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VARIOUS BOUNDS FOR LIAR'S DOMINATION NUMBER

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set if $\bigcup_{v \in S} N[v] = V$, where $N[v]$ is the closed neighborhood of v. Let $L \subseteq V$ be a dominating set, and let v be a designated vertex in V (an intruder vertex). Each vertex in $L \cap N[v]$ can report that v is the location of the intruder, but (at most) one $x \in L \cap N[v]$ can report any $w \in N[x]$ as the intruder location or x can indicate that there is no intruder in $N[x]$. A dominating set L is called a liar's dominating set if every $v \in V(G)$ can be correctly identified as an intruder location under these restrictions. The minimum cardinality of a liar's dominating set is called the liar's domination number, and is denoted by $\gamma_{LR}(G)$. In this paper, we present sharp bounds for the liar's domination number in terms of the diameter, the girth and clique covering number of a graph. We present two Nordhaus-Gaddum type relations for $\gamma_{LR}(G)$, and study liar's dominating set sensitivity versus edge-connectivity. We also present various bounds for the liar's domination component number, that is, the maximum number of components over all minimum liar's dominating sets.

Keywords: liar's domination, diameter, regular graph, Nordhaus-Gaddum.

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1. Introduction

Throughout this article, all graphs are simple, connected and undirected. For notation and terminology not given here, the reader is referred to [2]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *open neighborhood* $N(v)$ of a vertex v is defined as the set of vertices adjacent to v and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is defined as $\deg_G(v) = |N(v)|$. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)$, ($\Delta(G)$, respectively). A graph G is called *regular* if $\delta(G) = \Delta(G)$. For any pair of vertices x, y, $d(x, y)$ is the distance of the shortest path between x and y. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum $d(x, y)$ for all $x, y \in V(G)$. For a subset S of V, let $G[S]$ denote the subgraph of G induced by S. A subset D of vertices is a *clique* if $G[D]$ is a complete graph. The *clique number* of G, denoted by $\omega(G)$, is the maximum cardinality of a clique in G. A *vertex clique covering* of G is a set of cliques whose union is the entire vertex set of a graph G. The minimum cardinality of a vertex clique covering is called the *vertex clique covering number* $\Theta(G)$, which is equal to the chromatic number of $\overline{G}$, where $\overline{G}$ is the complement of G. A vertex v of a graph G is called a *cut vertex* of G if its removal produces a disconnected graph. A *cut edge* (also called a *bridge*) is defined similarly. An edge cut in a graph G is a set X of edges of G such that $G - X$ is disconnected [3]. The girth $g(G)$ of a graph G is the length of a shortest cycle contained in G.

A set $S \subseteq V$ is a *dominating set* if $\bigcup_{v \in S} N[v] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For an integer $k \geq 1$, a dominating set $D \subseteq V$ is a *k-tuple dominating set* if $|N_G[v] \cap D| \geq k$ for all $v \in V$. The minimum cardinality of a k-tuple dominating set is called the *k-tuple domination number* of G and is denoted by $\gamma_{\times k}(G)$. For the special cases $k = 2$ and $k = 3$, k-tuple domination is called *double-domination* and *triple domination*, respectively. For references on domination and some of its varieties, see for example [5, 4].

Slater in [16] introduced the concept of *liar's domination*. A graph could be used for many structures (like a computer network, a telecommunication network, a sensor network, map for a facility or a railroad network) where each vertex denotes some location in any network. In each network's location there could appear some intruder event, and its location must be determined.

In some locations there are detectors. Let $L \subseteq V$ be a liar's dominating set (as detectors) and v be a designated vertex in V. Each vertex in $L \cap N[v]$ can report that v is the location of the intruder, but (at most) one $x \in L \cap N[v]$ can report any $w \in N[x]$ as the intruder location or this x can indicate that there is no intruder in $N[x]$. A dominating set L is called a liar's dominating set if every $v \in V(G)$ can be correctly identified as an intruder location. So, it is a kind of fault-tolerance system. Such a dominating set L is called a *liar's dominating set.* The minimum cardinality of a liar's dominating set is called the *liar's domination number* and denoted by $\gamma_{LR}(G)$. A subset S is called a γ_{LR}*-set* if it is a liar's dominating set and $|S| = \gamma_{LR}(G)$. For references on liar's domination, see for example [6, 11, 12, 13, 14, 15, 16, 17].

It is obvious that a liar's dominating set is a double-dominating set. Furthermore, any triple dominating set is a liar's dominating set. Thus, if L is a liar's dominating set, then by definition each component of $G[L]$ has at least three vertices and $\gamma_{\times 2}(G) \leq \gamma_{LR}(G) \leq \gamma_{\times 3}(G)$ [16]. Checking for a subset L to be a liar's dominating set is difficult. Luckily we have the following theorem by Slater which make it much easier.

Theorem 1 (Slater, [16]). *A vertex set $L \subseteq V(G)$ is a liar's dominating set if and only if* (1) *L double dominates every $v \in V(G)$ and* (2) *for every pair u, v of distinct vertices we have $|(N[u] \cup N[v]) \cap L| \geq 3$.*

Slater [16] obtained the following sharp lower bounds for the liar's domination number of a graph.

Theorem 2 (Slater, [16]). *For a graph G of order $n = |V(G)|$ and size $m = |E(G)|$, $\gamma_{LR}(G) \geq \frac{3}{4}(2n - m)$.*

Theorem 3 (Slater, [16]). *If a graph G of order $n = |V(G)|$ has maximum degree $\Delta = \Delta(G)$, then $\gamma_{LR}(G) \geq (6/(3\Delta + 2))n$.*

Given a graph G with its liar's domination number $\gamma_{LR}(G)$, it might be possible that for a $\gamma_{LR}(G)$-set L_1, $G[L_1]$ has k_1 components, while, for a different $\gamma_{LR}(G)$-set L_2, $G[L_2]$ has $k_2 \neq k_1$ components. We define the *liar's domination component number* $k_{\gamma_{LR}(G)}$ as the maximum number of components over all the $\gamma_{LR}(G)$-sets. Thus $k_{\gamma_{LR}(G)} = \max\{k : G[L]$ has k components for a $\gamma_{LR}(G)-$ set $L\}$.

In this paper we first present sharp bounds for the liar's domination number in terms of the diameter, the girth, and clique covering number of a graph. Also two Nordhaus-Gaddum type relations are presented. Then we study liar's dominating set sensitivity versus edge-connectivity. In the last section, we present various bounds for the liar's domination component number of a graph.

2. BOUNDS

We begin with the following lemma of Roden and Slater.

Lemma 4 (Roden and Slater, [15]). *For path P_n of order $n \geq 3$, $\gamma_{LR}(P_n) = \left\lceil \frac{3}{4}(n+1) \right\rceil$.*

Theorem 5. *For any graph G of order $n \geq 3$, $\gamma_{LR}(G) \geq \frac{3}{4}(\mathrm{diam}(G)+2)$. This bound is sharp.*

Proof. Let L be a $\gamma_{LR}(G)$-set. We employee an induction on the number m of components of $G[L]$ to show that $\gamma_{LR}(G) \geq \mathrm{diam}(G) - m + 2$, and then the result follows immediately, since $m \leq \frac{\gamma_{LR}(G)}{3}$ by [15].

Before we go through the details, we should mention that by $G[L]$ and $d_{G[L]}$ we mean the induced subgraph by L and the shortest distance in $G[L]$, respectively. For the first step suppose that $G[L]$ has precisely one component, that is $G[L]$ is connected. Clearly $|L| \geq 3$. We show that the distance between any pair of vertices in G is at most $|L|-1$. Let x_1 and x_2 be two distinct vertices of G. If $x_1, x_2 \in L$, then clearly $d_G(x_1,x_2) \leq \mathrm{diam}(G[L]) \leq |L|-1$. Next assume that $x_1 \notin L$ and $x_2 \notin L$. It can be easily seen (as in [16]) that $|N[x_1] \cap L| \geq 2$. Since $x_1 \notin L$, there are two vertices x_i and x_j in $N(x_1) \cap L$. In the same way, it can be concluded that there are two vertices x'_i and x'_j in $N(x_2) \cap L$. If $\{x_i, x_j\} \cap \{x'_i, x'_j\} \neq \emptyset$, then $d(x_1,x_2) \leq |L|-1$, since $|L| \geq 3$. Thus, suppose that $\{x_i, x_j\} \cap \{x'_i, x'_j\} = \emptyset$. Assume, without loss of generality, that $d_{G[L]}(x_i, x'_i) = \min\{d_{G[L]}(u,v) : u \in \{x_i,x_j\}, v \in \{x'_i,x'_j\}\}$. It is easy to see that $d_{G[L]}(x_i,x'_i) \leq |L|-3$. Now $d_G(x_1,x_2) \leq d_G(x_1,x_i) + d_{G[L]}(x_i,x'_i) + d_G(x'_i,x_2) \leq 1 + (|L|-3) + 1 = |L|-1$. Next we take $x_1 \notin L$ and $x_2 \in L$. As before, there are two vertices x_i and x_j in $N(x_1) \cap L$. If $\{x_i,x_j\} \cap \{x_2\} \neq \emptyset$, then $d_G(x_1,x_2) = 1 \leq |L|-1$ (as $|L| \geq 3$). Therefore, let $\{x_i,x_j\} \cap \{x_2\} = \emptyset$. Assume, without loss of generality, that $d_{G[L]}(x_i,x_2) = \min\{d_{G[L]}(u,x_2) : u \in \{x_i,x_j\}\}$ Then, $d_G(x_1,x_2) \leq d_G(x_1,x_i) + d_{G[L]}(x_i,x_2) \leq 1 + (|L|-2) = |L|-1$. These provide the base step of the induction.

Suppose that the result holds if the number of components of $G[L]$ is less than m. Let $L = \bigcup_{i=1}^{m} L_i$, where $G[L_i]$ is the component of $G[L]$ for $i = 1,2,\ldots,m$. For $i = 1,2,\ldots,m$, let V_i be the set of all the vertices of $V(G) - L$ with at least two neighbors in L_i, and $G_i = G[V_i \cup L_i]$.

In order to maximize the diameter, we may assume, without loss of generality, that for $i = 1,2,\ldots,m-1$, $|N(G_i) \cap N(G_{i+1})| = 1$ and for every $j > i+1$, $|N(G_i) \cap N(G_j)| = 0$. Let $N(G_i) \cap N(G_{i+1}) = \{v_i\}$ for $i = 1,2,\ldots,m-1$. Let a,b be two distinct vertices of $V(G)$ with $d_G(a,b) = \mathrm{diam}(G)$. Then

$$\begin{aligned} d_G(a,b) &\leq \sum_{i=1}^{m} \mathrm{diam}(G_i) + d_G(G_1,v_1) + d_G(v_1,G_2) + d_G(G_2,v_2) + \cdots + d_G(v_{m-1},G_m) \\ &= \sum_{i=1}^{m} \mathrm{diam}(G_i) + \underbrace{(1+\cdots+1)}_{2(m-1)\text{ times}} = \sum_{i=1}^{m} \mathrm{diam}(G_i) + 2(m-1). \end{aligned}$$

From the base step of the induction we find that $\mathrm{diam}(G_i) \leq |L_i| - 1$ for all $1 \leq i \leq m$. Hence, $d_G(a,b) \leq \sum_{i=1}^{m}(|L_i|-1) + 2(m-1) = |L| - m + 2(m-1) =$

$|L| + m - 2$, as desired. To see the sharpness of the lower bound consider a path of order $4k + 3$ and apply Lemma 4. ■

The base step of the proof of Theorem 5 indicates the following.

Corollary 6. *If a graph G has a connected $\gamma_{LR}(G)$-set, then $\gamma_{LR}(G) \geq \operatorname{diam}(G) + 1$.*

It can also be easily seen that if $\gamma_{LR}(G) < 6$, then $\gamma_{LR}(G) \geq \operatorname{diam}(G) + 1$. Also by using the liar's domination number of a cycle, we can conclude that for any graph G of order $n \geq 3$, $\gamma_{LR}(G) \geq \left\lceil \frac{3g(G)}{4} \right\rceil$.

We next give a sharp upper bound for the liar's domination number in terms of clique covering of a graph.

Theorem 7. *For any connected graph G of order $n \geq 3$, $\gamma_{LR}(G) \leq 3|\Theta(G)|$ and this bound is sharp.*

Proof. Divide $V(G)$ into its cliques, i.e., $\Theta(G)$ cliques. Now form L as follows: choose any three arbitrary vertices from the cliques with the size greater than two and choose the two vertices from every clique with size two and also an arbitrary neighbor of either of those. It is straightforward to see that for every vertex $u \in V(G)$, $|N[u] \cap L| \geq 2$ and for any pair of vertices $u, v \in V(G)$ we have $|(N[u] \cup N[v]) \cap L| \geq 3$. To see the sharpness consider any complete graph of order $n \geq 3$. ■

2.1. Nordhaus-Gaddum type bounds

We next obtain bounds for $\gamma_{LR}(G) + \gamma_{LR}(\overline{G})$ and $\gamma_{LR}(G)\gamma_{LR}(\overline{G})$.

Theorem 8. *Let G be a graph of order $n \geq 3$. If G and $\overline{G}$ are connected, then*

$$\left\lceil \frac{4\sqrt{(3\delta(G)+1)(3\Delta(G)+2)} + 2(3\delta(G)+1)}{3\Delta(G)+2} \right\rceil + 2 \leq \gamma_{LR}(G) + \gamma_{LR}(\overline{G}) \leq 2n.$$

Proof. The upper bound is trivial. We prove the lower bound. By Theorem 3,

$$\gamma_{LR}(G) + \gamma_{LR}(\overline{G}) \geq \frac{6n}{3\Delta(G)+2} + \frac{6n}{3\Delta(\overline{G})+2} = \frac{6n}{3\Delta(G)+2} + \frac{6n}{3n-3\delta(G)-1}.$$

Let $\phi(n) = \frac{6n}{3\Delta(G)+2} + \frac{6n}{3n-3\delta(G)-1}$. By using calculus it is a routine matter to see that ϕ is maximized at $n_1 = \frac{\sqrt{(3\delta(G)+1)(3\Delta(G)+2)} + (3\delta(G)+1)}{3}$. Thus, the result follows. ■

As an example to the sharpness of the lower bound consider a cycle of order five, and to see the sharpness of the upper bound consider a path of order four.

Corollary 9. *If G is a regular graph, then $\gamma_{LR}(G) + \gamma_{LR}(\overline{G}) \geq 8$.*

Theorem 10. *If G and $\overline{G}$ are connected and $|V(G)| = n$, then*

$$15 \leq \gamma_{LR}(G)\gamma_{LR}(\overline{G}) \leq n^2.$$

Proof. The upper bound is obvious. We establish the lower bound. Let G be a connected graph of order n such that $\overline{G}$ is connected. Clearly $\gamma_{LR}(G) \geq 3$ and $\gamma_{LR}(\overline{G}) \geq 3$. The result is obvious if $\min\{\gamma_{LR}(G), \gamma_{LR}(\overline{G})\} \geq 4$. Hence let $\min\{\gamma_{LR}(G), \gamma_{LR}(\overline{G})\} = 3$. Without loss of generality assume that $\gamma_{LR}(G) = 3$. We show that $\gamma_{LR}(\overline{G}) \geq 5$. Let $L = \{v_1, v_2, v_3\}$ be a $\gamma_{LR}(G)$-set, and L' be a $\gamma_{LR}(\overline{G})$-set. We partition the set $V(G) \setminus L$ into two sets $A = \{x \in V(G) \setminus L : |N(x) \cap L| = 2\}$ and $B = \{x \in V(G) \setminus L : |N(x) \cap L| = 3\}$. Note that $|A| \leq \binom{|L|}{2} = 3$. Since $\overline{G}$ is connected, and none of the vertices in B are adjacent to any vertex in L in $\overline{G}$, we can deduce that $|A| \geq 3$, hence $|A| = 3$. Let $A = \{u_1, u_2, u_3\}$, where u_i is not adjacent to v_i in G for $i = 1, 2, 3$. Since $G[L]$ is connected, we may think that $\{v_1, v_3\} \subseteq N(v_2)$. Now we can figure out that the only vertex adjacent to v_2 in $\overline{G}$ is u_2, so $\{u_2, v_2\} \in L'$. In addition, according to Theorem 1 and what already has been discussed, $|\{v_1, u_1, v_3, u_3\} \cap L'| \geq 3$. Therefore, $\gamma_{LR}(\overline{G}) = |L'| \geq 5$. ∎

As an example of the sharpness of the upper bound consider a path of order four, and to see the sharpness of the lower bound consider the graph G_2 shown in Figure 2. It is very straightforward to check that $\gamma_{LR}(G_4) = 3$ and $\gamma_{LR}(\overline{G_4}) = 5$.

3. Liar's Domination, Edge-Connectivity and Identifying Codes

In this section we first investigate the liar's dominating set sensitivity versus cut-sets and present upper and lower bounds which can be obtained by separating graph G into its connected components. Then we investigate relations of liar's domination and identifying codes.

Theorem 11. *Let x be a cut vertex of a graph G, and $H_1, \ldots, H_t$ be the components of $G - x$. If $|H_i| \geq 2$ and $G_i = H_i \cup \{x\}$, for $1 \leq i \leq t$, then*

$$\sum_{i=1}^{t} \gamma_{LR}(G_i) - (2t-1) \leq \gamma_{LR}(G) \leq \sum_{i=1}^{t} \gamma_{LR}(G_i).$$

Proof. The right inequality follows from Theorem 1 . We prove the left inequality. Let L be a $\gamma_{LR}(G)$-set, and $L_i = L \cap V(G_i)$, for $i = 1, 2, \ldots, t$. If $x \in L$, we have $\sum_{i=1}^{t} |L_i| = \gamma_{LR}(G) + t$ and if $x \notin L$, then $\sum_{i=1}^{t} |L_i| = \gamma_{LR}(G)$. Clearly, for any vertex $a \in V(G_i) \setminus \{x\}$ $(i = 1, 2, \ldots, t)$, $N_{G_i}[a] \cap L_i = N_G[a] \cap L$, and thus $|N_{G_i}[a] \cap L_i| \geq 2$. Furthermore, for any pair $a, b \in V(G_i) \setminus \{x\}$ $(i = 1, 2, \ldots, t)$, $|(N[a] \cup N[b]) \cap L_i| = |(N[a] \cup N[b]) \cap L| \geq 3$.

Thus, we can have two different cases to consider, first if $x \in L$ and second if it does not. Assume that $x \in L$. First consider the case that for any $1 \leq i \leq t$, there is at least one vertex $r_i \in N(x) \cap L_i$, so x is double dominated by each L_i. Since each vertex $y \in V(G_i) \setminus (N_{G_i}(x) \cap N(r_i))$ must be double dominated by L_i, there should be at least a vertex $w \in N(y) \cap L_i$, $w \neq r_i$. Thus, for any vertex $y \in V(G_i) - (N_{G_i}(x) \cap N(r_i))$, $|(N_{G_i}[x] \cup N[y]) \cap L_i| \geq |\{x, r_i, w\}| = 3$.

Hence, let $y \in N_{G_i}(x) \cap N(r_i)$. Now let $L'_i = L_i \cup \{y\}$. So, we can conclude that for every vertex $z \in N_{G_i}(x) \cap N(r_i)$, $|(N_{G_i}[x]) \cup N[z]) \cap L'_i| \geq |\{x, r_i, y\}| = 3$. Therefore, we can say that in this case L'_i is a liar's dominating set for each G_i, so $\gamma_{LR}(G_i) \leq |L'_i| \leq |L_i| + 1$.

So, let say that in some components, for each vertex $r \in N(x) \cap V(G_i)$, $r \notin L_i$. Again, because every vertex $y \in V(G_i) \setminus \{x\}$ in such components must be double dominated by L_i, there should be at least one vertex $y' \in N(y) \cap L_i$. Hence, let $L'_i = L_i \cup \{r\}$ for the components that each vertex $r \in N(x) \cap V(G_i)$, $r \notin L_i$. Thus, for each vertex $y \in V(G_i) \setminus \{x\}$, $|(N_{G_i}[x] \cup N[y]) \cap L'_i| \geq |\{x, r, y'\}| = 3$. Also $|N_{G_i}[x] \cap L'_i| \geq |\{x, r\}| = 2$.

For the components that there is at least one vertex $r_i \in N(x) \cap L_i$, similar to the first part, for $y \in V(G_i) \setminus (N_{G_i}(x) \cap N(r_i))$ we have $|(N_{G_i}[x] \cup N[y]) \cap L_i| \geq |\{x, r_i, w\}| = 3$ and for $y \in N_{G_i}(x) \cap N(r_i)$. Let $L'_i = L_i \cup \{y\}$, for every vertex $z \in N_{G_i}(x) \cap N(r_i)$, $|(N_{G_i}[x] \cup N[z]) \cap L'_i| \geq |\{x, r_i, y\}| = 3$.

Hence, L'_i is a liar's dominating set and $\gamma_{LR}(G_i) \leq |L'_i| = |L_i| + 1$ in any case. Therefore, according to all the facts, we can deduce in any case, as $x \in L$, L'_i is a liar's dominating set for $G_i (1 \leq i \leq t)$. Thus, $\sum_{i=1}^{t} \gamma_{LR}(G_i) \leq \sum_{i=1}^{t} |L'_i| \leq \sum_{i=1}^{t} (|L_i| + 1) = \sum_{i=1}^{t} |L_i| + t = \gamma_{LR}(G) + (t-1) + t$. So, $\sum_{i=1}^{t} \gamma_{LR}(G_i) - (2t-1) \leq \gamma_{LR}(G)$.

Now suppose that $x \notin L$. Because x is double dominated by L, there is at least one component, without loss of generality assume it is G_1, that it has a vertex $r_1 \in N(x) \cap L_1$. Let $L'_i = L_i \cup \{x, r_i\}$, where r_i is an arbitrary vertex in $N(x) \cap V(G_i)$ $(i = 2, 3, \ldots, t)$. As every vertex $y \in V(G_i) \setminus \{x\}$ is double dominated by L_i, there are at least two vertices $w, z \in N[y] \cap L_i$. Considering these two facts leads us to conclusion which for every $y \in V(G_i) \setminus \{x\}$ we have $|(N[y] \cup N_{G_i}[x]) \cap L'_i \geq |\{w, z, x\}| = 3$ and $|N[x] \cap L'_i| \geq |\{x, r_i\}| = 2$, as desired. Therefore, $\sum_{i=1}^{t} \gamma_{LR}(G_i) \leq \sum_{i=1}^{t} |L'_i| \leq \sum_{i=1}^{t} (|L_i| + 2) - 1 = \sum_{i=1}^{t} |L_i| + 2t - 1 = \gamma_{LR}(G) + 2t - 1$. So, $\sum_{i=1}^{t} \gamma_{LR}(G_i) - (2t-1) \leq \gamma_{LR}(G)$. ■

Theorem 12. *Let $e = uv$ be a cut edge in a graph G, and G_1 and G_2 be the components of $G - e$. If $|V(G_1)| \geq 3$ and $|V(G_2)| \geq 3$, then*

$$\gamma_{LR}(G_1) + \gamma_{LR}(G_2) - 2 \leq \gamma_{LR}(G) \leq \gamma_{LR}(G_1) + \gamma_{LR}(G_2).$$

Proof. Let L be a $\gamma_{LR}(G)$-liar's dominating set and $L_1 = L \cap G_1$, $L_2 = L \cap G_2$. If neither u nor v belongs to L, deleting edge e will not change the size of liar's dominating sets of G_1 and G_2. Thus, let assume that $u \in L$ and $v \notin L$. Then, by applying Theorem 1 and considering the fact that each component of L has size at least three, we can assume that there are four vertices $u' \in N(u) \cap L_1$, $u'' \in (N(u) \cap L_1) \cup (N(u') \cap L_1) \backslash \{u, u'\}$, $v' \in N(v) \cap L_2$ and finally $v'' \in (N(v) \cap L_2) \cup (N(v') \cap L_2) \backslash \{v, v'\}$. Let $L'_1 = L_1$ and $L'_2 = L_2 \cup \{v\}$, hence L'_1 and L'_2 make a liar's domiating sets for G_1 and G_2, respectively and $|L'_1| + |L'_2| - 1 \leq |L|$. For the final part suppose that both u and v belong to L. Thus, we can assume that there is a vertex $u' \in N(u) \cap L_1$ or $v' \in N(v) \cap L_2$ (without loss of generality, suppose that it is u'). Let $L'_1 = L_1 \cup \{u''\}$ where $u'' \in (N(u) \cap G_1) \cup (N(u') \cap G_1) \backslash \{u, u'\}$ and $L'_2 = L_2 \cup \{v', v''\}$ where $v' \in N(v) \cap G_2$ and $v'' \in (N(v) \cap G_2) \cup (N(v') \cap G_2) \backslash \{v, v'\}$. Therefore, L'_1,L'_2 make liar's domiating sets for G_1 and G_2, respectively, and $|L'_1| + |L'_2| - 2 \leq |L|$. The proof of the right side of the inequality follows from the fact that the union of the $\gamma_{LR}(G_1)$-set and $\gamma_{LR}(G_2)$-set forms a liar's dominating set for G. ■

The following can be proved by an argument similar to Theorems 11 and 12, so we omit the details.

Theorem 13. *Let $E_c = \{e_1, \ldots, e_k\}$ be an edge cut in a graph G, and $G - E_c = G_1 \cup G_2$, where $|V(G_1)| \geq 3$ and $|V(G_2)| \geq 3$. If $A = \{e_i : \exists e_j \in E_c$ and $e_i \cap e_j \neq \emptyset\}$, $B = E_c \setminus A$ and $C = \{v_i : \exists e_j \in A$ and v_i is incidents with $e_j\}$, then $\gamma_{LR}(G_1) + \gamma_{LR}(G_2) - 3|B| - 2|C| \leq \gamma_{LR}(G) \leq \gamma_{LR}(G_1) + \gamma_{LR}(G_2)$.*

If $\lambda(G)$ is the edge connectivity of a graph G, then we obtain the following from Theorem 13, since $|B| \leq \lambda(G)$ and $|C| \leq \lambda(G)$.

Corollary 14. *A graph G has an edge cut E_c, and G_1 and G_2 are the components of $G - E_c$, then $\gamma_{LR}(G_1) + \gamma_{LR}(G_2) - 5\lambda(G) \leq \gamma_{LR}(G) \leq \gamma_{LR}(G_1) + \gamma_{LR}(G_2)$.*

In the rest of this section we investigate the relations of liar's domination and identifying codes. An identifying code of a graph G is a subset of vertices of G that allows one to distinguish each vertex of G by means of its neighborhood within the identifying code. This notation introduced by Karpovsky, Chakrabarty and Levitin in 1998 [9] and has been studied in many papers ([7, 8, 10], and etc.).

A separating code of a graph is a subset of vertices that allows one to distinguish all vertices from each other using their neighborhoods within the code $\mathcal{C}$ ([9]). So, we can define separating code as follows.

Definition [9]**.** A *separating code* is a subset $\mathcal{C}$ of vertices of G such that for any pair u, v of distinct vertices of G, we have $N[u] \cap \mathcal{C} \neq N[v] \cap \mathcal{C}$.

Definition [9]**.** For a graph G, a subset $\mathcal{C}$ of $V(G)$ is an *identifying code* of G, if $\mathcal{C}$ is both a dominating set and a separating code of G. $\mathcal{I}(G)$ is the size of the minimum identifying code of G.

An important consideration in identifying code, is that for any pair u, v of arbitrary vertices in our graph, we must have $N[u] \neq N[v]$. Otherwise this graph cannot admit any identifying code (like complete graphs). If there would be any vertices in graph in which their closed neighborhoods are the same, these vertices are called *twin vertices* ([1]). Graphs which have no twin vertices are called *twin-free graphs* ([1]). So, identifying code is defined only for twin-free graphs.

Proposition 15. *Let G be a twin-free and connected graph. If $\mathcal{C}$ is both identifying and double dominating set, then $\mathcal{C}$ is liar's dominating set as well, so in this case $\gamma_{LR}(G) \leq \mathcal{I}(G)$.*

Proof. Because $\mathcal{C}$ is a double dominating set, the first condition of Theorem 1 holds, and, since $\mathcal{C}$ is also an identifying code, for any two arbitrary vertices $x, y \in V(G)$, $N[x] \cap \mathcal{C} \neq N[y] \cap \mathcal{C}$, so $|(N[x] \cap \mathcal{C} - N[y] \cap \mathcal{C}) \cup (N[y] \cap \mathcal{C} - N[x] \cap \mathcal{C})|$ is at least 3. Hence the second condition of Theorem 1 holds too. ■

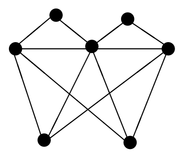

Figure 1. The graph H.

In the general case Proposition 15 can be expressed as follows: $\gamma_{LR}(G) \leq |\mathcal{C}| + |\{v_1, \ldots, v_r\}|$, where v_i $(1 \leq i \leq r)$ are the vertices for which $|N[v_i] \cap \mathcal{C}| = 1$. For some graphs like $G = C_4$, $\gamma_{LR}(G) = \mathcal{I}(G) = 3$. Also for general graphs, it seems that if G is twin-free and 2-connected, then $\gamma_{LR}(G) \geq \mathcal{I}(G)$ but we show that it is not always true. Let H be the graph, shown in Figure 1. Let H_i be a copy of H for $i = 1, 2, 3, 4$. Let x_i and y_i be the vertices of degree three of H_i. Let G be a graph formed by adding edges y_1x_2, y_2x_3, y_3x_4 and y_4x_1. It can be seen that G is twin-free and 2-connected, while $\gamma_{LR}(G) < \mathcal{I}(G)$.

4. BOUNDS ON THE LIAR'S DOMINATION COMPONENT NUMBER

Given a graph G with its liar's domination number $\gamma_{LR}(G)$, it maybe possible that for a $\gamma_{LR}(G)$-set L_1, $G[L_1]$ has k_1 components, while for a different $\gamma_{LR}(G)$-set L_2, $G[L_2]$ has $k_2 \neq k_1$ components. We define *liar's domination component number* $k_{\gamma_{LR}(G)}$ as the maximum number of components over all $\gamma_{LR}(G)$-sets. Thus, $k_{\gamma_{LR}(G)} = \max\{k : G[L] \text{ has } k \text{ components for a } \gamma_{LR}(G)\text{-set } L\}$. In this section we determine various bounds for $k_{\gamma_{LR}(G)}$.

Observation 16. *If a graph G has a $\gamma_{LR}(G)$-set L such that $G[L]$ has $k \geq 1$ components, then $k \geq |L| - |E(G[L])|$.*

Proof. Let L be a $\gamma_{LR}(G)$-set, and $G[L]$ has $k \geq 1$ components $G_1, \ldots, G_k$. Clearly for $i = 1, 2, \ldots, k$, $|V(G_i)| \geq 3$ and $|E(G_i)| \geq |V(G_i)| - 1$, since G_i is connected. Thus, $|E(G[L])| = \sum_{i=1}^{k} |E(G_i)| \geq \sum_{i=1}^{k} |V(G_i)| - 1 \geq |L| - k$. ■

Using Observation 16 we obtain the following.

Theorem 17. *For a graph G of order $n = |V(G)|$ and size $m = |E(G)|$, $k_{\gamma_{LR}(G)} \geq 2n - m - \gamma_{LR}(G)$.*

Proof. Let L be a $\gamma_{LR}(G)$-set, and $G[L]$ has $k \geq 1$ components. By Observation 16, $|E(G[L])| \geq |L| - k$. Since L is a liar's dominating set, each vertex of $V(G) \setminus L$ has at least two neighbors in L. Thus, we obtain $m \geq 2(n - |L|) + |L| - k$, and this implies that $\gamma_{LR}(G) \geq 2n - k - m$. ■

As an example to the sharpness of Theorem 17 consider a path P_6. Note that $V(P_6)$ is the unique $\gamma_{LR}(P_6)$-set.

Theorem 18. *For a graph G of order $n = |V(G)|$ and size $m = |E(G)|$, $k_{\gamma_{LR}(G)} \geq (2n - \Delta(G)\gamma_{LR}(G))/2$.*

Proof. Let L be a $\gamma_{LR}(G)$-set, and $G[L]$ has $k \geq 1$ components. By Observation 16, $|E(G[L])| \geq |L| - k$. It is obvious that $\sum_{v \in L} \deg(v) \leq |L|\Delta$. Thus, the number of edges between L and $V(G) \setminus L$ is at most $|L|\Delta - 2(|L| - k)$. Since each $v \in V(G) - L$ has at least two neighbors in L, we obtain that $|V(G) \setminus L| \leq (1/2)(|L|\Delta - 2(|L| - k))$. Thus, $n \leq |L| + (1/2)(|L|\Delta - 2(|L| - k))$, which implies the result. ■

As an example to the sharpness of Theorem 18 consider a cycle C_{12} with a liar's dominating set of cardinality $\gamma_{LR}(C_{12}) = 9$ in which each of it's three components is a path P_3.

Theorem 19. *For a graph G of order $n = |V(G)|$ and size $m = |E(G)|$,*

$$k_{\gamma_{LR}(G)} \geq \left\lceil \frac{6n - 2m - (\gamma_{LR}(G))^2 - 3\gamma_{LR}(G)}{2} \right\rceil.$$

Proof. Let L be a $\gamma_{LR}(G)$-set. We partition the vertex set of G into three subsets L, N_1 and N_2, where N_1 is the set of vertices of $V(G) \setminus L$ with more than two neighbors in L, and N_2 is the set of vertices of $V(G) \setminus L$ with exactly two neighbors in L. Clearly $n = |N_1| + |N_2| + |L|$ and $0 \leq |N_i| \leq |V(G) - L|$ for $i = 1, 2$. By Observation 16, $|E(G[L])| \geq |L| - k$. Thus, we obtain that $m \geq 3|N_1| + |L| - k + 2|N_2|$. Since $n = |N_1| + |L| + |N_2|$, and $|N_2| \leq \binom{|L|}{2}$ we obtain that $|L|^2 + 3|L| + 2m - 6n + 2k \geq 0$, which yields $k \geq \frac{6n - 2m - (\gamma_{LR}(G))^2 - 3\gamma_{LR}(G)}{2}$. ■

As an example of the sharpness of Theorem 19 consider the graph G_1 shown in Figure 2.

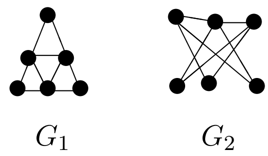

Figure 2. The graphs G_1 and G_2.

We next show that for every $k \geq 1$ there is a graph $G_{r,k}$ of order n in which $k = k_{\gamma_{LR}(G)} = (2n - \gamma_{LR}(G)\Delta)/2$.

Theorem 20. *For every $r \geq 2$ and $k \geq 1$ there is a graph $G_{r,k}$ of order n in which $k = k_{\gamma_{LR}(G)} = (2n - r\gamma_{LR}(G))/2$ and $\Delta(G_{r,k}) = r$.*

Proof. First assume that $k = 1$. Let P_{r+1} be a path of length $r + 1$ with vertex set $\{v_1, \ldots, v_{r+1}\}$. Let $G_{r,1}$ be a graph obtained from P_{r+1} by adding a vertex set $\{u_j\}$ $(1 \leq j \leq \frac{(r+1)(r-2)}{2} + 1)$, and an edge set $\{v_iu_j : |N(u_j)| = 2, |N(v_i)| = r$ for $1 \leq i \leq r + 1; 1 \leq j \leq \frac{(r+1)(r-2)}{2} + 1$, and $|N(u_a) \cap N(u_b)| \leq 1$ for all $a \neq b\}$. Let $L = V(P_{r+1})$. We show L is a $\gamma_{LR}(G_{r,1})$-set with the property satisfying in Theorem 20. Since $|L| = r + 1$, we find that $n = (r + 1) + \frac{(r+1)(r-2)}{2} + 1 = \frac{r(r+1)}{2} + 1$ which leads to $|L| = \frac{2(n-1)}{r}$. It is straightforward to see that L is a liar's dominating set for $V(G_{r,k})$, hence $\gamma_{LR}(G_{r,k}) \leq |L|$. By Theorem 3, $\gamma_{LR}(G_{r,k}) \geq \frac{6n}{3r+2}$, so $\gamma_{LR}(G_{r,k}) \geq \frac{3r(r+1)+6}{3r+2} \geq r + \frac{6}{3r+2}$, hence $\gamma_{LR}(G_{r,k}) \geq r + 1 = |L|$. Therefore, $\gamma_{LR}(G_{r,k}) = |L|$ and so $|L|$ is a $\gamma_{LR}(G_{r,k})$-set. We show that $\Delta(G_{r,k}) = r$. Since the vertices v_1 and v_{r+1} have degree 1 in P_{r+1} there are $r - 1$ edges coming out of these vertices. For each vertex v_i $(i \neq 1, r + 1)$, we can assign $r - 2$ extra edges to it as well, since it has degree 2 in P_{r+1}. Thus, we have $2(r - 1) + (r - 1)(r - 2)$ edges and as each pair of vertices can specify exactly a vertex according to the structure, we have $\frac{2(r-1)+(r-1)(r-2)}{2} = \frac{(r+1)(r-2)}{2} + 1$ vertices which has the exact amount of vertex set $\{u_j\}$ $(1 \leq j \leq \frac{(r+1)(r-2)}{2} + 1)$, and we conclude that $\Delta(G_{r,k}) = r$. Figure 3 shows the graph G with $r = 5$ and $k = 1$.

Next assume that $k \geq 2$. Let $r \geq 2$, and let P_{r+1} be a path on $r + 1$ vertices with vertex set $\{v_1, \ldots, v_{r+1}\}$. Let G_1 be a graph obtained from P_{r+1} by adding the vertex set $\{u_j\}$ $(1 \leq j \leq \frac{(r+1)(r-2)}{2})$, and the edge set $\{v_iu_j : |N(u_j)| = 2, |N(v_i)| = r(2 \leq i \leq r), |N(v_i)| = r - 1(i = 1, r + 1); \forall a \neq b |N(u_a) \cap N(u_b)| \leq 1; 1 \leq j \leq \frac{(r+1)(r-2)}{2}\}$.

Let $V(G_1) = V(P_{r+1}) \cup \{u_j\}(1 \leq j \leq \frac{(r+1)(r-2)}{2})$ $E(G_1) = E(P_{r+1}) \cup A$ and $G_1 = (V(G_1), E(G_1))$. Now consider k copies of the graph G_1 like $\{G_1^1, \ldots, G_1^k\}$.

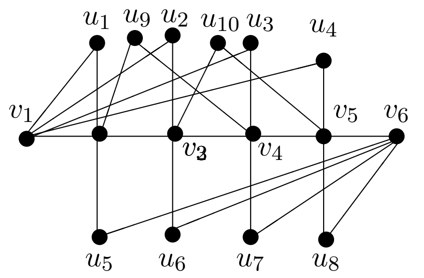

Figure 3. The graph G with $r = 5$ and $k = 1$.

Suppose v_1^i and v_{r+1}^i as the first and the last vertices of each P_{r+1}^i in G_1^i, respectively, and consider k vertices $\{x_i\}$ $(1 \leq i \leq k)$. Make a cycle with these k paths $\{P_{r+1}^i\}_{i=1}^k$ and vertices $\{x_i\}$ in a way each x_i will be the connection of two paths P_{r+1}^i and P_{r+1}^{i+1} (for $1 \leq i \leq k-1$) and x_k will be the connection of P_{r+1}^k and P_{r+1}^1. In other words consider the following edge set: $B = \{v_{r+1}^i x_i, v_1^{i+1} x_i : 2 \leq i \leq k-1\} \cup \{v_1^1 x_1, v_1^2 x_1, v_{r+1}^k x_k, v_{r+1}^1 x_k\}$. Now let $V(G_{r,k}) = \{V(G_1^i)\}_{i=1}^k \cup \{x_i\}_{i=1}^k$; $E(G_{r,k}) = \{E(G_1^i)\}_{i=1}^k \cup B$ and $G_{r,k} = (V(G_{r,k}), E(G_{r,k}))$. Consider $L = \{P_{r+1}^i\}_{i=1}^k$. Just like in the last part we can conclude L is a $\gamma_{LR}(G_{r,k})$-set and also we have: $|L| = k(r+1)$, $n = k(r+1) + k\frac{(r+1)(r-2)}{2} + k = \frac{kr(r+1)}{2} + k$, which leads to $|L| = \frac{2(n-k)}{r}$. The proof of the existence of the set A can be done in the same way as in the previous part. ■

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References

[1] D. Auger, *Induced paths in twin-free graphs*, Electron. J. Combin. **15** #N17 (2008).

[2] J.A. Bondy and U.S.R. Murty, Graph Theory, Graduate Texts in Mathematics **244** (Springer-Verlag, London, 2008).

[3] G. Chartrand and L. Lesniak, Graphs and Digraphs, 4th Ed. (CRC Press, Bocz Raton, 2004).

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs, Advanced Topics (Marcel Dekker, Inc., New York, 1998).

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graph (Marcel Dekker, Inc., New York, 1998).

[6] T.W. Haynes, P.J. Slater and C. Sterling, *Liar's domination in ladders*, Congr. Numer. **212** (2012) 45–56.

[7] I. Honkala, T. Laihonen and S. Ranto, *On codes identifying sets of vertices in Hamming spaces*, Des. Codes Cryptogr. **24** (2001) 193–204. doi:10.1023/A:1011256721935

[8] V. Junnila and T. Laihonen, *Optimal identifying codes in cycles and paths,* Graphs Combin. **28** (2012) 469–481. doi:10.1007/s00373-011-1058-6

[9] M.G. Karpovsky, K. Chakrabarty and L.B. Levitin, *On a new class of codes for identifying vertices in graphs,* IEEE Trans. Inform. Theory **44** (1998) 599–611. doi:10.1109/18.661507

[10] M. Nikodem, *False alarms in fault-tolerant dominating sets in graphs*, Opuscula Math. **32** (2012) 751–760. doi:10.7494/OpMath.2012.32.4.751

[11] B.S. Panda and S. Paul, *Hardness results and approximation algorithm for total liar's domination in graphs*, J. Comb. Optim. **27** (2014) 643–662. doi:10.1007/s10878-012-9542-3

[12] B.S. Panda and S. Paul, *Liar's domination in graphs: Complexity and algorithm*, Discrete Appl. Math. **161** (2013) 1085–1092. doi:10.1016/j.dam.2012.12.011

[13] B.S. Panda and S. Paul, *A linear time algorithm for liar's domination problem in proper interval graphs*, Inform. Process. Lett. **113** (2013) 815–822. doi:10.1016/j.ipl.2013.07.012

[14] M.L. Roden and P.J. Slater, *Liar's domination and the domination continuum*, Congr. Numer. **190** (2008) 77–85.

[15] M.L. Roden and P.J. Slater, *Liar's domination in graphs*, Discrete Math. **309** (2009) 5884–5890. doi:10.1016/j.disc.2008.07.019

[16] P.J. Slater, *Liar's domination*, Networks **54** (2009) 70–74. doi:10.1002/net.20295

[17] J. Zhou, Z. Zhang, W. Wu and K. Xing, *A greedy algorithm for the fault-tolerant connected dominating set in a general graph*, J. Comb. Optim. **28** (2014) 310–319. doi:10.1007/s10878-013-9638-4

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