ a graph. Let us denote the cardinality of V by n. The *neighbourhood* of a vertex $v \in V$ is the set $N_G(v)$ of all vertices adjacent to v in G. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges incident with v. For a set $X \subseteq V$, the *open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* $N_G[X] = N_G(X) \cup X$. If H is an induced subgraph of G, then we write $H \leqslant G$.

A subset D of V is *dominating* in G if every vertex of $V \setminus D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. A set $D \subseteq V$ of vertices is said to be a *distance k-dominating set* if every vertex of $V \setminus D$ is at distance at most k from D. Minimum cardinality of a distance k-dominating set of G is the *distance k-domination number* of G, denoted by $\gamma_k(G)$ (see also [6, 7]). By definition, $\gamma_1(G) = \gamma(G)$.

A subset S of vertices of a graph G is called a *k-path vertex cover* if every path of order k in G contains at least one vertex from S. Denote by $\psi_k(G)$ the minimum cardinality of a k-path vertex cover in G (for more details we refer the reader to [3]).

Let σ and ρ be two types of graph theoretical parameters. We say that a graph G is *$(\sigma-\rho)$-perfect* if $\sigma(H) = \rho(H)$ for every connected induced subgraph H of G. The most well-known example of $(\sigma - \rho)$-perfect graphs are perfect graphs (see e.g. [1]) which can be obtained by replacing σ and ρ by the chromatic number χ and the clique number ω.

In [9], for $\gamma_c(G)$ the connected domination number of G, the $(\gamma - \gamma_c)$-perfect graphs are called *perfect connected dominant* graphs. The author gives the following characterization.

Theorem 1 [9]**.** *A connected graph G is a perfect connected dominant graph if and only if G contains neither an induced path P_5 nor an induced cycle C_5.*

In [4], $(\gamma - \gamma_w)$-perfect graphs were characterized, where γ_w is a weakly connected domination number of a graph G. Here we consider $(\psi_k - \gamma_{k-1})$-perfect graphs for $k \geq 2$.

Our study is partially motivated by the following real life situation concerning availability of some important services in a town (e.g. installation of telephone in the developing countries). One wants to guarantee that each service is available at most at the distance k from every house or flat. We also can prefer that on every path of given length you can find at least one instance of the considered service. If we model this situation then it is quite easy to see that those two goals can be measured by k-distance domination number γ_k and k-path vertex cover ψ_k.

Since $\psi_k(G) = 0$ for $k > n$, we consider k such that $2 \leq k \leq n$. Obviously, $\gamma_{k-1}(G) \leq \psi_k(G)$ for any graph G with no isolates and such that $\psi_k(G) > 0$. Moreover, the difference between ψ_k and γ_{k-1} can be arbitrarily large (see Lemma 2). Nevertheless, we describe $(\psi_k - \gamma_{k-1})$-perfect graphs defined below.

Definition. For a graph G without isolated vertices let $\mathcal{H}_k(G) = \{H \leqslant G : \psi_k(H) \geq 1$ and H has no isolates$\}$. We say that a graph G is $(\psi_k - \gamma_{k-1})$-perfect if $\psi_k(H) = \gamma_{k-1}(H)$ for every induced subgraph $H \in \mathcal{H}_k(G)$.

In the next section, we study $(\psi_k - \gamma_{k-1})$-perfect paths, cycles and complete graphs for $k \geq 2$. In the last section, Theorems 10 and 11 give a full characterization of $(\psi_k - \gamma_{k-1})$-perfect graphs for $k = 2$.

2. Characterization of $(\psi_k - \gamma_{k-1})$-Perfect Graphs

We begin with the following lemma.

Lemma 2. *For any positive integer p there exists a graph G such that $\psi_k(G) - \gamma_{k-1}(G) = p$.*

Proof. Let p be a positive integer. We construct the graph G such that $\psi_k(G) - \gamma_{k-1}(G) = p$. We start with the star $K_{1,p+1}$ with central vertex v and end vertices $v_1, v_2, \ldots, v_{p+1}$. Then to every vertex v_i we add a cycle of the length $k-1$ by identifying any vertex of the cycle with the vertex v_i $(1 \leq i \leq p+1)$. In such a graph G vertex v forms a minimum distance k-dominating set of G and all neighbours of the vertex v form a minimum k-path vertex cover of G. Hence, $\psi_{k-1}(G) = p+1$ and $\gamma_k(G) = 1$. ■

It is easy to determine the values of ψ_k and γ_{k-1} for paths and cycles.

Remark 3. Let n be a positive integer. Then

(i) $\psi_k(P_n) = \lfloor \frac{n}{k} \rfloor$ and $\gamma_k(P_n) = \left\lceil \frac{n}{2k+1} \right\rceil$,

(ii) $\psi_k(C_n) = \lceil \frac{n}{k} \rceil$ and $\gamma_k(C_n) = \left\lceil \frac{n}{2k+1} \right\rceil$.

From Remark 3 we obtain Tables 1 and 2 which allow us to compare the behaviour of ψ_k and γ_{k-1} for paths and cycles.

By Table 1, if $k \leq n < 3k$, then $\psi_k(P_n) = \gamma_{k-1}(P_n)$. Moreover, if $n = 3k$, then $3 = \psi_k(P_{3k}) \neq \gamma_{k-1}(P_{3k}) = 2$. Since $P_{3k} \leqslant P_n$ for $n > 3k$, we have the following theorem.

Theorem 4. *The graph P_n is $(\psi_k - \gamma_{k-1})$-perfect if and only if $k \leq n < 3k$.*

We have a similar theorem for cycles.

Table 1. $\psi_k(P_n)$ and $\gamma_{k-1}(P_n)$.

s	$\psi_k(P_n)$	$\gamma_{k-1}(P_n)$
1	$k \le n < 2k$	$k \le n < 2k$
2	$2k \le n < 3k$	$2k \le n \le 4k-2$
3	$3k \le n < 4k$	$4k-1 \le n \le 6k-3$
$s \ge 4$	$sk \le n < (s+1)k$	$2k(s-1)-(s-2) \le n \le (2k-1)s$

Table 2. $\psi_k(C_n)$ and $\gamma_{k-1}(C_n)$.

s	$\psi_k(C_n)$	$\gamma_{k-1}(C_n)$
1	$n \le k$	$k \le n \le 2k-1$
2	$k < n \le 2k$	$2k-1 < n \le 4k-2$
$s \ge 3$	$(s-1)k < n \le sk$	$(s-1)(2k-1) < n \le (2k-1)s$

Theorem 5. *The graph C_n is $(\psi_k - \gamma_{k-1})$-perfect if and only if $n = k$ or $n = 2k$.*

Proof. • For $n \ge 3k+1$, $P_{3k} \leqslant C_n$; by the above theorem the cycle C_n is not a $(\psi_k - \gamma_{k-1})$-perfect graph.

• If $2k < n \le 3k$, then from Table 2, $3 = \psi_k(C_n) \ne \gamma_{k-1}(C_n) = 2$. Hence, C_n is not a $(\psi_k - \gamma_{k-1})$-perfect graph.

• If $k < n < 2k$, then from Table 2, $2 = \psi_k(C_n) \ne \gamma_{k-1}(C_n) = 1$. Hence, C_n is not $(\psi_k - \gamma_{k-1})$-perfect.

• If $n = k$, then from Table 2, $\psi_k(C_n) = 1 = \gamma_{k-1}(C_n)$ and C_n is the only induced subgraph $H \in \mathcal{H}_k(C_n)$, so C_n is $(\psi_k - \gamma_{k-1})$-perfect.

• If $n = 2k$, then from Table 2, $\psi_k(C_n) = 2 = \gamma_{k-1}(C_n)$. Moreover, every induced subgraph $H \in \mathcal{H}_k(C_n)$ is either C_n or P_l, where $k \le l \le 2k-1$. By the above theorem, $\psi_k(P_l) = \gamma_{k-1}(P_l)$ and C_n is $(\psi_k - \gamma_{k-1})$-perfect. ■

A chord is an edge joining two non-adjacent vertices in a cycle. Let F be a set of chords of a graph G. For any $F' \subseteq F$ by $G \cup F'$ we denote a graph obtained from G by adding edges from the set F'.

Corollary 6. *If $k < n < 2k$ or $n > 2k$, then $C_n \cup F'$ is not a $(\psi_k - \gamma_{k-1})$-perfect graph.*

Proof. Since $\psi_k(C_n) \le \psi_k(C_n \cup F')$ and $\gamma_{k-1}(C_n) \ge \gamma_{k-1}(C_n \cup F')$ for any subset F' of chords of C_n, we obtain the desired result. ■

Theorem 7. *For $n \ge 2$ a complete graph K_n is $(\psi_k - \gamma_{k-1})$-perfect if and only if $n = k$.*

Proof. Since $\psi_k(K_n) = n - k + 1$, $\gamma_{k-1}(K_n) = 1$ and every induced subgraph of a complete graph is also a complete graph, we obtain the desired result. ∎

3. Characterization of $(\psi_2 - \gamma)$-Perfect Graphs

In [8] Volkmann characterized graphs G for which $\psi_2(G) = \gamma(G)$. We now provide a complete characterization of $(\psi_2 - \gamma)$-perfect graphs in terms of the family $\mathcal{F}$ of forbidden subgraphs. The graphs belonging to $\mathcal{F}$ are depicted in Figure 1.

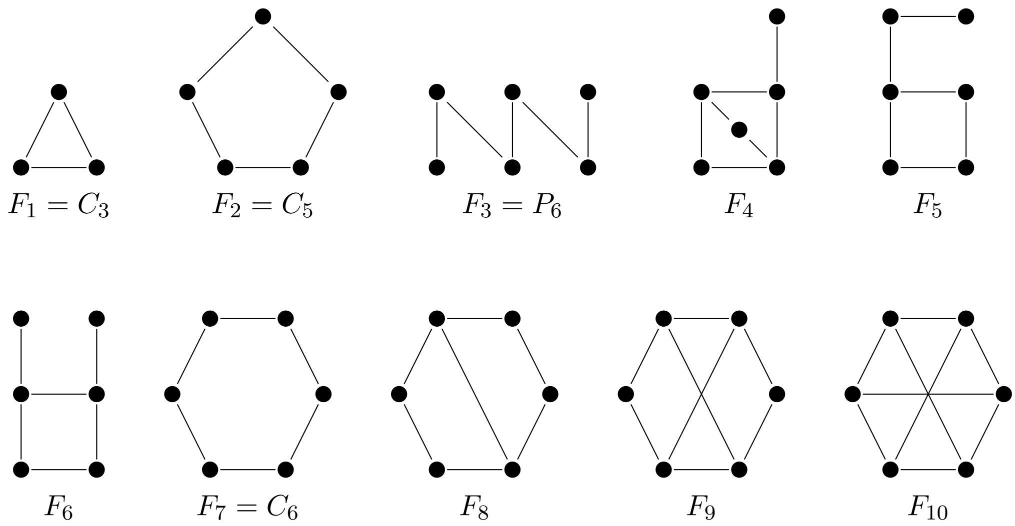

Figure 1. Family $\mathcal{F}$ of forbidden subgraphs.

Theorem 8. *Graph G is $(\psi_2 - \gamma)$-perfect if and only if G does not contain any graph from the family $\mathcal{F}$ as an induced subgraph.*

Proof. One can easily verify that for any graph F_i, $i = 1, 2, \ldots, 10$, belonging to $\mathcal{F}$ we have $\gamma(F_i) < \psi_2(F_i)$. It means that none of them is $(\psi_2 - \gamma)$-perfect. Moreover, each of their induced subgraphs, which is not an isolated vertex, is $(\psi_2 - \gamma)$-perfect.

Let us prove now the opposite implication. Suppose to the contrary that G does not contain any graph from the family $\mathcal{F}$ as an induced subgraph and G is not $(\psi_2 - \gamma)$-perfect. Let G be the minimum counterexample. Thus $\gamma(G) < \psi_2(G)$.

Let D be a minimum dominating set of G and, without loss of generality, chosen such that it has the minimum number of edges in $G[V - D]$ among all minimum dominating sets of G. Since $\gamma(G) < \psi_2(G)$, D is not a (2-path) vertex cover of G and there exists an edge ab such that $a, b \in V - D$.

If the vertices of the edge ab are dominated by the same vertex u belonging to D, then we find an induced C_3 in G, a contradiction. Therefore a, b must be dominated by two different vertices $u, v \in D$. Without loss of generality, let $ua, vb \in E(G)$. Moreover, among all possible u and v choose such a pair that their mutual distance in G is the smallest possible.

If one of u and v have degree one, say u, then $D' = D - \{u\} \cup \{a\}$ is a minimum dominating set of G with fewer edges in $G[V - D']$, a contradiction. Hence we can suppose that both u and v have degree at least 2 and we have edges uu', vv' for some vertices u', v'. Observe that $u' \neq v'$, otherwise C_3 or C_5 appears in the graph, which is a contradiction.

Now we distinguish two cases.

Case 1. Assume $uv \notin E(G)$. Since there is no induced C_3 in G, we have $u'a, v'b, va, ub \notin E(G)$. Suppose $u'v \in E(G)$. Since there is no induced C_3 in G, we obtain $G[\{u, u', b, v, a\}] = C_5$, a contradiction. So $u'v \notin E(G)$. By an application of analogous arguments we obtain $uv' \notin E(G)$.

Suppose now $u'b \in E(G)$. If $av' \notin E(G)$ and $u'v' \notin E(G)$, then $G[\{u, u', b, v, a, v'\}] = F_5$. If $av' \notin E(G)$ and $u'v' \in E(G)$, then $G[\{u, u', b, v, a, v'\}] = F_8$. It means that av' must belong to $E(G)$. Then if $u'v' \in E(G)$, then $G[\{u, u', b, v, a, v'\}] = F_9$ and if $u'v' \notin E(G)$, then $G[\{u, u', b, v, a, v'\}] = F_8$. Thus we can conclude that $u'b \notin E(G)$ and because of symmetry $v'a \notin E(G)$.

It is easy to see that if $u'v' \in E(G)$, then $G[\{u, u', b, v, a, v'\}] = C_6$ and if $u'v' \notin E(G)$, then $G[\{u, u', b, v, a, v'\}] = P_6$. Hence in all subcases we have an induced subgraph isomorphic to some graph from $\mathcal{F}$, a contradiction.

Case 2. Assume $uv \in E(G)$. Suppose now that $d_G(u) = 2$. Then $u' = v$ and $D' = D - \{u\} \cup \{a\}$ is a minimum dominating set with less edges in $G[V - D']$, a contradiction. Hence, $d_G(u) > 2$. Similarly we can show that $d_G(v) > 2$. Since G has no induced C_3, if none of the edges $u'b$, $u'v'$ and $v'a$ are in $E(G)$ we have $G[\{u, u', b, v, a, v'\}] = F_6$. So at least one of these edges must be in $E(G)$.

Evidently if $u'v' \in E(G)$, then $G[\{u, u', b, v, a, v'\}]$ is isomorphic either to F_8, F_9 of F_{10} depending on the fact whether $u'b$, $v'a$ are in $E(G)$.

Hence we may suppose $u'v' \notin E(G)$. Then the graph $G[\{u, u', b, v, a, v'\}]$ is isomorphic either to F_4 providing that exactly one of $u'b$, $v'a$ is in $E(G)$, or to F_9 whenever both edges $u'b$, $v'a$ are in $E(G)$. ∎

Now we can describe the family of $(\psi_2 - \gamma)$-perfect graphs. From the previous result it follows that this class is infinite but the structure of its members is relatively simple.

Theorem 9. *G is a connected $(\psi_2 - \gamma)$-perfect graph if and only if it is one of the graphs:*

(i) *a tree with diameter at most* 4;

(ii) *a bipartite graph* $G = (A, B; E)$*, where* $|A| = 2$ *and* $|B| \geq 2$.

Proof. It is easy to verify that if a graph satisfies condition (i) or (ii) of Theorem 9, then it is $(\psi_2 - \gamma)$-perfect.

Suppose now that G is a connected $(\gamma - \tau)$-perfect graph. If G is a tree, then since G is F_3-free, it must be a tree with diameter at most 4. Suppose that G has a cycle. It is known (cf. [2]), that C_3-, C_5-, C_6-, P_6-free graphs are chordal bipartite graphs (each cycle of length at least 6 has a chord). Since a connected $(\gamma - \tau)$-perfect graph is $(F_1 - F_3)$- and $(F_7 - F_{10})$-free, it contains only cycles on 4 vertices. It implies that G is a bipartite graph with vertex partition into sets A and B where $|A| \geq 2$ and $|B| \geq 2$. Assume that $|A| \geq 3$ and $|B| \geq 3$. Let (u_1, v_1, u_2, v_2) be a cycle in G, where $\{u_1, u_2\} \subset A$ and $\{v_1, v_2\} \subset B$. Since G is connected, $|A| \geq 3$ and $|B| \geq 3$; without loss of generality we assume that there is a vertex $x \in A$ adjacent to v_1 or to v_2. Moreover, $B - \{v_1, v_2\} \neq \emptyset$ and from this set we choose a vertex y such that the number $d_G(y, \{x, u_1, u_2\})$ is the smallest one from all possible values. Hence, y must be adjacent to at least one vertex from the set $\{x, u_1, u_2\}$. We consider the following cases:

Case 1. $xv_1 \in E(G)$ and $xv_2 \in E(G)$.

Case 1.1. If y is adjacent to only one vertex from $\{x, u_1, u_2\}$, then $F_4 \leqslant G$, a contradiction;

Case 1.2. if y is adjacent to exactly two vertices from $\{x, u_1, u_2\}$, then $F_9 \leqslant G$, a contradiction;

Case 1.3. if y is adjacent to each vertex from $\{x, u_1, u_2\}$, then $F_{10} \leqslant G$, a contradiction.

Case 2. Without loss of generality, $xv_1 \in E(G)$ and $xv_2 \notin E(G)$. Consider the following subcases.

Case 2.1. If y is adjacent to only one vertex from $\{x, u_1, u_2\}$, then

- if $yx \in E(G)$, then $F_5 \leqslant G$, a contradiction;
- if $yu_2 \in E(G)$, then $F_6 \leqslant G$, a contradiction;
- if $yu_2 \in E(G)$, then $F_6 \leqslant G$, a contradiction.

Case 2.2. If y is adjacent to exactly two vertices from $\{x, u_1, u_2\}$, then

- if $yu_1 \in E(G)$ and $yu_2 \in E(G)$, then $F_4 \leqslant G$, a contradiction;
- if $yx \in E(G)$ and $yu_2 \in E(G)$, then $F_8 \leqslant G$, a contradiction;
- if $yu_1 \in E(G)$ and $yx \in E(G)$, then $F_8 \leqslant G$, a contradiction.

Case 2.3. If y is adjacent to each vertex from $\{x, u_1, u_2\}$, then $F_9 \leqslant G$, a contradiction.

In all cases we obtained a contradiction. Hence one partition has exactly two vertices and the other one has at least two vertices. ∎

Acknowledgement

The third and fourth authors are thankful to Gdańsk University of Technology whose hospitality they enjoyed during the preparation of this paper.

The research of the third author was also supported under the grant numbers APVV-0035-10 and projects ITMS 26220120007 and FP7 EU CELIM 316310. The rest of the authors were supported under the grant PAPIIT-IN114415-UNAM.

References

[1] C. Berge, *Färbung von Graphen, deren samtliche bzw. deren ungerade Kreise starr sind*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **10** (1961) 114.

[2] A. Brandstädt, V.B. Le and J.P. Spinrad, Graph Classes: A Survey (Monographs on Discrete Math. Appl.) (SIAM, Philadelphia, 1999). doi:10.1137/1.9780898719796

[3] B. Brešar, F. Kardoš, J. Katrenič and G. Semanišin, *Minimum k-path vertex cover*, Discrete Appl. Math. **159** (2011) 1189–1195. doi:10.1016/j.dam.2011.04.008

[4] G.S. Domke, J.H. Hattingh and L.R. Markus, *On weakly connected domination in graphs* II, Discrete Math. **305** (2005) 112–122. doi:10.1016/j.disc.2005.10.006

[5] T. Haynes, S. Hedetniemi and P. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, 1998).

[6] M.A. Henning, O.R. Oellermann and H.C. Swart, *Bounds on distance domination parameters*, J. Combin. Inform. System Sci. **16** (1991) 11–18.

[7] M.A. Henning, O.R. Oellermann and H.C. Swart, *Relationships between distance domination parameters*, Math. Pannon. **5** (1994) 69–79.

[8] L. Volkmann, *On graphs with equal domination and covering numbers*, Discrete Appl. Math. **51** (1994) 211–217. doi:10.1016/0166-218X(94)90110-4

[9] I.E. Zverovich, *Perfect-connected-dominant graphs*, Discuss. Math. Graph Theory **23** (2003) 159–162. doi:10.7151/dmgt.1192

Received 3 March 2015
Revised 23 September 2015
Accepted 9 October 2015