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CYCLE DOUBLE COVERS OF INFINITE PLANAR GRAPHS

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Abstract

In this paper, we study the existence of cycle double covers for infinite planar graphs. We show that every infinite locally finite bridgeless k-indivisible graph with a 2-basis admits a cycle double cover.

Keywords: cycle double cover, infinite planar graph.

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1. Introduction

A *cycle double cover* (CDC) of a graph G is a collection of cycles of G that altogether cover every edge of G exactly twice. The CDC conjecture, formulated independently by Szekeres [23] and Seymour [21], states that every bridgeless graph has a cycle double cover. Here, a bridgeless graph is a graph where the removal of any edge increases the number of connected components. A bridgeless connected graph is called 2-edge connected, while a 2-vertex connected (or 2-connected) graph is a graph that stays connected after the removal of any vertex.

The CDC conjecture is a consequence of the Strong Embedding conjecture, which states that every 2-connected graph can be drawn on a surface without its edges crossing each other, each face is homeomorphic to an open disc, and each face boundary is a cycle of the graph.

Despite many efforts and a variety of techniques, the CDC conjecture has remained open. One approach is to study the smallest counterexample (with the smallest number of edges) to the CDC conjecture. It is known that such a smallest counterexample is:

(i) simple i.e., it has no loops or double edges;
(ii) cubic i.e., every vertex has degree 3;
(iii) cyclically 4-edge-connected i.e., every cut separating the graph into non-acyclic components has at least 4 edges;
(iv) with chromatic index 4;
(v) with girth at least 12.

In particular, such a counterexample is a *snark* (a 2-edge connected cubic graph with chromatic index 4). Snarks were first introduced by Tait, who showed that the four color theorem is equivalent to the statement that snarks are not planar; [24]. Almost a hundred years after their introduction in 1880, mathematicians had a list of only five snarks. Hence it was conjectured that there are only finitely many of them, until Isaacs constructed infinite families of snarks; [17].

The *girth* of a graph is the length of its shortest circuit. Jaeger and Swart conjectured that the largest girth of a snark is 6; if true, the CDC conjecture follows, since Goddyn [13] gave the lower bound of 10 (improved by Huck [16] to 12) for the girth of a smallest counterexample of the CDC conjecture.

Another approach in tackling the CDC conjecture is the study of nowhere-zero k-flows introduced by Tutte. Tutte showed that in a finite plane graph G the nowhere-zero k-flows correspond to the k-colorings of the faces of G; [27].

A *nowhere-zero k-flow* on a directed graph G with the edge set $E(G)$ is a function $\phi : E(G) \to \{1, 2, \ldots, k-1\}$ such that for every vertex v in G the sum of $\phi(e)$ for the edges e ending in v is equal (modulo k) to the sum of $\phi(e)$ for the edges e starting at v. Tutte [27] conjectured that every bridgeless graph has a nowhere-zero 5-flow. Jaeger [19] proved the 8-flow theorem, which states that every bridgeless graph admits a nowhere-zero 8-flow. Seymour [22] proved the 6-flow theorem, which states that every bridgeless graph admits a nowhere-zero 6-flow.

Nowhere-zero k flows have been be used to construct certain types of cycle covers. An m-cycle k-cover is a collection of m Eulerian subgraphs that altogether cover each edge exactly k times. Bermond *et al.* [1] used the 8-flow theorem to prove that every bridgeless graph admits a 7-cycle 4-cover. Fan [9] proved a similar result for 10-cycle 6-covers using the 6-flow theorem.

More recently, necessary and sufficient conditions for the existence of 5-cycle double covers have been investigated in connection with the strong cycle double conjecture. The strong cycle double cover conjecture states that given a circuit C in a bridgeless cubic graph, there exists a cycle double cover containing C. It is proved in [15] that the existence of a 5-cycle double cover of a cubic graph G containing a circuit C is equivalent to the existence of a matching M such that removing M yields a graph with a nowhere-zero 4-flow, and in addition, M is the intersection of the edge sets of two cycles C_1 and C_2 with $C \subseteq C_1$.

The existence of a CDC can be deduced from the existence of large cycles; [12, 29]. For example, a cycle in a cubic graph that misses only one vertex can be extended to a CDC [12]. A computer-aided investigation shows that if a cycle in a bridgeless cubic graph misses at most ten vertices, then the graph admits a CDC [2]; see also [14, 30] for other closely related results.

The CDC conjecture is related to a variety of topological conjectures on graph embeddings in surfaces. For a survey of the CDC conjecture and other related conjectures and results, see [4] and [18].

In this paper, we study the CDC of infinite planar graphs. Recall that a *plane* graph $G = (V, E)$ is a set $V \subseteq \mathbb{R}^2$ of vertices together with a set E of edges, where each edge is a simple arc with endpoints in V that does not intersect V, itself, or other edges, except possibly at its endpoints. A graph is *planar*, if it is isomorphic to a plane graph. If a graph G is planar, any plane graph isomorphic to G is called a *plane representation* of G.

Fary proved that every finite planar graph has a straight line representation, where all of the edges are straight segments [10]. Finite planar graphs are characterized by Kuratowski's theorem, which states that a finite graph is planar unless it contains a subdivision of K_5 or $K_{3,3}$. Another characterization of planar graphs is given by the MacLane's planarity criterion as explained in the sequel.

1.1. The cycle space

A *circuit* is a sequence of distinct vertices $v_1, \ldots, v_n$, together with a sequence of edges $e_1, \ldots, e_n$, such that the endpoints of e_i are v_i and v_{i+1} for all $1 \leq i \leq n$, where the indices are computed modulo n. A *cycle* is a finite union of circuits with mutually disjoint edge sets.

The *cycle space* of a graph G is the space of all cycles of G. The cycle space of G forms a vector space, where the addition of two cycles is the cycle whose edge set is the symmetric difference of the edge sets of the two cycles. A collection B of cycles of G is called a 2-*basis* (or a *simple generating subset*) of the cycle space of G, if every cycle can be written as a finite sum of the cycles in B, and in addition, every edge is contained in at most two cycles in B.

Every circuit C in a plane graph divides the rest of the plane into two disjoint open and connected sets, its (bounded) *interior* and (unbounded) *exterior*. If the interior (respectively, exterior) of C does not intersect G, then C is called a *facial cycle*, and the closure of its interior (respectively, exterior) of C is called a *face*.

In a finite 2-connected plane graph, facial cycles form a 2-basis. In fact, MacLane's planarity criterion states that a 2-connected finite graph is planar if and only if it has a 2-basis [20].

The facial cycles of a finite 2-connected plane graph provide a CDC of the graph; however, in an infinite plane graph, the facial cycles do not necessarily form a CDC (see Figure 1).

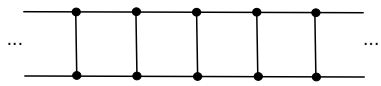

Figure 1. The facial cycles of the infinite ladder do not form a CDC.

1.2. Infinite planar graphs

Both Kuratowski's characterization and Fary's theorem remain valid for infinite planar graphs; the former was generalized by Wagner [28], and the latter was proved by Thomassen [26] for infinite graphs. However, the existence of a 2-basis for a 2-connected graph turns out to be more restrictive in the infinite case.

A point p in the plane is called a VAP (*vertex accumulation point*) of a plane graph G, if every neighborhood of p in the plane contains infinitely many vertices of G. The graph G is called VAP-*free*, if it has no VAP. Similarly, a point p is called an EAP (*edge accumulation point*) of G if every neighborhood of p intersects infinitely many edges of G; the graph G is called EAP-*free*, if it has no EAP.

The facial cycles of an infinite 2-connected VAP-free plane graph form a 2-basis. Conversely, Thomassen [25] proved that any 2-connected infinite graph G with a 2-basis has a VAP-free plane representation; in addition, G is countable and has a plane representation that is VAP-free and EAP-free.

More recently, Bruhn and Stein [3] showed that a countable locally finite graph is planar if and only if its topological cycle space has a simple generating set. The topological cycle space was introduced by Diestel and Kühl to study infinite cycles [7, 8]. In this paper, we do not consider infinite cycles. For an expository paper on topological cycle spaces of infinite graphs, see [6].

1.3. Main results

For a plane graph G, let $\partial G \subseteq \mathbb{R}^2$ be the topological union of edges that are contained in exactly one face of G. If G is a finite 2-connected plane graph, then $\partial G = \emptyset$ (since G has an unbounded face). However, in the infinite case, ∂G can be nonempty and have finitely or infinitely many connected components. The following theorem is the main result of this paper.

Theorem 1. *Let G be an infinite bridgeless plane graph without accumulation points. In addition, suppose that ∂G has finitely many connected components. Then G has a CDC.*

Every 2-connected graph is bridgeless, and every bridgeless graph G can be decomposed into 2-connected subgraphs $\{G_i\}_{i\in J}$ such that G_i and G_j intersect at most at one vertex for distinct indices $i, j \in J$. If every G_i, $i \in J$, has a CDC,

then G has a CDC. Therefore, in proving Theorem 1, we can assume, without loss of generality, that G is 2-connected.

Let v be a vertex of degree $k \neq 3$ in G connected to $v_1, \ldots, v_k$. If $k = 2$, remove v and combine the two edges connected to v into one edge. If $k > 3$, remove v and replace it by a k-gon with vertices $w_1, \ldots, w_k$ and connect v_i to w_i for each $1 \leq i \leq k$. Let G' be the graph obtained from G by applying this process to all vertices of degree $\neq 3$. If G satisfies the conditions of Theorem 1, then so does G'. If G' has a CDC, then so does G. Therefore, in proving Theorem 1, without loss of generality, we assume that G is cubic as well. Finally, if $\partial G = \emptyset$, then the facial cycles do form a CDC (since every edge is contained in exactly two faces). Therefore, without loss of generality, we will also assume that $\partial G \neq \emptyset$.

An infinite graph G is called *k-indivisible*, if the deletion of any finite number of vertices leaves at most $k - 1$ infinite connected components. The following theorem is a corollary of Theorem 1, but it is formulated independent of plane representations.

Theorem 2. *Let G be an infinite locally finite bridgeless graph with a 2-basis. In addition, suppose that G is k-indivisible for some $k \geq 1$. Then G has a CDC.*

A graph G admits a nowhere-zero 4-flow if and only if it has a 3-cycle double cover [18]. In particular, the size of the largest cycle appearing in the cover is at least $2|E(G)|/3$. We propose the following conjecture, which gives an upper bound on the size of cycles in the cover in the case of plane graphs.

Conjecture 3. *Let G be a finite plane graph, where the degree of every vertex and the number of edges in every face are bounded by K. Then there exists a cycle double cover of G by cycles of length at most $C(K)$, where $C(K)$ is a constant that depends only on K.*

Theorem 1 (and consequently, Theorem 2) follows from Conjecture 3 for plane graphs with an upper bound on the vertex degrees and face sizes (via a limiting argument).

The four color theorem holds true for infinite plane graphs, since by de Bruijn-Erdős theorem [5], if every finite subgraph of an infinite graph is k-colorable, then the infinite graph itself is k-colorable. Therefore, an infinite plane graph admits a nowhere-zero 4-flow. The four color theorem implies the strong cycle double cover conjecture for finite planar graphs i.e., if $S = \{C_1, \ldots, C_k\}$ is a set of edge-disjoint cycles of a finite bridgeless planar graph G, then there exists a CDC of G that extends S (see [11] for a proof that does not use the four color theorem). However, in the infinite case, it does not follow from the existence of a nowhere-zero 4-flow that the graph has a cycle double cover, because the construction that uses the nowhere-zero 4-flow to yield cycles in the finite case might not terminate

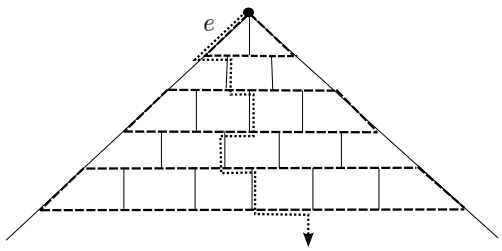

Figure 2. The dashed lines represent the cycles, while the dotted line represents a hypothetical second cycle passing through the edge e, which cannot exist because it will constantly move downward in this infinite graph.

in the infinite case. Hence, Theorem 1 does not immediately follow from the four-color theorem or the (finite) CDC conjecture.

It is worth mentioning that if S is an infinite set of edge-disjoint cycles in an infinite plane graph G, then there does not necessarily exist a CDC of G that extends S. Figure 2 shows a counterexample. The second cycle containing e in such a cycle double cover of the graph extending $S = \{C_1, C_2, \ldots\}$ must extend in a downward direction at each level, hence it cannot form a cycle. However, we prove in Theorem 15 that if S is a finite set of edge-disjoint cycles that do not intersect ∂G, then S can be extended to a CDC of G.

This is how this paper is organized. In Section 2, we introduce A-cuts and perfect partitionings. In Section 3, we prove the existence of perfect partitionings for infinite graphs satisfying the conditions of Theorem 1. In Section 4, we show how a perfect partitioning can be used to construct a CDC. Both Theorems 1 and 2 are proved in Section 4.

2. Partitioning Infinite Plane Graphs

In this section, G is an infinite 2-connected VAP-free and EAP-free cubic plane graph.

2.1. Topological considerations

Let $\mathcal{G}$ be the set of faces of G. For $\mathcal{C} \subseteq \mathcal{G}$, let $\bigcup \mathcal{C} \subseteq \mathbb{R}^2$ be the topological union of those faces of G that belong to $\mathcal{C}$. Since G is VAP-free, the interior of a facial cycle does not contain any vertices of G, hence each face of G (defined as the closure of the interior of the corresponding facial cycle) is a compact subset of the plane. Moreover, for $\mathcal{C} \subseteq \mathcal{G}$, the set $\bigcup \mathcal{C}$ is closed (compact, if $\mathcal{C}$ is finite). To see this, let $x = \lim_{i\to\infty} x_i$ with $x_i \in \bigcup \mathcal{C}$, $i \geq 1$. If there exists a face in $\mathcal{C}$

that contains x_i for i large enough, then that face must also include x (since each face is closed). Otherwise, x is an accumulation point for an infinite sequence of distinct faces, which contradicts our assumption that G is without accumulation points. It follows that $\bigcup \mathcal{C}$ is closed, and in particular, $\bigcup \mathcal{G}$ is closed. A similar argument shows that $\partial G \subseteq \mathbb{R}^2$ is also closed (recall that ∂G is the topological union of those edges that are contained in exactly one face).

A vertex or edge of G is called a *boundary vertex* or *edge*, if it is contained in ∂G; otherwise, it is called an *interior vertex* or *edge*. Therefore, each interior edge is contained in exactly two faces, while each boundary edge is contained in exactly one face. Also, an *interior face* is a face whose edges are all interior edges, while a *boundary face* is a face that contains at least one boundary edge.

We say two subsets of $\mathcal{G}$ are *adjacent*, if some face in one is adjacent to some face in the other.

A *one-way* (respectively, 2-*way*) *infinite path* in G is a sequence $\{v_i\}_{i\in J}$ of distinct vertices of G together with a sequence of edges $\{e_i\}_{i\in J}$ of G so that the endpoints of e_i are v_i and v_{i+1} for all $i \in J$, where $J = \mathbb{N}$ (respectively, $J = \mathbb{Z}$).

Since G is EAP-free, ∂G coincides with the topological boundary of $\bigcup \mathcal{G}$ (by the topological boundary of a set $X \subseteq \mathbb{R}^2$, we mean the set $X - X^\circ$, where X° is the set of interior points of X). Moreover, each connected component of ∂G is a 2-way infinite path. To see this, suppose ∂G has a finite connected component α. It follows that α is a closed path. Since the region inside α is compact and G is VAP-free, there must exist vertices of G in the exterior of α. Let f_1 be a face contained in the closure of the interior of α and f_2 be a face contained in the closure of the exterior of α. Since G is connected, there must exist a facial path connecting f_1 to f_2. But such a path includes two adjacent faces, one inside and one outside α; these two adjacent faces share an edge of α, which is a contradiction, since every edge of α is a boundary edge.

2.2. A-cuts and perfect partitionings

To construct a CDC for an infinite graph, we divide the graph into bounded regions by using A-cuts.

Definition 1. An ordered set of faces $\mathcal{C} = (f_0, \ldots, f_n)$ is called an A-cut if $\mathcal{G} \setminus \mathcal{C}$ is disconnected and either $n = 0$ or all of the following conditions hold:

(i) there exists $2 \le k \le n$ such that for all $0 \le i < j \le n$

$$f_i \text{ and } f_j \text{ are adjacent } \iff (j = i + 1) \vee (i = 0 \wedge 2 \le j \le k);$$

(ii) f_i is an interior face if and only if $1 < i < n$;

(iii) $\mathcal{G} \setminus \mathcal{C}$ has at most two infinite connected components. Moreover, each infinite connected component of $\mathcal{G} \setminus \mathcal{C}$ is adjacent to both $\{f_0, f_1\}$ and f_n.

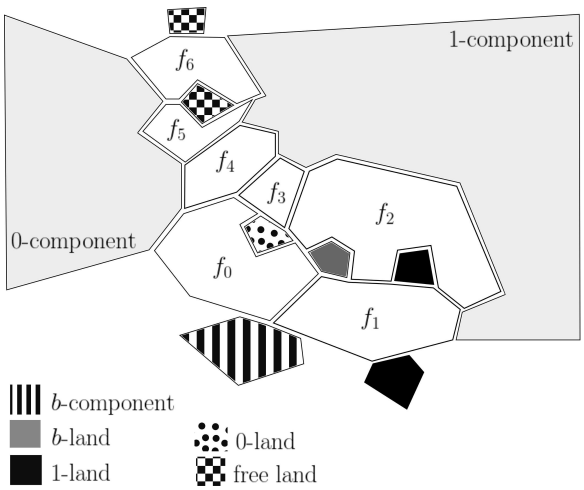

Figure 3. An A-cut and its components with $n = 6$ and $k = 4$.

Note that there are no A-cuts with only two faces. In other words, in Definition 1 either $n = 0$ or $n \geq 2$. In the next definitions, we define admissible and perfect partitionings.

Definition 2. A collection Λ of A-cuts in G is called an admissible partitioning of G if

(i) distinct elements of Λ are not adjacent, and
(ii) every connected component of $\mathcal{G} \setminus \bigcup \Lambda$ is finite.

Let $\mathcal{D} \subseteq \mathcal{G}$ be finite and connected. The outer boundary of $\mathcal{D}$, denoted by $\partial_0 \mathcal{D}$, is the connected component of the topological boundary of $\bigcup \mathcal{D}$ that contains all of the faces in $\mathcal{D}$ in the closure of its interior. We say a set $U \subseteq \mathbb{R}^2$ is *trapped* by $\mathcal{D}$ if U is included in the open interior of $\partial_0 \mathcal{D}$.

Definition 3. Let Λ be an admissible partitioning of G and $\mathcal{C} = (f_0, \ldots, f_n) \in \Lambda$. Let L be a connected component of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to $\mathcal{C}$.

(i) Suppose L is adjacent to both f_0 and f_1. If L is not trapped by $\mathcal{C}$, we call L a b-component of $\mathcal{C}$; otherwise, L is a b-land of $\mathcal{C}$.
(ii) Suppose L is adjacent to f_i but not to f_{1-i}, $i = 0, 1$. If L is adjacent to f_n and not trapped by $\mathcal{C}$, we call L an i-component of $\mathcal{C}$; otherwise, L is an i-land of $\mathcal{C}$.
(iii) If $n = 0$, any two distinct connected components of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to $\mathcal{C}$ can be declared as the unique 0-component and the unique 1-component.

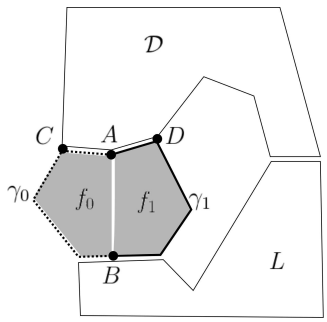

Figure 4. The b-component of an A-cut does not intersect f_i for $i > 1$.

Lemma 4. *Let Λ be an admissible partitioning of G and $\mathcal{C} = (f_0, \ldots, f_n) \in \Lambda$ with $n \geq 2$. Then*

(i) *each i-component of $\mathcal{C}$ contains boundary faces, $i \in \{0, 1, b\}$;*
(ii) *the b-component is not adjacent to f_i for $i > 1$;*
(iii) *there exists at most one i-component of $\mathcal{C}$ for each $i \in \{0, 1, b\}$.*

Proof. (i) Let L be an i-component of $\mathcal{C}$, $i \in \{0, 1, b\}$. If L does not contain boundary faces, then $\partial_0 L$ is contained in a union of A-cuts in Λ. Since the A-cuts in Λ are mutually non-adjacent, it follows that $\partial_0 L$ is a subset of $\bigcup \mathcal{C}$ alone, hence L is trapped by $\mathcal{C}$, which is a contradiction.

(ii) Let $\gamma = \partial_0\{f_0, f_1\}$ and $\mathcal{D} = \mathcal{C} \setminus \{f_0, f_1\}$. There are exactly two points A and B on γ that belong to $f_0 \cap f_1$. Points A and B divide γ into two parts $\gamma_0 \subseteq f_0$ and $\gamma_1 \subseteq f_1$. One of these points, say A, is trapped by $\mathcal{C}$. Let $C \in \gamma_0$ and $D \in \gamma_1$ be the unique points on $\partial_0\mathcal{C}$ that belong to both $f_0 \cup f_1$ and $\bigcup \mathcal{D}$. Let L be a b-component of $\mathcal{C}$. We need to show that L is not adjacent to $\mathcal{D}$. Since L intersects both f_0 and f_1, then B is trapped by $L \cup \{f_0, f_1\}$. If L is adjacent to $\mathcal{D}$, then $L \cup \mathcal{C}$ traps either C or D, say D (see Figure 4). But then $L \cup \mathcal{C}$ traps γ_0, which is a contradiction, since γ_0 contains the boundary edges of f_0.

(iii) Let L_1 and L_2 be distinct b-components. Let μ_i be the part of γ trapped by $L_i \cup \{f_0, f_1\}$, $i \in \{0, 1\}$. From (i), we know that $B \in \mu_i$, $i \in \{0, 1\}$. Since L_1 and L_2 are not adjacent, it follows that $\mu_i \subseteq \mu_{1-i}$ for some $i \in \{0, 1\}$. But then L_i is trapped by $L_{1-i} \cup \{f_0, f_1\}$, which is a contradiction since L_i contains boundary faces.

Next, we show that there can be at most one 0-component (the proof for 1-components is similar). Let L_i be a 0-component, $i = 1, 2$; hence it is adjacent to both f_0 and f_n but not f_1. Let η be the outer boundary of $\mathcal{C}$, and $\eta_i \subseteq \eta$ be the maximal subset trapped by $L_i \cup \mathcal{C}$, $i = 1, 2$. Since L_1 and L_2 are not adjacent, we have two cases.

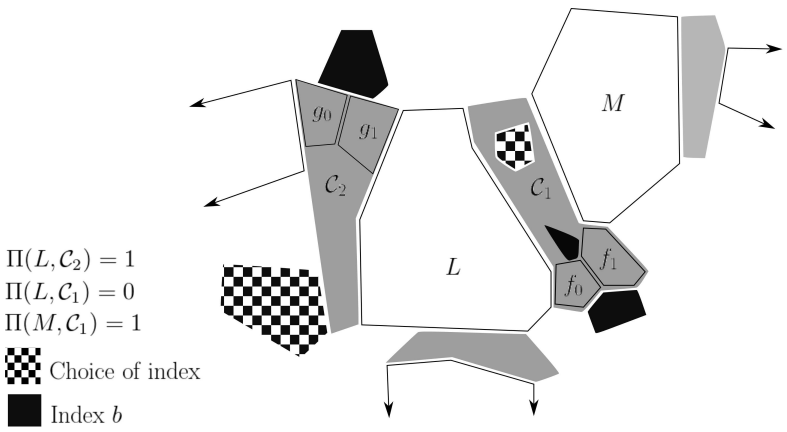

Figure 5. Components and their indices relative to neighboring A-cuts.

Case 1. $\eta_1 \cap \eta_2 = \emptyset$. In this case, f_1 is trapped by $\mathcal{C} \cup L_1 \cup L_2$, which is a contradiction, since f_1 is a boundary face.

Case 2. $\eta_1 \subseteq \eta_2$ or $\eta_2 \subseteq \eta_1$. Suppose $\eta_1 \subseteq \eta_2$ (the other case is similar). Then L_1 is trapped by $\mathcal{C} \cup L_2$, which is a contradiction, since L_1 contains boundary faces. It follows that there is at most one i-component for each $i \in \{0, 1, b\}$. ■

Definition 4. A perfect partitioning of G is an admissible partitioning Λ of G such that

(i) for each $i \in \{0, 1\}$ and each $\mathcal{C} \in \Lambda$, a unique i-component exists (or declared, if $\mathcal{C}$ is a single face);
(ii) each component of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to any $\mathcal{C} \in \Lambda$ is either an i-component of $\mathcal{C}$, $i \in \{0, 1\}$, or a connected component of $\mathcal{G} \setminus \mathcal{C}$.

Let L be a connected component of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to $\mathcal{C} \in \Lambda$. We define $\Pi(L, \mathcal{C})$, the index of L relative to $\mathcal{C}$, as follows. If L is an i-component or i-land of $\mathcal{C}$, $i \in \{0, 1, b\}$, then let $\Pi(L, \mathcal{C}) = i$. Otherwise, choose $i \in \{0, 1\}$ (to be fixed thereafter) and let $\Pi(L, \mathcal{C}) = i$; see Figure 5.

2.3. An equivalence relation

Let Λ be a perfect partitioning of G. Two connected components of $\mathcal{G} \setminus \bigcup \Lambda$ are called *related*, if they are adjacent to some member of Λ and have the same index relative to it. Let L and M be two connected components of $\mathcal{G} \setminus \bigcup \Lambda$. Then L is said to be *equivalent* to M (and we write $L \sim M$), if there exist connected components K_i of $\mathcal{G} \setminus \bigcup \Lambda$, $1 \leq i \leq m$, such that $K_1 = L$, $K_m = M$, and K_i is related to K_{i+1} for all $1 \leq i \leq m-1$. Clearly, $\sim$ is an equivalence relation on the

set of connected components of $\mathcal{G} \setminus \bigcup \Lambda$. In Proposition 7, we prove that every equivalence class is finite. We first need two lemmas.

Lemma 5. *Let Λ be a perfect partitioning of G, and let L be a connected component of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to distinct A-cuts $\mathcal{C}_1, \mathcal{C}_2 \in \Lambda$. Then L must be an i-component of both $\mathcal{C}_1$ and $\mathcal{C}_2$ with possibly different values of $i \in \{0,1\}$.*

Proof. If L is not an i-component of $\mathcal{C}_1$ for some $i \in \{0,1\}$, then L must be a connected component of $\mathcal{G} \setminus \mathcal{C}_1$. But then L cannot be adjacent to any other A-cut in Λ, which is a contradiction. ■

Lemma 6. *Let Λ be a perfect partitioning of G, and let L be a connected component of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to $\mathcal{C} \in \Lambda$. Suppose L is not the 0 or 1-component of $\mathcal{C}$. If M is related to L, then M is adjacent to $\mathcal{C}$ and $\Pi(L, \mathcal{C}) = \Pi(M, \mathcal{C})$.*

Proof. By Lemma 5, the only member of Λ adjacent to L is $\mathcal{C}$. It then follows from the definition of equivalence that M must be adjacent to $\mathcal{C}$ with the same index. ■

Proposition 7. *Let Λ be a perfect partitioning of G. Then each equivalence class of $\sim$ is finite.*

Proof. Let Ω be an equivalence class of $\sim$. Suppose Ω contains an i-component L of $\mathcal{C}_1$ for some $i \in \{0,1\}$ and some $\mathcal{C}_1 \in \Lambda$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_l \in \Lambda$, be all the A-cuts in Λ that are adjacent to L. By Lemma 5, L is a 0 or 1-component of each $\mathcal{C}_j$, $1 \le j \le l$. Let D_j be the set of all connected components of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to $\mathcal{C}_j$ and related to L. We claim that

$$\Omega = \bigcup_{j=1}^{l} D_j. \tag{1}$$

Clearly, $D_j \subseteq \Omega$. Therefore, to prove (1), let $M \in \Omega$ with $M \neq L$, and we show that $M \in D_j$ for some $1 \le j \le l$. Since $L \sim M$, there must exist distinct elements $K_1, \ldots, K_m$ in Ω with $K_1 = L$ and $K_m = M$, where K_i and K_{i+1} are related for all $1 \le i \le m-1$. Let $1 \le j \le l$ be such that $\Pi(L, \mathcal{C}_j) = \Pi(K_2, \mathcal{C}_j)$; in particular $K_2 \in D_j$. Since L is an i-component of $\mathcal{C}_j$ for some $i \in \{0,1\}$, it follows from the uniqueness of i-components that K_2 is not an i-component of $\mathcal{C}_j$. By Lemma 6, K_3 is adjacent to $\mathcal{C}_j$ and $\Pi(K_2, \mathcal{C}_j) = \Pi(K_3, \mathcal{C}_j)$ and so $K_3 \in D_j$. Similarly, if $K_i \in D_j$, then $K_{i+1} \in D_j$. It follows by induction that $M = K_m \in D_j$, and (1) follows. Now, by (1), Ω is a finite union of finite sets, hence it is finite.

Next, suppose that Ω does not contain any 0 or 1-components of any $\mathcal{C} \in \Lambda$. Let $L \in \Omega$, and choose $\mathcal{C} \in \Lambda$ adjacent to L. We must have $\Pi(L, \mathcal{C}) = b$, otherwise L is related to 0 or 1-component of $\mathcal{C}$, contradicting our assumption. Therefore,

L is a b-component or a b-land of $\mathcal{C}$. It follows from Lemma 5 that any member of Ω is either a b-component or a b-land of $\mathcal{C}$, which implies that Ω is a finite set. ■

3. Existence of Perfect Partitionings

In this section, we show that every infinite 2-connected VAP-free and EAP-free cubic plane graph with finitely many boundary components admits a perfect partitioning. In the next section, we use perfect partitionings to construct cycle double covers.

We first show that the number of infinite boundary components does not depend on the plane embedding. Recall that a graph is called k-indivisible if the deletion of a finite subgraph leaves at most $k-1$ infinite connected components.

Lemma 8. *Let G be an infinite 2-connected VAP-free and EAP-free plane graph such that ∂G is connected and nonempty. If $\mathcal{H} \subseteq \mathcal{G}$ is finite, then $\bigcup(\mathcal{G} \setminus \mathcal{H})$ has exactly one unbounded component G_1; moreover, ∂G_1 is connected and $\partial G_1 \oplus \partial G$ is finite.*

Proof. First, note that ∂G divides the plane into two unbounded connected components homeomorphic to the half-plane, one of which is $\bigcup \mathcal{G}$. To see this, let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a continuous embedding, where the image of γ is ∂G. Let p be an interior point of $\bigcup \mathcal{G}$. The inversion at p maps ∂G to a closed continuous curve passing through p; the curve is closed and passes through p, because $dist_G(p, \gamma(t)) \to \infty$ as $t \to \pm\infty$; this latter fact is true, since G is VAP-free and EAP-free. The claim then follows from the Jordan curve theorem.

Removing any compact subset of the half-plane $\mathbb{R} \times [0, \infty)$ leaves exactly one component that contains all of points $(t, 0)$ for $|t|$ large enough. It follows that removing $\bigcup \mathcal{H}$, which is compact subset of $\bigcup \mathcal{G}$, leaves exactly one connected component G_1 that contains all but finitely many edges of ∂G. ■

Proposition 9. *Let G be an infinite 2-connected VAP-free and EAP-free plane graph such that ∂G has k boundary components. Let $\mathcal{H} \subseteq \mathcal{G}$ be a finite subset of faces of G. Then $\bigcup(\mathcal{G} \setminus \mathcal{H})$ has at most k unbounded components. Moreover, there exists a finite $\mathcal{H} \subseteq \mathcal{G}$ such that $\bigcup(\mathcal{G} \setminus \mathcal{H})$ has k unbounded components.*

Proof. Proof is by induction on k. The base case $k = 1$ follows from Lemma 8. Let γ be the shortest path in G connecting two distinct components of ∂G. Then γ divides $\mathcal{G}$ into two subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\bigcup \mathcal{E}_i$ has k_i boundary components, $i = 1, 2$, and $k_1 + k_2 = k$. Let $F_i = \cup(\mathcal{H} \cap \mathcal{E}_i)$, $i = 1, 2$. By the inductive hypothesis, removing F_i from $\bigcup \mathcal{E}_i$ leaves at most k_i unbounded

connected components, $i = 1, 2$. Therefore, removing $\mathcal{H}$ from $\mathcal{G}$ leaves at most $k_1 + k_2 = k$ unbounded connected components.

To prove the second part of the theorem, let l be the largest number such that there exists a finite subset $\mathcal{H} \subseteq \mathcal{G}$ with the property that $\bigcup(\mathcal{G} \setminus \mathcal{H})$ has l unbounded components. We need to show that $l = k$. If $k = 2$, then $l = 2$, since both $\bigcup(\mathcal{E}_1 \setminus \mathcal{H})$ and $\bigcup(\mathcal{E}_2 \setminus \mathcal{H})$ are unbounded, where $\mathcal{H}$ is the set of faces that intersect γ. From the first part of the proof, we know that $l \leq k$. If $l < k$, a component S of $\bigcup(\mathcal{G} \setminus \mathcal{H})$ must have at least two boundary components. From the case $k = 2$, we know that there must exist a finite subset of faces $\mathcal{S}$ in S that divides S into two unbounded components. But then $\bigcup \mathcal{G} \setminus (\mathcal{H} \cup \mathcal{S})$ has at least $l + 1$ components. This is a contradiction, and the proof is completed. ■

Corollary 10. *Let G be a 2-connected VAP-free and EAP-free plane representation of an infinite graph G'. Then G' is $(k+1)$-indivisible if and only if ∂G has at most k connected components, where $k \geq 1$.*

The following technical lemma is essential in proving the existence of perfect partitionings.

Lemma 11. *Let G be an infinite 2-connected VAP-free and EAP-free cubic plane graph such that ∂G is nonempty and connected. Let $\mathcal{H} \subseteq \mathcal{G}$ be a finite subset of faces of G. Then there exists an A-cut $\mathcal{C} = (f_0, \ldots, f_n)$ in G non-adjacent to $\mathcal{H}$ such that:*

(i) *$\mathcal{G} \setminus \mathcal{C}$ has a finite connected component that contains $\mathcal{H}$. If $n > 0$, this finite connected component is the only connected component of $\mathcal{G} \setminus \mathcal{C}$ that is adjacent to both f_0 and f_n.*

(ii) *The infinite connected component of $\mathcal{G} \setminus \mathcal{C}$ is unique. If $n > 0$, this infinite connected component is the only connected component of $\mathcal{G} \setminus \mathcal{C}$ that is adjacent to both f_1 and f_n.*

Proof. Without loss of generality, we can assume $\mathcal{H}$ is connected and contains a boundary face (otherwise, one adds more faces to $\mathcal{H}$ to meet these conditions). The boundary of G is a 2-way infinite path. Therefore, we can label each boundary edge of G by a unique integer so that edges i and j are adjacent if and only if $|j - i| \leq 1$. For a subgraph J, let $\theta(J) \subseteq \mathbb{Z}$ be the set of labels of its edges in ∂G. Let $\mathcal{H}'$ be the set of all faces in G that intersect some face in $\mathcal{H}$ (in particular, $\mathcal{H} \subseteq \mathcal{H}'$). Let

$$\mathcal{I}_1 = \left\{ f \in \mathcal{G} : \min \theta(f) < \min \theta(\mathcal{H}') \right\},$$

and

$$\mathcal{I}_2 = \left\{ f \in \mathcal{G} : \max \theta(f) > \max \theta(\mathcal{H}') \right\}.$$

By Lemma 8, the unique unbounded component of $\bigcup(\mathcal{G} \setminus \mathcal{H}')$ contains all but finitely many edges of ∂G, hence it must contain $\mathcal{I}_1$ and $\mathcal{I}_2$. It follows that every

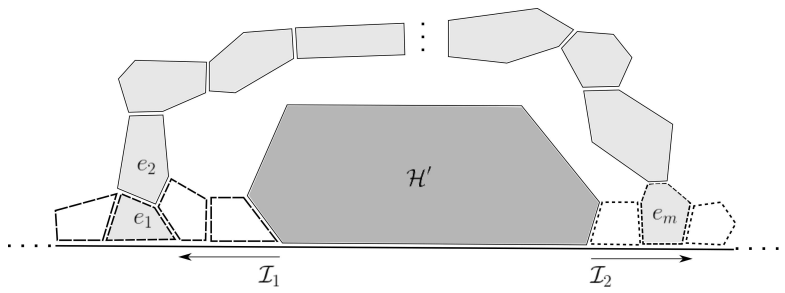

Figure 6. Construction of an A-cut bounding $\mathcal{H}'$ by a finite component.

face in $\mathcal{I}_1$ can be connected to every face in $\mathcal{I}_2$ via a path of faces in $\bigcup(\mathcal{G} \setminus \mathcal{H}')$. Let $e_1, \ldots, e_m \in \mathcal{G} \setminus \mathcal{H}'$ be the shortest facial path connecting a face in $\mathcal{I}_1$ to a face in $\mathcal{I}_2$. If $m = 1$, then $e_1 \in \mathcal{I}_1 \cap \mathcal{I}_2$ and (e_1) is the desired A-cut. To see this, note that by Proposition 8, $\mathcal{G} \setminus \{e_1\}$ has exactly one infinite component. Moreover, $\mathcal{G} \setminus \{e_1\}$ is disconnected because e_1 has non-adjacent boundary edges.

Therefore, assume that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ and $m > 1$. Clearly e_1 and e_m are boundary faces. Next, we show that e_i is an interior face for all $1 < i < m$. On contrary, suppose there exists $1 < k < m$ such that e_k is a boundary face. By the minimality of m, and since $e_k \notin \mathcal{H}'$, there must exist an integer l labeling an edge of e_k such that

$$\min \theta(\mathcal{H}') < l < \max \theta(\mathcal{H}').$$

Choose the least integer n_1 and the largest integer n_2 such that $n_1 \leq l \leq n_2$ and none of the edges labelled by $n_1, n_1 + 1, \ldots, l, \ldots, n_2$ are contained in any of the faces in $\mathcal{H}'$. It follows that the initial and terminal vertices of the path $\mathcal{P}_1$ of edges labelled from n_1 to n_2 are contained in $\bigcup \mathcal{H}'$. Since $\bigcup \mathcal{H}'$ is connected, there exists a simple path $\mathcal{P}_2$ comprised of interior edges contained in $\bigcup \mathcal{H}'$ connecting the endpoints of $\mathcal{P}_1$. It follows that $\mathcal{P}_2$ divides $\bigcup \mathcal{G}$ into two components one containing e_k and one containing both e_1 and e_m. But then the path $e_1, \ldots, e_m$ of faces connecting e_1 to e_m must intersect $\mathcal{P}_2$, which contradicts the assumption that this path does not contain any faces from $\mathcal{H}'$.

The boundary face containing the edge labelled $\min \theta(e_1) - 1$ is adjacent to e_1 and belongs to $\mathcal{G} \setminus \mathcal{H}'$. Let d_1 be a boundary face adjacent to e_1 such that $\min \theta(d_1)$ is the smallest possible. The faces d_1 and e_2 are distinct, otherwise $m = 2$ and $e_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$, which is a contradiction.

Moreover, e_2 and d_1 belong to the same unbounded connected component of $\bigcup(\mathcal{G} \setminus \{e_1\})$. Therefore, they can be connected via a path $d_1, \ldots, d_s = e_2$, $s > 1$, of faces adjacent to e_1. Choose the largest i for which $d_i \in \mathcal{I}_1$. We must have $i < s$, otherwise e_2 is a boundary face, hence $m = 2$ and $e_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$, which is a contradiction.

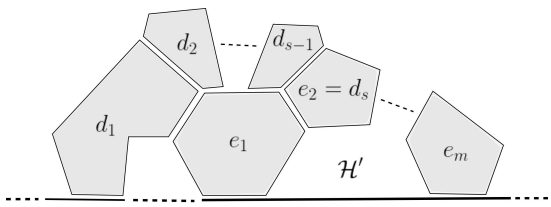

Figure 7. Construction of an A-cut separating $\mathcal{H}'$.

Let $f_0 = e_1$ and $f_1 = d_i$. Moreover, let

$$f_k = d_{i+k-1}$$

for all $1 \leq k \leq s+1-i$, and

$$f_k = e_{k-s+i+1}$$

for all $s+1-i \leq k \leq m+s-i-1 = n$.

One can modify the facial path $(f_2, \ldots, f_n)$ by deriving the shortest subpath that connects f_2 to f_n. Such a shortest subpath cannot contain adjacent faces with non-consecutive labels (otherwise it can be made yet shorter). The resulting sequence $(f_0, \ldots, f_n)$ satisfies conditions (i) and (ii) of Definition 1. Next, we show that condition (iii) also holds. This completes the construction of the A-cut. By the construction, $\mathcal{G} \setminus \mathcal{C}$ has only one infinite component that contains all but finitely many boundary edges. In particular, it contains all of the edges with a label less than $\min \theta(f_1)$ or greater than $\max \theta(f_n)$. In particular, the infinite component of $\mathcal{G} \setminus \mathcal{C}$ is adjacent to both f_1 and f_n. The uniqueness of the connected component adjacent to both f_1 and f_n follows from Lemma 4.

Next, we need to show that the component of $\mathcal{G} \setminus \mathcal{C}$ that contains $\mathcal{H}'$ is finite. Consider the set of edges labelled l with

$$\min \theta(f_1) \leq l \leq \max \theta(f_n).$$

These edges form a path γ_1 on the boundary of G that contains all of boundary edges contained in $\mathcal{H}'$. Let u and v be the endpoints of this path. There exists a simple path γ_2 connecting u to v that is contained in $\bigcup \mathcal{C}$ and which intersects γ_1 only at u and v. The curve γ_2 divides $\bigcup \mathcal{G}$ into an unbounded region and a bounded region containing $\bigcup \mathcal{H}$. It follows that $\mathcal{G} \setminus \mathcal{C}$ has a bounded component containing $\mathcal{H}$. The edges labelled $\max \theta(f_0) + 1$ and $\min \theta(f_n) - 1$ are contained in this bounded component. It follows that this bounded component is adjacent to both f_0 and f_n. The uniqueness of the component adjacent to both f_0 and f_n follows from Lemma 4. This completes the proof of Lemma 11. ■

Lemma 12. *Let G be an infinite 2-connected VAP-free and EAP-free cubic plane graph such that ∂G is nonempty and connected. Then there exists a perfect partitioning of G.*

Proof. We define a sequence Λ of A-cuts $\mathcal{C}_i$, $i \geq 1$, inductively as follows. By Lemma 11, there exists an A-cut $\mathcal{C}_1$ in G (by letting $\mathcal{H}$ to be an arbitrary finite subset of $\mathcal{G}$). Suppose, we have defined A-cuts $\mathcal{C}_1, \ldots, \mathcal{C}_k$ such that:

(i) For $1 \leq i < j \leq k$, the A-cuts $\mathcal{C}_i$ and $\mathcal{C}_j$ are not adjacent.

(ii) The unique infinite component of $\mathcal{G} \setminus (\bigcup_{i=1}^{k} \mathcal{C}_i)$ is adjacent to $\mathcal{C}_k$ but not adjacent to $\mathcal{C}_i$ for $1 \leq i < k$.

Let $\mathcal{G}_k$ be the union of $\mathcal{C}_k$ with the unique infinite component of $\mathcal{G} \setminus (\bigcup_{i=1}^{k} \mathcal{C}_i)$. It follows that $\mathcal{G}_k$ is connected and has exactly one boundary component. By Lemma 11, there exists an A-cut $\mathcal{C}_{k+1}$ in $\mathcal{G}_k$ non-adjacent to $\mathcal{C}_k$. Moreover, $\mathcal{G}_k \setminus \mathcal{C}_{k+1}$ has a finite connected component containing $\mathcal{C}_k$. Since $\mathcal{C}_{k+1} \subseteq \mathcal{G}_k$, it follows that $\mathcal{C}_{k+1}$ is not adjacent to $\mathcal{C}_i$ for all $1 \leq i \leq k$. In addition, the infinite connected component of $\mathcal{G} \setminus (\bigcup_{i=1}^{k+1} \mathcal{C}_i)$ coincides with the infinite connected component of $\mathcal{G}_k \setminus \mathcal{C}_{k+1}$, which is adjacent to $\mathcal{C}_{k+1}$ but not adjacent to $\mathcal{C}_i$ for all $1 \leq i \leq k$. This completes the construction of a sequence of A-cuts satisfying conditions (i) and (ii) above.

Next, we show that each connected component of $\mathcal{G} \setminus \bigcup \Lambda$ is finite. Let L be any connected component of $\mathcal{G} \setminus \bigcup \Lambda$. The topological boundary of L is not entirely comprised of boundary edges; therefore, there exists a smallest integer $i \geq 1$ such that L is adjacent to $\mathcal{C}_i$. But then L is contained in the finite connected component of $\mathcal{G} \setminus \mathcal{C}_{i+1}$ that contains $\mathcal{C}_i$, which implies that L is finite.

Let L be a connected component of $\mathcal{G} \setminus \bigcup \Lambda$. By construction of Λ, either L is adjacent to a unique $\mathcal{C}_i$ or exactly two consecutive A-cuts $\mathcal{C}_i = (f_0, \ldots, f_n)$ and $\mathcal{C}_{i+1} = (g_0, \ldots, g_m)$ for some $i \geq 1$. In the former case, L is a connected component of $\mathcal{G} \setminus \mathcal{C}_i$. We show that in the latter case, L is a 1-component of $\mathcal{C}_i$ and a 0-component of $\mathcal{C}_{i+1}$. Recall that by Lemma 11, the infinite component of $\mathcal{G} \setminus \mathcal{C}_i$ is adjacent to both f_1 and f_n if $n > 0$ (if $n = 0$, then we declare L to be the 1-component of $\mathcal{C}_i$). Since L is included in the infinite component of $\mathcal{G} \setminus \mathcal{C}_i$ and is adjacent to $\mathcal{C}_i$, it follows that L is adjacent to f_1 and f_n, hence L is a 1-component of $\mathcal{C}_i$. Similarly, Lemma 11 implies that L is a 0-component of $\mathcal{C}_{i+1}$, since the finite component of $\mathcal{G} \setminus \mathcal{C}_{i+1}$ is adjacent to both g_0 and g_m if $m > 0$. If $m = 0$, then we declare this finite component to be the 0-component of $\mathcal{C}_{n+1}$.

We have shown that each $\mathcal{C} \in \Lambda$ has a unique 0 and a 1-component. We have also shown that each connected component of $\mathcal{G} \setminus \bigcup \Lambda$ adjacent to $\mathcal{C} \in \Lambda$ is either a connected component of $\mathcal{G} \setminus \mathcal{C}$ or a 0 or 1-component of $\mathcal{C}$. So Λ is a perfect partitioning of G. ∎

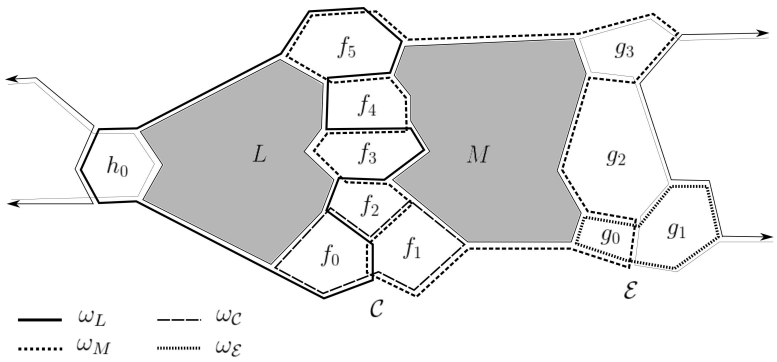

Figure 8. A-cuts, components, and corresponding cycles.

Proposition 13. *Let G be an infinite 2-connected cubic plane graph without accumulation points such that ∂G has a finite nonzero number of connected components. Let $\mathcal{H} \subseteq \mathcal{G}$ be finite. Then there exists a perfect partitioning Λ of G such that $\mathcal{H}$ is included in the 0-component of a member of Λ.*

Proof. Proof is by induction on n, the number of connected components of ∂G. The case $n = 1$ was considered in Lemma 12. Suppose the claim is true for n. Let G be a graph with $n+1$ boundary components. Let γ be the shortest path in G connecting two distinct components of ∂G. Without loss of generality, we can assume that $\mathcal{H}$ includes all of the faces that intersect γ as well. Then γ divides $\mathcal{G}$ into two subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\bigcup \mathcal{E}_i$ has $k_i > 0$ boundary components and $k_1 + k_2 = n + 1$. Let $\mathcal{H}_i = \mathcal{E}_i \cap \mathcal{H}$, $i = 1, 2$.

By the inductive hypothesis, each $\mathcal{E}_i$ admits a perfect partitioning Λ_i such that $\mathcal{H}_i$ is included in the 0-component of a member of Λ_i, $i = 1, 2$. It follows that $\Lambda = \Lambda_1 \cup \Lambda_2$ is a perfect partitioning of G and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is included in the 0-component of a member of Λ. ∎

4. Construction of CDC

In this section Λ is a perfect partitioning of a 2-connected VAP-free and EAP-free cubic plane graph G. The cycle space of G, comprised of all finite cycles in G, is viewed as a $\mathbb{Z}_2$-vector space, where the sum of two cycles is the symmetric difference of their edges. For $f \in \mathcal{G}$, we denote the facial cycle of f by f again.

Let $\mathcal{C} = (f_0, \ldots, f_n) \in \Lambda$ and M be an equivalence class of $\sim$ adjacent to $\mathcal{C}$ with index $i \in \{0, 1, b\}$. We define the cycles $\omega_{\mathcal{C}}$, ω_M, η_M, and δ_M as follows.

If $n = 0$, let $\omega_{\mathcal{C}} = f_0$, and if $n > 1$, let $\omega_{\mathcal{C}} = f_0 + f_1$.

If $i \in \{0, 1\}$ and n is even, let $\eta_M(\mathcal{C})$ be the sum of all faces in all of the

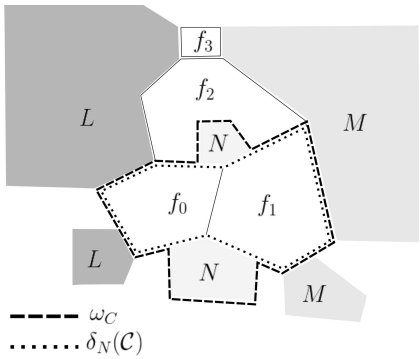

Figure 9. The cycle bounding the b-components and b-lands.

b-lands of $\mathcal{C}$; otherwise, let $\eta_M(\mathcal{C}) = 0$.

If $i = b$, let $\delta_M(\mathcal{C}) = f_0 + f_1$; otherwise, let $\delta_M(\mathcal{C})$ be the sum of all faces f_j, $0 \leq j \leq n$, such that f_j is adjacent to M with the requirement that $j - n$ is even if $j > 1$. If M is not adjacent to $\mathcal{C}$, we simply set $\delta_M(\mathcal{C}) = 0$. Finally, we define

$$\omega_M = \sum_{f \in M} f + \sum_{\mathcal{C} \in \Lambda} \delta_M(\mathcal{C}) + \sum_{\mathcal{C} \in \Lambda} \eta_M(\mathcal{C}).$$

Proposition 14. *Let Λ be a perfect partitioning of G with equivalence classes M_i, $i \geq 1$. Then the following collection of cycles provides a cycle double cover of G*

(i) *the facial cycles of faces in M_i, $i \geq 1$;*
(ii) *ω_{M_i}, $i \geq 1$;*
(iii) *$\omega_{\mathcal{C}}$, if $\mathcal{C} \in \Lambda$ has no b-components and no b-lands.*

Proof. Let a be an edge of G. We prove that the given collection covers e exactly twice.

Case 1. Suppose a is a boundary edge, and let f be the unique face that contains a. Every face is either contained in an equivalence class M or in an A-cut $\mathcal{C} \in \Lambda$. If a is contained in an equivalence class M, then the facial cycle f and ω_M both contain a, hence covering a twice. No other cycle in the collection contains a.

Next, suppose f is contained in $\mathcal{C} = (f_0, \ldots, f_n)$, and so $f = f_j$ for some $j \in \{0, 1, n\}$ (we allow $n = 0$ here). Let M_i be the equivalence class containing the i-component of $\mathcal{C}$, $i \in \{0, 1, b\}$. If $f = f_n$, then a is covered by ω_{M_0} and ω_{M_1} (but not ω_{M_b} or $\omega_{\mathcal{C}}$). Thus, suppose $n > 1$ and $f = f_i$ for some $i \in \{0, 1\}$. If M_b exists, then a is covered by ω_{M_b} and ω_{M_i} (but not $\omega_{M_{1-i}}$); otherwise, a is covered by $\omega_{\mathcal{C}}$ and ω_{M_i}.

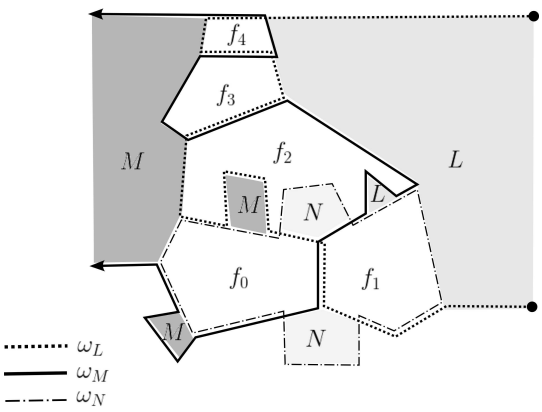

Figure 10. Equivalence classes and corresponding cycles.

In the rest of the proof, we assume that a is not a boundary edge, hence there exist exactly two faces e_1 and e_2 that share a. There are these cases.

Case 2. Both e_1 and e_2 are contained in the same equivalence class. Then the facial cycles e_1 and e_2 cover a, while no other cycle in the collection covers a.

Case 3. There exists $\mathcal{C} = (f_0, \ldots, f_n) \in \Lambda$ and an equivalence class M such that e_1 is contained in M and e_2 is contained in $\mathcal{C}$. Let M_i, $i \in \{0, 1, b\}$, be the equivalence class containing the i-component of $\mathcal{C}$.

If a is contained in f_i, $i \in \{0, 1\}$, and $M = M_i$, then a is covered by the facial cycle e_1 and ω_{M_b} ($\omega_{\mathcal{C}}$, if M_b does not exist).

If a is contained in f_i, $i \in \{0, 1\}$, and $M = M_b$ but a is not trapped by $\mathcal{C}$, then a is covered by facial cycle e_1 and ω_{M_i}.

If a is contained in f_i, $i \in \{0, 1\}$, and $M = M_b$ and a is trapped in $\mathcal{C}$, then a is covered by facial cycle e_1 and ω_{M_j}, where $j = i$ if n is odd and $j = 1 - i$ if n is even.

Suppose a is contained in f_j for some $j > 1$. If $j - n$ even, then a is covered by the facial cycle e_1 and $\omega_{M_{1-i}}$; otherwise, a is covered by the facial cycle e_1 and ω_{M_i}.

Case 4. Both e_1 and e_2 belong to $\mathcal{C}$. Since e_1 and e_2 are adjacent, we must have one of the following cases.

$e_1 = f_0$ and $e_2 = f_1$. In this case ω_{M_0} and ω_{M_1} cover a twice.

$e_1 = f_1$ and $e_2 = f_2$. In this case ω_{M_b} (or $\omega_{\mathcal{C}}$, if M_b does not exist) and ω_{M_i} cover a twice, where $j \in \{0, 1\}$ is such that $n - i$ is even.

$e_1 = f_i$ and $e_2 = f_{i+1}$ for some $i > 1$. Then ω_{M_0} and ω_{M_1} cover a twice.

$e_1 = f_0$ and $e_2 = f_j$ for some $j > 1$. Then ω_{M_b} (or $\omega_{\mathcal{C}}$, if M_b does not exist) and ω_{M_i} cover a twice, where $i \in \{0, 1\}$ is such that $n - i - j$ is odd.

Therefore, in each case, the edge a is contained in exactly two cycles in the collection, and the proof is completed. ■

***Proof of Theorems* 1 *and* 2.** By Proposition 13, G admits a perfect partitioning, and consequently it has a CDC by Proposition 14. Theorem 2 then follows from Theorem 1 and Corollary 10. ■

Finally, we prove a theorem regarding the strong CDC conjecture for infinite plane graphs.

Theorem 15. *Let G be as in Theorem* 1, *and let S be a finite set of edge-disjoint cycles that do not intersect ∂G. Then S can be extended to a CDC of G.*

Proof. Let C be a cycle in G that contains all of the cycles in S in its interior. Let G' be a graph obtained from G by removing all of the vertices of G that are trapped by C. Let Λ be a perfect partitioning of G'. By removing a finite number of the A-cuts in Λ if necessary, we can assume, without the loss of generality, that C is contained in an equivalence class. By Proposition 14, there exists a CDC Ω_1 of G' that contains C as a cycle in the cover. Next, we modify this CDC of G' to obtain a CDC for G that extends S. Let H denote the maximal subgraph of G with vertices in the interior or on C. Then H is a finite plane graph and $S' = S \cup \{C\}$ is a collection of edge-disjoint cycles in H. The set S' can be extended to a CDC Ω_2 of H; [11]. Now, the set of cycles $\Omega = (\Omega_1 \cup \Omega_2)\backslash\{C\}$ is a CDC of G that extends S. ■

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