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ON THE WEIGHT OF MINOR FACES IN TRIANGLE-FREE 3-POLYTOPES

Oleg V. Borodin[1]

Institute of Mathematics Siberian Branch
Russian Academy of Sciences, Novosibirsk, 630090, Russia

e-mail: brdnoleg@math.nsc.ru

and

Anna O. Ivanova[2]

Ammosov North-Eastern Federal University
Yakutsk, 677000, Russia

e-mail: shmgnanna@mail.ru

Abstract

The weight $w(f)$ of a face f in a 3-polytope is the degree-sum of vertices incident with f. It follows from Lebesgue's results of 1940 that every triangle-free 3-polytope without 4-faces incident with at least three 3-vertices has a 4-face with $w \leq 21$ or a 5-face with $w \leq 17$. Here, the bound 17 is sharp, but it was still unknown whether 21 is sharp.

The purpose of this paper is to improve this 21 to 20, which is best possible.

Keywords: plane map, plane graph, 3-polytope, structural property, weight of face.

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1. Introduction

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [34], the 3-polytopes are in one-to-one correspondence with the 3-connected planar graphs.

The *degree* $d(x)$ of a vertex or face x in a 3-polytope M is the number of incident edges. A *k-vertex* and *k-face* is one of degree k, a *k^+-vertex* has degree at least k, and so on. The *weight* $w(f)$ of a face f in M is the degree-sum of vertices incident with f. By $w(M)$, or simply w, we denote the minimum weight of 5^--faces in M. By Δ and δ denote the maximum and minimum vertex degree of M, respectively.

We say that f is a *face of type* $(k_1, k_2, \ldots)$ or simply a $(k_1, k_2, \ldots)$*-face* if the set of degrees of the vertices incident with f is majorized by the vector $(k_1, k_2, \ldots)$. A 4-face of the type $(3, 3, 3, \infty)$ is *pyramidal.* Note that in the $(3, 3, 3, n)$-Archimedean solid each face f is pyramidal and satisfies $w(f) = n + 9$.

We now recall some results on the structure of 5^--faces in 3-polytopes. Back in 1940, Lebesgue [26] gave an approximate description of types of 5^--faces in normal plane maps.

Theorem 1 (Lebesgue [26])**.** *Every normal plane map has a 5^--face of one of the following types:*

$$(3,6,\infty),\ (3,7,41),\ (3,8,23),\ (3,9,17),\ (3,10,14),\ (3,11,13),$$
$$(4,4,\infty),\ (4,5,19),\ (4,6,11),\ (4,7,9),\ (5,5,9),\ (5,6,7),$$
$$(3,3,3,\infty),\ (3,3,4,11),\ (3,3,5,7),\ (3,4,4,5),\ (3,3,3,3,5).$$

The classical Theorem 1, along with other ideas in Lebesgue [26], has a lot of applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [10, 31, 33]).

Some parameters of Lebesgue's Theorem were improved for narrow classes of plane graphs. In 1963, Kotzig [24] proved that every plane triangulation with $\delta = 5$ satisfies $w \le 18$ and conjectured that $w \le 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 2 (Borodin [2])**.** *Every normal plane map with $\delta = 5$ has a $(5, 5, 7)$-face or a $(5, 6, 6)$-face, where all parameters are tight.*

Theorem 2 also confirmed a conjecture of Grünbaum [19] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [32]).

We note that a 3-polytope with $(4, 4, \infty)$-faces can have unbounded w, as follows from the n-pyramid, double n-pyramid, and a related construction in which every 3-face is incident with a 3-vertex, 4-vertex, and n-vertex. As mentioned above, the same is true concerning $(3, 3, 3, \infty)$-faces.

For plane triangulations without 4-vertices, Kotzig [25] proved $w \leq 39$, and Borodin [4], confirming Kotzig's conjecture in [25], proved $w \leq 29$, which is best possible due to the dual of the twice-truncated dodecahedron. Borodin [5] further showed that each triangulated 3-polytope without $(4, 4, \infty)$-faces satisfies $w \leq 29$, and that for triangulations without adjacent 4-vertices there is a sharp bound $w \leq 37$.

For an arbitrary 3-polytope, Theorem 1 yields $w \leq \max\{51, \Delta + 9\}$. Horňák and Jendrol' [20] strengthened this as follows: if there are neither $(4, 4, \infty)$-faces nor $(3, 3, 3, \infty)$-faces, then $w \leq 47$. Borodin and Woodall [7] proved that forbidding $(3, 3, 3, \infty)$-faces implies $w \leq \max\{29, \Delta + 8\}$.

For quadrangulated 3-polytopes, Avgustinovich and Borodin [1] improved the description of 4-faces implied by Lebesgue's Theorem as follows: $(3, 3, 3, \infty)$, $(3, 3, 4, 10)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$.

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in a recent survey by Jendrol' and Voss [22], and also in [3, 6, 8, 17, 18, 21, 23, 27–30, 35].

In 2002, Borodin [9] strengthened Lebesgue's Theorem 1 as follows (the entries marked by an asterisk are proved in [9] to be best possible).

Theorem 3 (Borodin [9])**.** *Every normal plane map has a* 5^-*-face of one of the following types:*

$$(3, 6, \infty^*),\ (3, 8^*, 22),\ (3, 9^*, 15),\ (3, 10^*, 13),\ (3, 11^*, 12),$$
$$(4, 4, \infty^*),\ (4, 5^*, 17),\ (4, 6^*, 11),\ (4, 7^*, 8),\ (5, 5^*, 8),\ (5, 6, 6^*),$$
$$(3, 3, 3, \infty^*),\ (3, 3, 4^*, 11),\ (3, 3, 5^*, 7),\ (3, 4, 4, 5^*),\ (3, 3, 3, 3, 5^*).$$

Recently, precise descriptions of the structure of faces were obtained for 3-polytopes with $\delta \geq 4$ and for triangulated 3-polytopes.

Theorem 4 (Borodin, Ivanova [11])**.** *Every* 3*-polytope without* 3*-vertices has a* 3*-face of one of the following types:*

$$(4, 4, \infty),\ (4, 5, 14),\ (4, 6, 10),\ (4, 7, 7),\ (5, 5, 7),\ (5, 6, 6),$$

where all parameters are sharp.

Theorem 5 (Borodin, Ivanova, Kostochka [12])**.** *Every triangulated* 3*-polytope has a face of one of the following types:*

$$(3, 4, 31),\ (3, 5, 21),\ (3, 6, 20),\ (3, 7, 13),\ (3, 8, 14),\ (3, 9, 12),\ (3, 10, 12),$$
$$(4, 4, \infty),\ (4, 5, 11),\ (4, 6, 10),\ (4, 7, 7),\ (5, 6, 6),\ (5, 5, 7),$$

where all parameters are sharp.

It follows from Lebesgue's Theorem 1 that every triangle-free 3-polytope without pyramidal faces has a 4-face with $w \leq 21$ or a 5-face with $w \leq 17$. For a long time, it was not known whether Lebesgue's bound $w \leq 21$ is sharp. The purpose of our paper is to answer this question by proving

Theorem 6. *Every triangle-free* 3*-polytope without pyramidal* 4*-faces has a* 4*-face of weight at most* 20 *or a* 5*-face of weight at most* 17*, where both bounds* 20 *and* 17 *are sharp.*

2. Proving Theorem 6

To prove the sharpness of the bound 20, it suffices to insert the configuration shown in Figure 1 into every face of the icosahedron, which provides a triangle-free 3-polytope without pyramidal 4-faces in which every 4-face has weight 20. The sharpness of the bound 17 follows from the $(3, 3, 3, 3, 5)$-Archimedean solid.

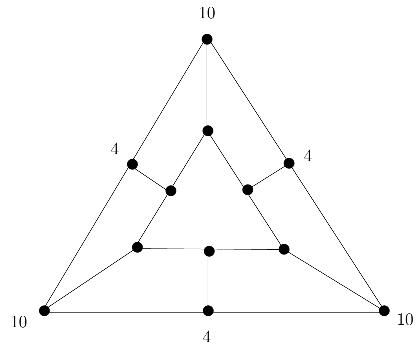

Figure 1. A fragment of an extremal construction derived from the icosahedron.

Now suppose M is a counter-example to the upper bounds in Theorem 6. Euler's formula $|V| - |E| + |F| = 2$ for M implies

$$\sum_{x \in V \cup F} (d(x) - 4) = -8, \tag{1}$$

where V, E, and F are the sets of vertices, edges, and faces of M.

We assign an *initial charge* $\mu(x) = d(x) - 4$ to every $x \in V \cup F$; so only the 3-vertices in V have a negative charge. Using the properties of M as a counterexample, we will define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x)$ is nonnegative whenever $x \in V \cup F$. This will contradict the fact that the sum of new charges, according to (1), is -8.

2.1. Basic properties of the counterexample M

We need a few definitions and comments.

A face f is *strong* if either $d(f) \geq 6$, or $d(f) = 5$ and f is incident with a 6^+-vertex, or else $d(f) = 4$ and f is incident with at least two 6^+-vertices. Otherwise, f is *weak*.

Clearly, a weak 5-face f can be incident with at most three 3-vertices, since $w(f) > 4 \times 3 + 5$. A weak 5-face is *helpful* if it is incident with at most two 3-vertices. A $(3, 3, 3, 4, 5)$-face is a *transmitter*. A transmitter is a *transmitter*-1 if its 4-vertex is adjacent to its 5-vertex; otherwise, it is a *transmitter*-2. We will not be concerned about weak 5-faces incident with three 3-vertices and two 5-vertices.

A weak 4-face $f = v_1 \cdots v_4$ is *sharp* if $d(v_1) = d(v_3) = 3$, $4 \leq d(v_2) \leq 5$, and $d(v_4) = 11$; here, v_2 is the *summit* of f. Depending on the degree of the summit, we have 4-*sharp* and 5-*sharp* faces. A weak 4-face is *special* if it is incident with two 3-vertices, a 5-vertex, and 10-vertex.

We also need a few more specialized definitions and remarks. An 11-vertex is *poor* if it is completely surrounded by $(3, 3, 5, 11)$-faces. Note that a poor vertex may be incident with $(3, 3, 4, 11)$-faces but not with $(3, 3, 3, 11)$-faces. We now explore the structural properties of poor vertices in some detail.

Remark 7. Every poor 11-vertex is incident with an odd number of sharp 4-faces due to the alteration of 3-neighbors with those of degree four of five. In particular, each poor vertex belongs to at least one sharp face.

Now we look what happens around a summit 4-vertex. Suppose a poor 11-vertex v has a 4-sharp face $f_1 = vv_1w_1v_2$ with the summit w_1; so $d(w_1) = 4$. Recall that $d(v_1) = d(v_2) = 3$ by definition. Furthermore, there are $(3, 3, 5, 11)$-faces $f_{11} = vv_{11}w_{11}v_1$ and $f_2 = vv_2w_2v_3$ (see Figure 2).

Here, we have *lying* faces $f_1^* = \cdots w_{11}v_1w_1x_1$ and $f_2^* = \cdots w_2v_2w_1x_2$, and also a *standing* face $\overline{f_1} = \cdots x_1w_1x_2$, which lies opposite to the sharp face f_1 with respect to the summit 4-vertex w_1.

Remark 8. A standing face can well be sharp, but no lying face is sharp. Indeed, for f_2^* in Figure 2 to be sharp, we should have $d(w_2) = 11$, whereas actually $d(w_2) \leq 5$ since v is poor.

Remark 9. If a 4-vertex w_1 is the summit of a sharp 4-face at a poor 11-vertex v such that the standing face $\overline{f_1}$ at w_1 is weak, then at least one of the lying faces at w_1 is a 5^+-face. Indeed, if both lying faces at w_1 are 4-faces (again, we follow the notation in Figure 2), then it follows from $d(w_{11}) \leq 5$ and $d(w_2) \leq 5$ that $d(x_1) \geq 21 - 5 - 3 - 4 = 9$ and $d(x_2) \geq 9$, so $\overline{f_1} = \cdots x_1w_1x_2$ is strong; a contradiction.

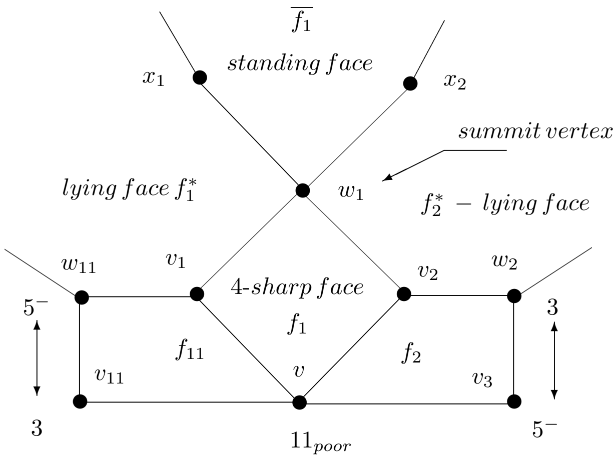

Figure 2. Objects related to a 4-sharp face f_1.

We say that a weak 5-face $f_2^* = x_2w_1v_2w_2z$ lying at a poor 11-vertex v (we follow Figure 2, so $d(w_1) = 4$ and $d(v_2) = 3$) *sees* v through the 3-vertex v_2. Note that a weak 5-face f can see at most two poor vertices since the boundary of f has either at most two 3-vertices or at most one 4-vertex, for otherwise $d(f) = 3 \times 3 + 2 \times 4 = 17$, which is impossible. Moreover, any transmitter-1 can see at most one poor vertex, since it has only one 4-vertex adjacent to 3-vertex along the boundary, which is necessary for a poor 11-vertex to be seen through a 3-vertex.

Now we are ready to introduce the key notion in our proof. A poor 11-vertex v is *bad* if it satisfies the following properties:

(B1) v has no 5-neighbors;
(B2) v has neither standing nor lying strong faces;
(B3) v has neither helpful nor transmitter-1 lying 5-faces;
(B4) v has precisely one face that is either 5-sharp or lying transmitter-2.

Remark 10. A transmitter-2 $f_2^* = x_2w_1v_2w_2z$ can see at most one bad 11-vertex v, which happens through the 3-vertex v_2 when $d(w_1) = 4$ and $d(w_2) = 3$ (see Figure 2 again). Indeed, the possibility $d(w_2) = 5$ contradicts (B4) for v.

2.2. Rules of discharging

We use the following rules of discharging (see Figure 3). Some notation in the statements of our rules is borrowed from Figure 2.

R1. *Each face gives* $\frac{1}{3}$ *to every incident* 3*-vertex.*

R2. *Each strong face gives* $\frac{1}{3}$ *to each incident vertex* v *with* $4 \leq d(v) \leq 5$.

R3. *Each vertex* v *gives to each incident face:*

(a) $\frac{1}{3}$, *if* $6 \leq d(v) \leq 9$, *or*
(b) $\frac{3}{5}$, *if* $d(v) = 10$.

R4. *Each* 11*-vertex gives each incident face* f:

(a) $\frac{2}{3}$, *if* $d(f) = 4$ *and* f *is incident with two* 3*-vertices and a vertex of degree* 4 *or* 5, *or*
(b) $\frac{1}{3}$, *otherwise.*

R5. *Each* 12^{+}*-vertex gives* $\frac{2}{3}$ *to each incident face.*

R6. *If a* 5*-vertex* v *is incident with* 4*-faces* $f_1 = vx_1y_1z$ *and* $f_2 = vx_2y_2z$, *where* $d(z) = 11$ *and* $d(x_1) = d(x_2) = d(y_1) = d(y_2) = 3$, *then* v *gives* $\frac{1}{6}$ *to* z *through each of* f_1 *and* f_2.

R7. *If a* 4*-vertex* w_1 *is a summit of a sharp face* $f_1 = vv_1w_1v_2$ *and the standing face* $\overline{f_1}$ *at* w_1 *is strong, then* w_1 *transfers the* $\frac{1}{3}$ *received from* $\overline{f_1}$ *to the poor* 11*-vertex* v *through* f_1.

R8. *Suppose a* 4*-vertex* w_1 *is a summit of a sharp face* $f_1 = vv_1w_1v_2$ *and the lying face* f_1^* *at* w_1 *is strong. Then* w_1 *transfers the* $\frac{1}{3}$ *received from* f_1^* *evenly through the incident sharp faces. As a result, the poor* 11*-vertex* v *receives from* f_1^* *via* w_1:

(a) $\frac{1}{6}$, *if* w_1 *is a summit for two sharp faces, or*
(b) $\frac{1}{3}$, *otherwise.*

R9. *Suppose a* 4*-vertex* w_1 *is a summit of a sharp face* $f_1 = vv_1w_1v_2$ *and the lying face* $f_1^* = y_1x_1w_1v_1w_{11}$ *at* w_1 *is helpful. Then* f_1^* *gives the poor* 11*-vertex* v:

(a) $\frac{1}{3}$, *if* $d(w_{11}) = 3$, *or*
(b) $\frac{1}{6}$, *otherwise.*

R10. *Suppose a* 4*-vertex* w_1 *is a summit of a sharp face* $f_1 = vv_1w_1v_2$ *and the lying face* $f_1^* = y_1x_1w_1v_1w_{11}$ *at* w_1 *is a transmitter-*1*, which means that* $d(x_1) = 5$ *and* $d(y_1) = d(w_{11}) = 3$. *Then* x_1 *gives* $\frac{1}{3}$ *through* f_1^* *to the poor* 11*-vertex* v.

R11. *Suppose a* 4*-vertex* w_1 *is a summit of a sharp face* $f_1 = vv_1w_1v_2$ *and the lying face* $f_1^* = y_1x_1w_1v_1w_{11}$ *at* w_1 *is a transmitter-*2*, which means that* $d(y_1) = 5$ *and* $d(x_1) = d(w_{11}) = 3$. *Then* y_1 *gives through* f_1^* *to the poor* 11*-vertex* v:

(a) $\frac{1}{3}$, *if* v *is bad, or*
(b) $\frac{1}{6}$, *otherwise.*

R12. *If a* 5*-vertex* w_1 *is a summit of a sharp face* $f_1 = vv_1w_1v_2$, *then* w_1 *gives to the poor* 11*-vertex* v:

(a) $\frac{1}{3}$, *if v is bad, or*
(b) $\frac{1}{6}$, *otherwise.*

R13. *Each* 5*-vertex gives* $\frac{1}{15}$ *to each incident special* 4*-face.*

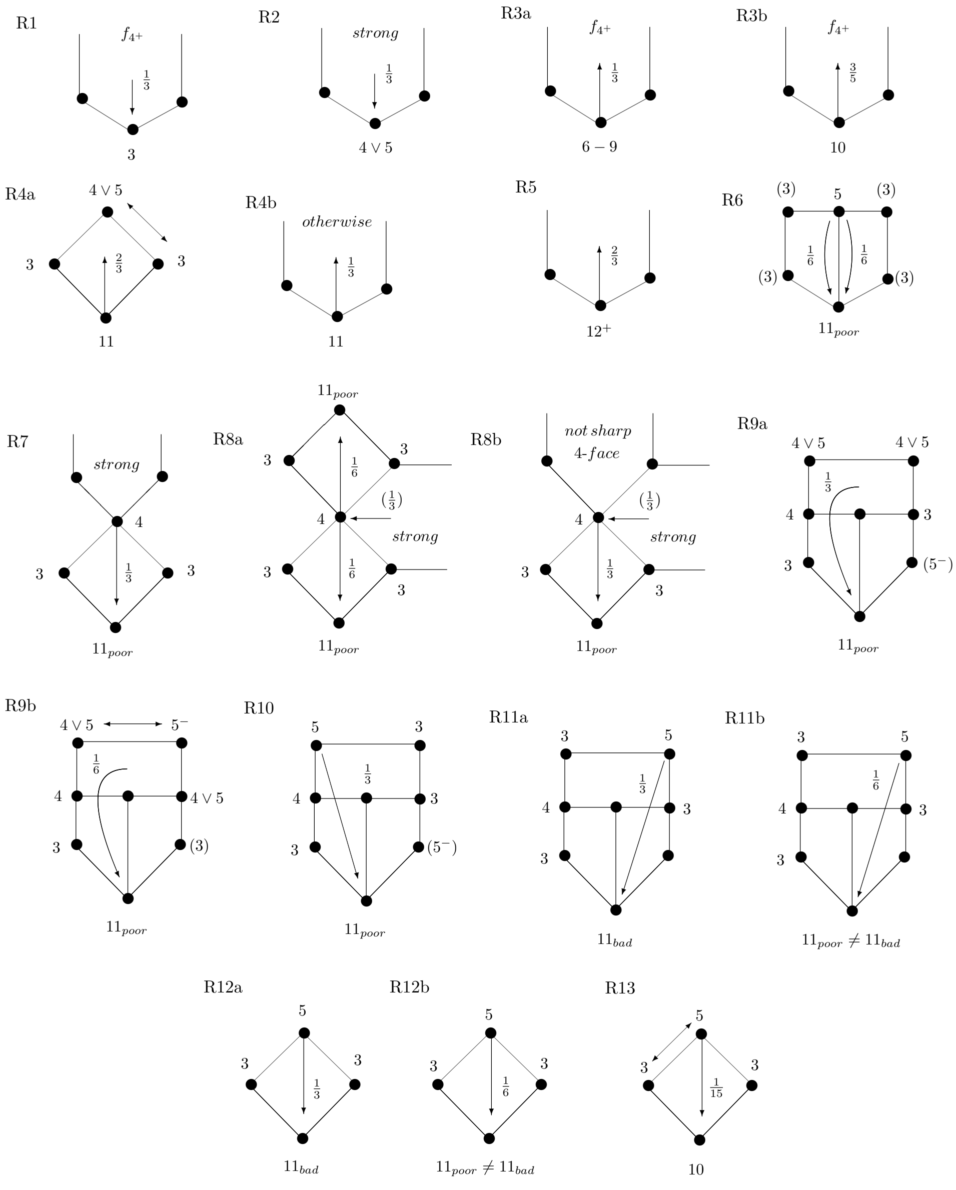

Figure 3. Rules of discharging.

2.3. Proving $\mu'(x) \geq 0$ whenever $x \in V \cup F$

Case 1. $f \in F$.

Subcase 1.1. $d(f) \geq 6$. Here, f is strong and gives $\frac{1}{3}$ to each incident 5^--vertex by R1 and R2, so $\mu'(f) \geq d(f) - 4 - d(f) \times \frac{1}{3} = \frac{2(d(f)-6)}{3} \geq 0$.

Subcase 1.2. Suppose $d(f) = 5$. If f is strong, then f receives at least $\frac{1}{3}$ by R3–R5 and gives $\frac{1}{3}$ to each incident 5^--vertex by R1 and R2, which results in $\mu'(f) \geq 5 - 4 + \frac{1}{3} - 4 \times \frac{1}{3} = 0$.

Now suppose f is weak. We note that f is incident with at most three 3-vertices since $w(f) \geq 18 > 5+4\times 3$ by assumption. If f is incident with precisely three 3-vertices, then f is either incident with two 5-vertices or is a transmitter (that is, has also a 4-vertex and a 5-vertex in its boundary). In both cases, f participates only in R1, so we have $\mu'(f) \geq 5 - 4 - 3 \times \frac{1}{3} = 0$.

Finally, if f is incident with at most two 3-vertices, then each of them receives $\frac{1}{3}$ from f by R1. Furthermore, such an f, called helpful, can participate only in R9, by giving $\frac{1}{3}$ or $\frac{1}{6}$ to each poor 11-vertex seen by f. More specifically, the donation of $\frac{1}{3}$ occurs only in R9a, in the case when the two 3-vertices in the boundary of f are adjacent, which easily implies that only one of them sees a poor vertex. This results in $\mu'(f) \geq 5 - 4 - 3 \times \frac{1}{3} = 0$. Otherwise, R9b works, and we have $\mu'(f) \geq 5 - 4 - 2 \times \frac{1}{3} - 2 \times \frac{1}{6} = 0$.

Subcase 1.3. $d(f) = 4$. Recall that f is incident with at most two 3-vertices due to the absence of pyramidal faces. First suppose that f is incident with precisely two 3-vertices.

If f is special, then it receives $\frac{1}{15}$ from a 5-vertex by R13 and $\frac{3}{5}$ from a 10-vertex by R3b, so $\mu'(f) \geq 4 - 4 - 2 \times \frac{1}{3} + \frac{1}{15} + \frac{3}{5} = 0$ in view of R1. From now on we assume that f is not special.

If f is strong, that is incident with two 6^+-vertices, then f receives at least $\frac{1}{3} + \frac{1}{3}$ by R3–R5, so $\mu'(f) \geq 0$. Otherwise, f is incident with an 11^+-vertex and a vertex of degree 4 or 5, in which case R4a or R5 is applicable, and we again have $\mu'(f) \geq 0$.

Now suppose f is incident with at most one 3-vertex. Recall that f is incident with at least one 6^+-vertex since $w(f) \geq 21 > 4 \times 5$ by assumption. If f is strong, then it can afford giving $\frac{1}{3}$ to each of at most two incident 5^--vertices by R1 and R2 as R3–R5 also apply. Otherwise, f gives $\frac{1}{3}$ at most once by R1, so $\mu'(f) \geq 4 - 4 + \frac{1}{3} - \frac{1}{3} = 0$.

Case 2. $v \in V$.

Subcase 2.1. $d(v) = 3$. Since v receives $\frac{1}{3}$ from each incident face by R1, we have $\mu'(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$.

Subcase 2.2. $d(v) = 4$. We note that v receives $\frac{1}{3}$ from each incident strong face by R2 and can transfer each such a donation either in full or as $\frac{1}{6} + \frac{1}{6}$ to poor 11-vertices by R7 and R8. Thus $\mu'(v) \geq \mu(v) = 4 - 4 = 0$.

Subcase 2.3. $d(v) = 5$. Examining our rules, we see that v can either give charge away to poor 11-vertices (in particular, to bad 11-vertices) by R6 and R10–R12, where the donation through each incident face is either $\frac{1}{3}$ or $\frac{1}{6}$, or can give $\frac{1}{15}$ to a 10-vertex by R13. Furthermore, the donation of $\frac{1}{3}$ along an edge e may be looked at as two donations of $\frac{1}{6}$ through two faces incident with e. As a result of such averaging, v gives through each incident 4-face either $\frac{1}{3}$ if R8 is applicable or at most $\frac{1}{6}$ otherwise.

If v is incident with a strong face f, then v, in turn, receives $\frac{1}{3}$ from f by R2, which results in $\mu'(v) \geq 5 - 4 + \frac{1}{3} - 4 \times \frac{1}{3} = 0$. So we assume from now on that v is completely surrounded by weak faces.

If v gives $\frac{1}{3}$ at most once, then $\mu'(v) \geq 5 - 4 - \frac{1}{3} - 4 \times \frac{1}{6} = 0$. We will prove that v actually gives away through its five faces at most 1 in total, which implies $\mu'(v) \geq 0$. To reach this goal, we need three lemmas.

Let $v_1 \dots v_5$ be the neighbors of v in a cyclic order, and let $f_i = \cdots v_i v v_{i+1}$, for $1 \leq i \leq 5$ (addition modulo 5).

Lemma 11. *If a* 5*-vertex v gives $\frac{1}{3}$ through a transmitter-*1 *face f_1 to a poor* 11*-vertex by* R10*, then v gives nothing through the face f_2 having a common* 4*-vertex with f_1.*

Proof. Suppose $f_1 = vv_1xyv_3$, where $d(v_1) = 4$ and $d(x) = d(y) = d(v_3) = 3$, and there is a poor 11-vertex z that receives $\frac{1}{3}$ from v by R10. Since z is poor, there is a 4-face zxv_1w with $d(w) = 3$ (see Figure 4).

The face $f_2 = v_1vv_2\cdots$ can conduct something from v by our rules only if $f_2 = v_1vv_2y'x'$, where $d(x') = d(y') = d(v_2) = 3$, and there is a poor 11-vertex z'.

However, this implies a strong face $\cdots zwz'$ at a poor 11-vertex z, a contradiction. ■

Lemma 12. *If a* 5*-vertex v gives $\frac{1}{3}$ through a transmitter-*2 *face f_1 to a bad* 11*-vertex by* R11a*, then v gives nothing through f_2 or f_5.*

Proof. Suppose $f_1 = vv_1xyv_2$, where $d(x) = 4$ and $d(v_1) = d(y) = d(v_2) = 3$, and there is a bad 11-vertex z that receives $\frac{1}{3}$ from v by R11a. Since z is bad, there is a 4-face zyv_2x' with $d(y) = d(v_2) = 3$, which implies by (B1) that $d(x') = 4$ (see Figure 5).

The face $f_2 = \cdots v_2vv_3$ can conduct a positive charge by our rules from v only if f_2 is a transmitter-2. This means that we are done unless $f_2 = v_2vv_3y'x'$ with $d(y') = d(v_3) = 3$, and there is a poor 11-vertex z' seen by v via f_2 and y'.

This implies that there is a 4-face $wx'y'z'$ at z' with $d(w) = 3$. Now looking again at the bad 11-vertex z, we see that there is a 4-face $zx'wu$ with $d(u) = 3$. This gives rise to a 4-face $tuwz'$ at a poor vertex z', where $4 \leq d(t) \leq 5$ since there are no pyramidal 4-faces in M. However, a bad vertex z cannot have a 5-sharp

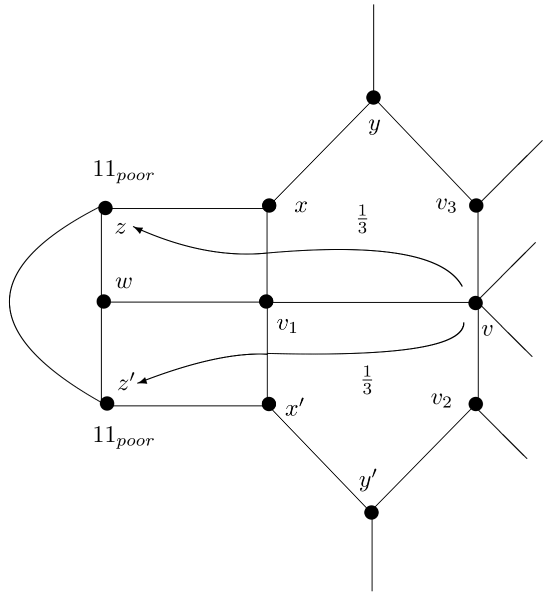

Figure 4. To Lemma 11.

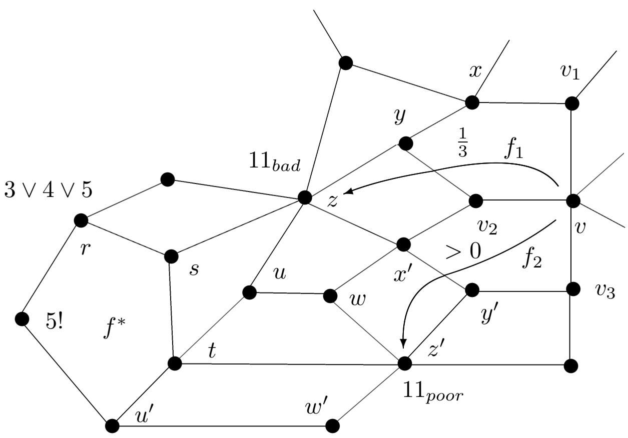

Figure 5. To Lemma 12.

face $zuts$ in addition to the transmitter-2 face f_1 by (B4), and so $d(t) = 4$. We note that $d(s) = 3$ since z is bad.

Now since z' is poor, we have a 4-face $z'tu'w'$ with $u' \neq u$ such that $d(u') = 3$ (and $d(w') = 3$, which is not important for us).

Finally, we consider the face $f^* = \cdots rstu'$ lying at z. We note that $d(r) \leq 4$, for otherwise a bad vertex z would have a 5-sharp face, which is impossible by (B4) again. It is also impossible for f^* to be strong by (B2) or helpful by (B3) since z is bad.

Thus $d(f^*) = 5$, and f^* is incident with three 3-vertices and a 4-vertex, so the fifth incident vertex must have degree 5 since $w(f^*) \geq 18$ in our counterexample M. This implies that $d(r) = 3$. We note that f^* is not transmitter-1 since its 4-vertex is not adjacent to its 5-vertex. So f^* is a transmitter-2 face, which contradicts (B4) applied to z. ■

Lemma 13. *If a* 5*-vertex* v *gives* $\frac{1}{3}$ *through a* 5*-sharp face* f_1 *by* R12a*, then the total donation of* v *through* f_2 *and* f_3 *is at most* $\frac{1}{3}$*.*

Proof. Suppose $f_1 = vv_1zv_2$, where $d(v_1) = d(v_2) = 3$, and z is a bad 11-vertex that receives $\frac{1}{3}$ from v by R12a (see Figure 6).

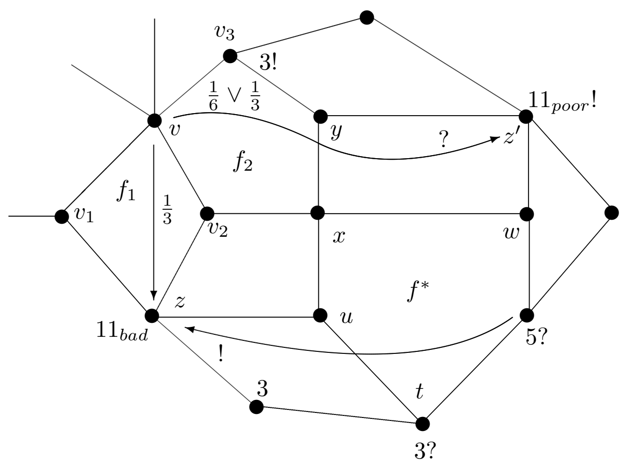

Figure 6. To Lemma 13.

We note that f_2 can conduct a positive charge from v by our rules only when $d(v_3) \in \{3, 4, 10, 11\}$. If $10 \leq d(v_3) \leq 11$, then v gives at most $\frac{1}{6} + \frac{1}{6}$ through f_2 and f_3 by R6 and R13, and we are done. If $d(v_3) = 4$, then the only possibility for f_2 and f_3 to conduct a positive charge from v is to be transmitters-1, which can happen with at most one of them due to Lemma 11. So we can assume that $d(v_3) = 3$.

Since z is bad, there is a 4-face zv_2xu with $d(x) \leq 4$. If $d(x) = 3$, then the only way for f_2, being incident with three 3-vertices and a 5-vertex, to conduct a positive charge from v is to be a transmitter-2 for a poor 11-vertex. However, the 3-vertex x, which is the only "suspicious" 3-vertex lying between a 4-vertex and a 3-vertex v_2 in the boundary of f_2, in fact cannot see an 11-vertex since x is in a common 4-face with the bad vertex z. Therefore, R11 is not applicable to f_2, and so we can assume from now on that $d(x) = 4$.

Now we are done unless $f_2 = vv_2xyv_3$, and f_2 sees a poor 11-vertex z' through the 3-vertex y. In this case, we have a 4-face $z'yxw$ with $d(w) = 3$.

On the other hand, the bad 11-vertex z has a 4-face zv_2xu with $d(u) = 3$ and another 4-face $zuts$ with $t \neq x$. We note that $d(t) \leq 4$, since z cannot belong to two 5-sharp faces according to (B4). Thus there is a 5^+-face $f^* = \cdots tuxw$ lying at z.

We see that f^* cannot be strong due to the property (B2) in the definition of z. If $d(t) = 4$, then f^* can have only two 3-vertices, u and w, on its boundary, as $w(f^*) \geq 18$. Thus f^* is helpful for z, which violates (B3), and so we can assume that $d(t) = 3$. Hence f^* is a transmitter-2 for z, but this contradicts the property (B4) for z. ■

We are now ready to complete the proof of Subcase 2.3. If v gives $\frac{1}{3}$ through f_3 by R12a, then it gives at most $\frac{1}{3}$ through f_1 and f_2 together due to Lemma 13. By symmetry, Lemma 13 is applied also to f_4 and f_5. This implies $\mu'(v) \geq 5 - 4 - 3 \times \frac{1}{3} = 0$.

It remains to assume that R12a is not applied to v. If v gives $\frac{1}{3}$ by R10 and R11a at most once, then we have $\mu'(v) \geq 5 - 4 - \frac{1}{3} - 4 \times \frac{1}{6} = 0$. If these rules are applied to two consecutive faces, say f_2 and f_3, then f_1 and f_4 conduct nothing from v due to Lemmas 11 and 12, which yields $\mu'(v) \geq 5 - 43 \times \frac{1}{3} = 0$.

Otherwise, each of at most two faces takes $\frac{1}{3}$ from v, and there is a face taking nothing from v by the two lemmas, so we have $\mu'(v) \geq 5 - 4 - 2 \times \frac{1}{3} - 2 \times \frac{1}{6} = 0$, as desired.

Subcase 2.4. $6 \leq d(v) \leq 9$. Now v gives $\frac{1}{3}$ to each incident face according to R3a and does not participate in the other rules, whence $\mu'(v) \geq d(v) - 4 - d(v) \times \frac{1}{3} = \frac{2(d(v)-6)}{3} \geq 0$.

Subcase 2.5. $d(v) = 10$. This time, v gives $\frac{3}{5}$ to each incident face by R3b, so we have $\mu'(v) \geq 10 - 4 - 10 \times \frac{3}{5} = 0$.

Subcase 2.6. $d(v) = 11$. Note that v gives either $\frac{1}{3}$ or $\frac{2}{3}$ to each incident face by R4. If v gives $\frac{1}{3}$ at least once, then we have $\mu'(v) \geq 11 - 4 - \frac{1}{3} - 10 \times \frac{2}{3} = 0$. So suppose that v gives $\frac{2}{3}$ to each incident face, which means due to R4 that each incident face is a 4-face incident with two 3-vertices and one vertex of degree 4 or 5. Hence, our v from now on is poor.

If v has a 5-neighbor, then $\mu'(v) \geq 11 - 4 - 12 \times \frac{2}{3} + 2 \times \frac{1}{6} = 0$ by R6, so suppose otherwise in what follows. Thus v satisfies (B1) in the definition of a bad 11-vertex.

If v belongs to at least two 5-sharp faces, then v is not bad due to (B4), hence $\mu'(v) \geq 11 - 4 - 11 \times \frac{2}{3} + 2 \times \frac{1}{6} = 0$ by R12b.

Next suppose v is incident with precisely one 5-sharp face. If v is bad, then $\mu'(v) \geq 11 - 7 - 11 \times \frac{2}{3} + \frac{1}{3} = 0$ by R12a. Otherwise, v must violate at least one of the properties (B2)–(B4), which implies that v receives $\frac{1}{6}$ by R12b and at least $\frac{1}{6}$ by one of the rules R7–R11 due to Remark 9, so again $\mu'(v) \geq 0$.

Finally, suppose v does not belong to 5-sharp faces. Due to Remark 7, there is a 4-sharp face $f_1 = vv_1w_1v_2$ with $d(v_1) = d(v_2) = 3$ and $d(w_1) = 4$. For further notation, we return to Figure 2.

It is not hard to check that each lying 5^+-face $f_1^* = \cdots x_1w_1v_1w_{11}$ brings v at least $\frac{1}{6}$ by R8–R11 since $d(w_{11}) \leq 4$ due to the absence of 5-sharp faces. This is obvious if f_1^* is strong, so suppose f_1 is weak and hence $d(f_1^*) = 5$. If $d(w_{11}) = 4$, then f_1^* is helpful and hence participates in R9. Otherwise, f_1^* gives v at least $\frac{1}{6}$ by R10 or R11.

Therefore, from now on we can assume that f_1^* is the only one lying 5^+-face at v. Since $d(f_2^*) = 4$, we have $d(x_2) \geq 21 - 3 - 4 - 4 = 10$, so R8a is not applicable to w_1. This means that the violation of (B2) by v implies $\mu'(v) \geq 0$, and we have nothing to prove. So from now on we can assume that (B2) is satisfied; in particular, f_1^* is a weak 5-face.

Note that f_1^* is either helpful or a transmitter since it is incident with a 4-vertex w_1 and satisfies $w(f_1^*) \geq 18$. Recall that still $d(w_{11}) \leq 4$ since v is poor.

If $d(w_{11}) = 4$, then f_1^* is helpful. This implies by Remark 7 that there is a 4-sharp face whose summit w_i differs from w_1 and w_{11}. However, then due to Remark 9 there is a lying 5^+-face at v other than f_1^*, which contradicts the assumptions made.

So suppose $d(w_{11}) = 3$. If f_1^* is helpful or transmitter-1, then v receives $\frac{1}{3}$ by R9a or R10, respectively, and we are done. So we can assume that (B3) is also satisfied by v. This means that f_1^* is a transmitter-2 for v, so (B4) is also true for v. Thus v is bad, and it remains to observe that v receives $\frac{1}{3}$ by R11a, which yields $\mu'(v) \geq 0$, as desired.

Subcase 2.7. $d(v) \geq 12$. Since v gives at most $\frac{2}{3}$ to each incident face by R5, we have $\mu'(v) \geq d(v) - 4 - d(v) \times \frac{2}{3} = \frac{d(v)-12}{3} \geq 0$.

Thus we have proved that $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and thus completes the proof of Theorem 6.

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