# ON THE WEIGHT OF MINOR FACES IN TRIANGLE-FREE 3-POLYTOPES 

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#### Abstract

The weight $w(f)$ of a face $f$ in a 3 -polytope is the degree-sum of vertices incident with $f$. It follows from Lebesgue's results of 1940 that every triangle-free 3 -polytope without 4 -faces incident with at least three 3 -vertices has a 4 -face with $w \leq 21$ or a 5 -face with $w \leq 17$. Here, the bound 17 is sharp, but it was still unknown whether 21 is sharp.

The purpose of this paper is to improve this 21 to 20 , which is best possible.


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## 1. Introduction

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [34], the 3-polytopes are in one-to-one correspondence with the 3connected planar graphs.

The degree $d(x)$ of a vertex or face $x$ in a 3 -polytope $M$ is the number of incident edges. A $k$-vertex and $k$-face is one of degree $k$, a $k^{+}$-vertex has degree at least $k$, and so on. The weight $w(f)$ of a face $f$ in $M$ is the degree-sum of vertices incident with $f$. By $w(M)$, or simply $w$, we denote the minimum weight of $5^{-}$-faces in $M$. By $\Delta$ and $\delta$ denote the maximum and minimum vertex degree of $M$, respectively.

We say that $f$ is a face of type $\left(k_{1}, k_{2}, \ldots\right)$ or simply a $\left(k_{1}, k_{2}, \ldots\right)$-face if the set of degrees of the vertices incident with $f$ is majorized by the vector $\left(k_{1}, k_{2}, \ldots\right)$. A 4 -face of the type $(3,3,3, \infty)$ is pyramidal. Note that in the $(3,3,3, n)$-Archimedean solid each face $f$ is pyramidal and satisfies $w(f)=n+9$.

We now recall some results on the structure of $5^{-}$-faces in 3 -polytopes. Back in 1940, Lebesgue [26] gave an approximate description of types of $5^{-}$-faces in normal plane maps.

Theorem 1 (Lebesgue [26]). Every normal plane map has a 5-face of one of the following types:

$$
\begin{gathered}
(3,6, \infty),(3,7,41),(3,8,23),(3,9,17),(3,10,14),(3,11,13) \\
(4,4, \infty),(4,5,19),(4,6,11),(4,7,9),(5,5,9),(5,6,7) \\
(3,3,3, \infty),(3,3,4,11),(3,3,5,7),(3,4,4,5),(3,3,3,3,5)
\end{gathered}
$$

The classical Theorem 1, along with other ideas in Lebesgue [26], has a lot of applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in $[10,31,33]$ ).

Some parameters of Lebesgue's Theorem were improved for narrow classes of plane graphs. In 1963, Kotzig [24] proved that every plane triangulation with $\delta=5$ satisfies $w \leq 18$ and conjectured that $w \leq 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 2 (Borodin [2]). Every normal plane map with $\delta=5$ has a (5,5,7)-face or a $(5,6,6)$-face, where all parameters are tight.

Theorem 2 also confirmed a conjecture of Grünbaum [19] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5 -connected planar graph is at most 11 , which is tight (a bound of 13 was earlier obtained by Plummer [32]).

We note that a 3 -polytope with $(4,4, \infty)$-faces can have unbounded $w$, as follows from the $n$-pyramid, double $n$-pyramid, and a related construction in which
every 3 -face is incident with a 3 -vertex, 4 -vertex, and $n$-vertex. As mentioned above, the same is true concerning $(3,3,3, \infty)$-faces.

For plane triangulations without 4-vertices, Kotzig [25] proved $w \leq 39$, and Borodin [4], confirming Kotzig's conjecture in [25], proved $w \leq 29$, which is best possible due to the dual of the twice-truncated dodecahedron. Borodin [5] further showed that each triangulated 3 -polytope without $(4,4, \infty)$-faces satisfies $w \leq 29$, and that for triangulations without adjacent 4 -vertices there is a sharp bound $w \leq 37$.

For an arbitrary 3 -polytope, Theorem 1 yields $w \leq \max \{51, \Delta+9\}$. Horňák and Jendrol' [20] strengthened this as follows: if there are neither ( $4,4, \infty$ )-faces nor $(3,3,3, \infty)$-faces, then $w \leq 47$. Borodin and Woodall [7] proved that forbidding $(3,3,3, \infty)$-faces implies $w \leq \max \{29, \Delta+8\}$.

For quadrangulated 3-polytopes, Avgustinovich and Borodin [1] improved the description of 4 -faces implied by Lebesgue's Theorem as follows: $(3,3,3, \infty)$, $(3,3,4,10),(3,3,5,7),(3,4,4,5)$.

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in a recent survey by Jendrol' and Voss [22], and also in [3, 6, $8,17,18,21,23,27-30,35]$.

In 2002, Borodin [9] strengthened Lebesgue's Theorem 1 as follows (the entries marked by an asterisk are proved in [9] to be best possible).

Theorem 3 (Borodin [9]). Every normal plane map has a $5^{-}$-face of one of the following types:

$$
\begin{gathered}
\left(3,6, \infty^{*}\right),\left(3,8^{*}, 22\right),\left(3,9^{*}, 15\right),\left(3,10^{*}, 13\right),\left(3,11^{*}, 12\right) \\
\left(4,4, \infty^{*}\right),\left(4,5^{*}, 17\right),\left(4,6^{*}, 11\right),\left(4,7^{*}, 8\right),\left(5,5^{*}, 8\right),\left(5,6,6^{*}\right) \\
\left(3,3,3, \infty^{*}\right),\left(3,3,4^{*}, 11\right),\left(3,3,5^{*}, 7\right),\left(3,4,4,5^{*}\right),\left(3,3,3,3,5^{*}\right)
\end{gathered}
$$

Recently, precise descriptions of the structure of faces were obtained for 3polytopes with $\delta \geq 4$ and for triangulated 3-polytopes.

Theorem 4 (Borodin, Ivanova [11]). Every 3-polytope without 3-vertices has a 3 -face of one of the following types:

$$
(4,4, \infty),(4,5,14),(4,6,10),(4,7,7),(5,5,7),(5,6,6)
$$

where all parameters are sharp.
Theorem 5 (Borodin, Ivanova, Kostochka [12]). Every triangulated 3-polytope has a face of one of the following types:

$$
\begin{gathered}
(3,4,31),(3,5,21),(3,6,20),(3,7,13),(3,8,14),(3,9,12),(3,10,12) \\
(4,4, \infty),(4,5,11),(4,6,10),(4,7,7),(5,6,6),(5,5,7)
\end{gathered}
$$

where all parameters are sharp.

It follows from Lebesgue's Theorem 1 that every triangle-free 3-polytope without pyramidal faces has a 4 -face with $w \leq 21$ or a 5 -face with $w \leq 17$. For a long time, it was not known whether Lebesgue's bound $w \leq 21$ is sharp. The purpose of our paper is to answer this question by proving

Theorem 6. Every triangle-free 3-polytope without pyramidal 4-faces has a 4face of weight at most 20 or a 5-face of weight at most 17, where both bounds 20 and 17 are sharp.

## 2. Proving Theorem 6

To prove the sharpness of the bound 20 , it suffices to insert the configuration shown in Figure 1 into every face of the icosahedron, which provides a trianglefree 3 -polytope without pyramidal 4-faces in which every 4 -face has weight 20 . The sharpness of the bound 17 follows from the ( $3,3,3,3,5$ )-Archimedean solid.


Figure 1. A fragment of an extremal construction derived from the icosahedron.

Now suppose $M$ is a counter-example to the upper bounds in Theorem 6. Euler's formula $|V|-|E|+|F|=2$ for $M$ implies

$$
\begin{equation*}
\sum_{x \in V \cup F}(d(x)-4)=-8 \tag{1}
\end{equation*}
$$

where $V, E$, and $F$ are the sets of vertices, edges, and faces of $M$.
We assign an initial charge $\mu(x)=d(x)-4$ to every $x \in V \cup F$; so only the 3 -vertices in $V$ have a negative charge. Using the properties of $M$ as a counterexample, we will define a local redistribution of charges, preserving their sum, such that the new charge $\mu^{\prime}(x)$ is nonnegative whenever $x \in V \cup F$. This will contradict the fact that the sum of new charges, according to (1), is -8 .

### 2.1. Basic properties of the counterexample $M$

We need a few definitions and comments.
A face $f$ is strong if either $d(f) \geq 6$, or $d(f)=5$ and $f$ is incident with a $6^{+}$-vertex, or else $d(f)=4$ and $f$ is incident with at least two $6^{+}$-vertices. Otherwise, $f$ is weak.

Clearly, a weak 5 -face $f$ can be incident with at most three 3 -vertices, since $w(f)>4 \times 3+5$. A weak 5 -face is helpful if it is incident with at most two 3 -vertices. A (3, 3, 3, 4, 5)-face is a transmitter. A transmitter is a transmitter-1 if its 4 -vertex is adjacent to its 5 -vertex; otherwise, it is a transmitter-2. We will not be concerned about weak 5 -faces incident with three 3 -vertices and two 5 -vertices.

A weak 4-face $f=v_{1} \cdots v_{4}$ is sharp if $d\left(v_{1}\right)=d\left(v_{3}\right)=3,4 \leq d\left(v_{2}\right) \leq 5$, and $d\left(v_{4}\right)=11$; here, $v_{2}$ is the summit of $f$. Depending on the degree of the summit, we have 4 -sharp and 5 -sharp faces. A weak 4 -face is special if it is incident with two 3 -vertices, a 5 -vertex, and 10 -vertex.

We also need a few more specialized definitions and remarks. An 11-vertex is poor if it is completely surrounded by $(3,3,5,11)$-faces. Note that a poor vertex may be incident with $(3,3,4,11)$-faces but not with ( $3,3,3,11$ )-faces. We now explore the structural properties of poor vertices in some detail.

Remark 7. Every poor 11-vertex is incident with an odd number of sharp 4faces due to the alteration of 3 -neighbors with those of degree four of five. In particular, each poor vertex belongs to at least one sharp face.

Now we look what happens around a summit 4 -vertex. Suppose a poor 11vertex $v$ has a 4 -sharp face $f_{1}=v v_{1} w_{1} v_{2}$ with the summit $w_{1}$; so $d\left(w_{1}\right)=4$. Recall that $d\left(v_{1}\right)=d\left(v_{2}\right)=3$ by definition. Furthermore, there are ( $3,3,5,11$ )faces $f_{11}=v v_{11} w_{11} v_{1}$ and $f_{2}=v v_{2} w_{2} v_{3}$ (see Figure 2).

Here, we have lying faces $f_{1}^{*}=\cdots w_{11} v_{1} w_{1} x_{1}$ and $f_{2}^{*}=\cdots w_{2} v_{2} w_{1} x_{2}$, and also a standing face $\overline{f_{1}}=\cdots x_{1} w_{1} x_{2}$, which lies opposite to the sharp face $f_{1}$ with respect to the summit 4 -vertex $w_{1}$.

Remark 8. A standing face can well be sharp, but no lying face is sharp. Indeed, for $f_{2}^{*}$ in Figure 2 to be sharp, we should have $d\left(w_{2}\right)=11$, whereas actually $d\left(w_{2}\right) \leq 5$ since $v$ is poor.

Remark 9. If a 4 -vertex $w_{1}$ is the summit of a sharp 4 -face at a poor 11 -vertex $v$ such that the standing face $\overline{f_{1}}$ at $w_{1}$ is weak, then at least one of the lying faces at $w_{1}$ is a $5^{+}$-face. Indeed, if both lying faces at $w_{1}$ are 4 -faces (again, we follow the notation in Figure 2), then it follows from $d\left(w_{11}\right) \leq 5$ and $d\left(w_{2}\right) \leq 5$ that $d\left(x_{1}\right) \geq 21-5-3-4=9$ and $d\left(x_{2}\right) \geq 9$, so $\overline{f_{1}}=\cdots x_{1} w_{1} x_{2}$ is strong; a contradiction.


Figure 2. Objects related to a 4 -sharp face $f_{1}$.

We say that a weak 5 -face $f_{2}^{*}=x_{2} w_{1} v_{2} w_{2} z$ lying at a poor 11 -vertex $v$ (we follow Figure 2, so $d\left(w_{1}\right)=4$ and $d\left(v_{2}\right)=3$ ) sees $v$ through the 3 -vertex $v_{2}$. Note that a weak 5 -face $f$ can see at most two poor vertices since the boundary of $f$ has either at most two 3 -vertices or at most one 4 -vertex, for otherwise $d(f)=3 \times 3+2 \times 4=17$, which is impossible. Moreover, any transmitter- 1 can see at most one poor vertex, since it has only one 4 -vertex adjacent to 3 -vertex along the boundary, which is necessary for a poor 11-vertex to be seen through a 3 -vertex.

Now we are ready to introduce the key notion in our proof. A poor 11-vertex $v$ is bad if it satisfies the following properties:
(B1) $v$ has no 5 -neighbors;
(B2) $v$ has neither standing nor lying strong faces;
(B3) $v$ has neither helpful nor transmitter-1 lying 5 -faces;
(B4) $v$ has precisely one face that is either 5 -sharp or lying transmitter- 2 .
Remark 10. A transmitter- $2 f_{2}^{*}=x_{2} w_{1} v_{2} w_{2} z$ can see at most one bad 11-vertex $v$, which happens through the 3 -vertex $v_{2}$ when $d\left(w_{1}\right)=4$ and $d\left(w_{2}\right)=3$ (see Figure 2 again). Indeed, the possibility $d\left(w_{2}\right)=5$ contradicts (B4) for $v$.

### 2.2. Rules of discharging

We use the following rules of discharging (see Figure 3). Some notation in the statements of our rules is borrowed from Figure 2.
R1. Each face gives $\frac{1}{3}$ to every incident 3 -vertex.

R2. Each strong face gives $\frac{1}{3}$ to each incident vertex $v$ with $4 \leq d(v) \leq 5$.
R3. Each vertex $v$ gives to each incident face:
(a) $\frac{1}{3}$, if $6 \leq d(v) \leq 9$, or
(b) $\frac{3}{5}$, if $d(v)=10$.

R4. Each 11-vertex gives each incident face f:
(a) $\frac{2}{3}$, if $d(f)=4$ and $f$ is incident with two 3 -vertices and a vertex of degree 4 or 5 , or
(b) $\frac{1}{3}$, otherwise.

R5. Each $12^{+}$-vertex gives $\frac{2}{3}$ to each incident face.
R6. If a 5-vertex $v$ is incident with 4-faces $f_{1}=v x_{1} y_{1} z$ and $f_{2}=v x_{2} y_{2} z$, where $d(z)=11$ and $d\left(x_{1}\right)=d\left(x_{2}\right)=d\left(y_{1}\right)=d\left(y_{2}\right)=3$, then $v$ gives $\frac{1}{6}$ to $z$ through each of $f_{1}$ and $f_{2}$.

R7. If a 4-vertex $w_{1}$ is a summit of a sharp face $f_{1}=v v_{1} w_{1} v_{2}$ and the standing face $\overline{f_{1}}$ at $w_{1}$ is strong, then $w_{1}$ transfers the $\frac{1}{3}$ received from $\overline{f_{1}}$ to the poor 11-vertex $v$ through $f_{1}$.

R8. Suppose a 4-vertex $w_{1}$ is a summit of a sharp face $f_{1}=v v_{1} w_{1} v_{2}$ and the lying face $f_{1}^{*}$ at $w_{1}$ is strong. Then $w_{1}$ transfers the $\frac{1}{3}$ received from $f_{1}^{*}$ evenly through the incident sharp faces. As a result, the poor 11-vertex $v$ receives from $f_{1}^{*}$ via $w_{1}$ :
(a) $\frac{1}{6}$, if $w_{1}$ is a summit for two sharp faces, or
(b) $\frac{1}{3}$, otherwise.

R9. Suppose a 4-vertex $w_{1}$ is a summit of a sharp face $f_{1}=v v_{1} w_{1} v_{2}$ and the lying face $f_{1}^{*}=y_{1} x_{1} w_{1} v_{1} w_{11}$ at $w_{1}$ is helpful. Then $f_{1}^{*}$ gives the poor 11-vertex $v$ :
(a) $\frac{1}{3}$, if $d\left(w_{11}\right)=3$, or
(b) $\frac{1}{6}$, otherwise.

R10. Suppose a 4-vertex $w_{1}$ is a summit of a sharp face $f_{1}=v v_{1} w_{1} v_{2}$ and the lying face $f_{1}^{*}=y_{1} x_{1} w_{1} v_{1} w_{11}$ at $w_{1}$ is a transmitter -1 , which means that $d\left(x_{1}\right)=5$ and $d\left(y_{1}\right)=d\left(w_{11}\right)=3$. Then $x_{1}$ gives $\frac{1}{3}$ through $f_{1}^{*}$ to the poor 11-vertex $v$.
R11. Suppose a 4-vertex $w_{1}$ is a summit of a sharp face $f_{1}=v v_{1} w_{1} v_{2}$ and the lying face $f_{1}^{*}=y_{1} x_{1} w_{1} v_{1} w_{11}$ at $w_{1}$ is a transmitter- 2 , which means that $d\left(y_{1}\right)=5$ and $d\left(x_{1}\right)=d\left(w_{11}\right)=3$. Then $y_{1}$ gives through $f_{1}^{*}$ to the poor 11-vertex $v$ :
(a) $\frac{1}{3}$, if $v$ is bad, or
(b) $\frac{1}{6}$, otherwise.

R12. If a 5-vertex $w_{1}$ is a summit of a sharp face $f_{1}=v v_{1} w_{1} v_{2}$, then $w_{1}$ gives to the poor 11-vertex $v$ :
(a) $\frac{1}{3}$, if $v$ is bad, or
(b) $\frac{1}{6}$, otherwise.

R13. Each 5-vertex gives $\frac{1}{15}$ to each incident special 4-face.


Figure 3. Rules of discharging.

### 2.3. Proving $\mu^{\prime}(x) \geq 0$ whenever $x \in V \cup F$

Case 1. $f \in F$.
Subcase 1.1. $d(f) \geq 6$. Here, $f$ is strong and gives $\frac{1}{3}$ to each incident $5^{-}$vertex by R 1 and R 2 , so $\mu^{\prime}(f) \geq d(f)-4-d(f) \times \frac{1}{3}=\frac{2(d(f)-6)}{3} \geq 0$.

Subcase 1.2. Suppose $d(f)=5$. If $f$ is strong, then $f$ receives at least $\frac{1}{3}$ by R3-R5 and gives $\frac{1}{3}$ to each incident $5^{-}$-vertex by R1 and R2, which results in $\mu^{\prime}(f) \geq 5-4+\frac{1}{3}-4 \times \frac{1}{3}=0$.

Now suppose $f$ is weak. We note that $f$ is incident with at most three 3vertices since $w(f) \geq 18>5+4 \times 3$ by assumption. If $f$ is incident with precisely three 3 -vertices, then $f$ is either incident with two 5 -vertices or is a transmitter (that is, has also a 4 -vertex and a 5 -vertex in its boundary). In both cases, $f$ participates only in R1, so we have $\mu^{\prime}(f) \geq 5-4-3 \times \frac{1}{3}=0$.

Finally, if $f$ is incident with at most two 3 -vertices, then each of them receives $\frac{1}{3}$ from $f$ by R1. Furthermore, such an $f$, called helpful, can participate only in R9, by giving $\frac{1}{3}$ or $\frac{1}{6}$ to each poor 11-vertex seen by $f$. More specifically, the donation of $\frac{1}{3}$ occurs only in R9a, in the case when the two 3 -vertices in the boundary of $f$ are adjacent, which easily implies that only one of them sees a poor vertex. This results in $\mu^{\prime}(f) \geq 5-4-3 \times \frac{1}{3}=0$. Otherwise, R9b works, and we have $\mu^{\prime}(f) \geq 5-4-2 \times \frac{1}{3}-2 \times \frac{1}{6}=0$.

Subcase 1.3. $d(f)=4$. Recall that $f$ is incident with at most two 3 -vertices due to the absence of pyramidal faces. First suppose that $f$ is incident with precisely two 3 -vertices.

If $f$ is special, then it receives $\frac{1}{15}$ from a 5 -vertex by R13 and $\frac{3}{5}$ from a $10-$ vertex by R3b, so $\mu^{\prime}(f) \geq 4-4-2 \times \frac{1}{3}+\frac{1}{15}+\frac{3}{5}=0$ in view of R1. From now on we assume that $f$ is not special.

If $f$ is strong, that is incident with two $6^{+}$-vertices, then $f$ receives at least $\frac{1}{3}+\frac{1}{3}$ by R3-R5, so $\mu^{\prime}(f) \geq 0$. Otherwise, $f$ is incident with an $11^{+}$-vertex and a vertex of degree 4 or 5 , in which case R4a or R5 is applicable, and we again have $\mu^{\prime}(f) \geq 0$.

Now suppose $f$ is incident with at most one 3 -vertex. Recall that $f$ is incident with at least one $6^{+}$-vertex since $w(f) \geq 21>4 \times 5$ by assumption. If $f$ is strong, then it can afford giving $\frac{1}{3}$ to each of at most two incident $5^{-}$-vertices by R1 and R2 as R3-R5 also apply. Otherwise, $f$ gives $\frac{1}{3}$ at most once by R1, so $\mu^{\prime}(f) \geq 4-4+\frac{1}{3}-\frac{1}{3}=0$.

Case 2. $v \in V$.
Subcase 2.1. $d(v)=3$. Since $v$ receives $\frac{1}{3}$ from each incident face by R1, we have $\mu^{\prime}(v)=3-4+3 \times \frac{1}{3}=0$.

Subcase 2.2. $d(v)=4$. We note that $v$ receives $\frac{1}{3}$ from each incident strong face by R2 and can transfer each such a donation either in full or as $\frac{1}{6}+\frac{1}{6}$ to
poor 11-vertices by R7 and R8. Thus $\mu^{\prime}(v) \geq \mu(v)=4-4=0$.
Subcase 2.3. $d(v)=5$. Examining our rules, we see that $v$ can either give charge away to poor 11-vertices (in particular, to bad 11-vertices) by R6 and R10-R12, where the donation through each incident face is either $\frac{1}{3}$ or $\frac{1}{6}$, or can give $\frac{1}{15}$ to a 10 -vertex by R13. Furthermore, the donation of $\frac{1}{3}$ along an edge $e$ may be looked at as two donations of $\frac{1}{6}$ through two faces incident with $e$. As a result of such averaging, $v$ gives through each incident 4 -face either $\frac{1}{3}$ if R8 is applicable or at most $\frac{1}{6}$ otherwise.

If $v$ is incident with a strong face $f$, then $v$, in turn, receives $\frac{1}{3}$ from $f$ by R2, which results in $\mu^{\prime}(v) \geq 5-4+\frac{1}{3}-4 \times \frac{1}{3}=0$. So we assume from now on that $v$ is completely surrounded by weak faces.

If $v$ gives $\frac{1}{3}$ at most once, then $\mu^{\prime}(v) \geq 5-4-\frac{1}{3}-4 \times \frac{1}{6}=0$. We will prove that $v$ actually gives away through its five faces at most 1 in total, which implies $\mu^{\prime}(v) \geq 0$. To reach this goal, we need three lemmas.

Let $v_{1} \ldots v_{5}$ be the neighbors of $v$ in a cyclic order, and let $f_{i}=\cdots v_{i} v v_{i+1}$, for $1 \leq i \leq 5$ (addition modulo 5 ).

Lemma 11. If a 5-vertex $v$ gives $\frac{1}{3}$ through a transmitter- 1 face $f_{1}$ to a poor 11vertex by R10, then $v$ gives nothing through the face $f_{2}$ having a common 4-vertex with $f_{1}$.

Proof. Suppose $f_{1}=v v_{1} x y v_{3}$, where $d\left(v_{1}\right)=4$ and $d(x)=d(y)=d\left(v_{3}\right)=3$, and there is a poor 11-vertex $z$ that receives $\frac{1}{3}$ from $v$ by R10. Since $z$ is poor, there is a 4 -face $z x v_{1} w$ with $d(w)=3$ (see Figure 4 ).

The face $f_{2}=v_{1} v v_{2} \cdots$ can conduct something from $v$ by our rules only if $f_{2}=v_{1} v v_{2} y^{\prime} x^{\prime}$, where $d\left(x^{\prime}\right)=d\left(y^{\prime}\right)=d\left(v_{2}\right)=3$, and there is a poor 11-vertex $z^{\prime}$.

However, this implies a strong face $\cdots z w z^{\prime}$ at a poor 11-vertex $z$, a contradiction.

Lemma 12. If a 5-vertex $v$ gives $\frac{1}{3}$ through a transmitter-2 face $f_{1}$ to a bad 11-vertex by R11a, then $v$ gives nothing through $f_{2}$ or $f_{5}$.

Proof. Suppose $f_{1}=v v_{1} x y v_{2}$, where $d(x)=4$ and $d\left(v_{1}\right)=d(y)=d\left(v_{2}\right)=3$, and there is a bad 11-vertex $z$ that receives $\frac{1}{3}$ from $v$ by R11a. Since $z$ is bad, there is a 4-face $z y v_{2} x^{\prime}$ with $d(y)=d\left(v_{2}\right)=3$, which implies by (B1) that $d\left(x^{\prime}\right)=4$ (see Figure 5).

The face $f_{2}=\cdots v_{2} v v_{3}$ can conduct a positive charge by our rules from $v$ only if $f_{2}$ is a transmitter-2. This means that we are done unless $f_{2}=v_{2} v v_{3} y^{\prime} x^{\prime}$ with $d\left(y^{\prime}\right)=d\left(v_{3}\right)=3$, and there is a poor 11 -vertex $z^{\prime}$ seen by $v$ via $f_{2}$ and $y^{\prime}$.

This implies that there is a 4-face $w x^{\prime} y^{\prime} z^{\prime}$ at $z^{\prime}$ with $d(w)=3$. Now looking again at the bad 11-vertex $z$, we see that there is a 4 -face $z x^{\prime} w u$ with $d(u)=3$. This gives rise to a 4 -face $t u w z^{\prime}$ at a poor vertex $z^{\prime}$, where $4 \leq d(t) \leq 5$ since there are no pyramidal 4 -faces in $M$. However, a bad vertex $z$ cannot have a 5 -sharp


Figure 4. To Lemma 11.


Figure 5. To Lemma 12.
face $z u t s$ in addition to the transmitter-2 face $f_{1}$ by $(\mathrm{B} 4)$, and so $d(t)=4$. We note that $d(s)=3$ since $z$ is bad.

Now since $z^{\prime}$ is poor, we have a 4 -face $z^{\prime} t u^{\prime} w^{\prime}$ with $u^{\prime} \neq u$ such that $d\left(u^{\prime}\right)=3$ (and $d\left(w^{\prime}\right)=3$, which is not important for us).

Finally, we consider the face $f^{*}=\cdots r s t u^{\prime}$ lying at $z$. We note that $d(r) \leq 4$, for otherwise a bad vertex $z$ would have a 5 -sharp face, which is impossible by (B4) again. It is also impossible for $f^{*}$ to be strong by (B2) or helpful by (B3) since $z$ is bad.

Thus $d\left(f^{*}\right)=5$, and $f^{*}$ is incident with three 3 -vertices and a 4 -vertex, so the fifth incident vertex must have degree 5 since $w\left(f^{*}\right) \geq 18$ in our counterexample $M$. This implies that $d(r)=3$. We note that $f^{*}$ is not transmitter -1 since its 4 -vertex is not adjacent to its 5 -vertex. So $f^{*}$ is a transmitter-2 face, which contradicts (B4) applied to $z$.

Lemma 13. If a 5 -vertex $v$ gives $\frac{1}{3}$ through a 5 -sharp face $f_{1}$ by R12a, then the total donation of $v$ through $f_{2}$ and $f_{3}$ is at most $\frac{1}{3}$.

Proof. Suppose $f_{1}=v v_{1} z v_{2}$, where $d\left(v_{1}\right)=d\left(v_{2}\right)=3$, and $z$ is a bad 11-vertex that receives $\frac{1}{3}$ from $v$ by R12a (see Figure 6).


Figure 6. To Lemma 13.

We note that $f_{2}$ can conduct a positive charge from $v$ by our rules only when $d\left(v_{3}\right) \in\{3,4,10,11\}$. If $10 \leq d\left(v_{3}\right) \leq 11$, then $v$ gives at most $\frac{1}{6}+\frac{1}{6}$ through $f_{2}$ and $f_{3}$ by R 6 and R13, and we are done. If $d\left(v_{3}\right)=4$, then the only possibility for $f_{2}$ and $f_{3}$ to conduct a positive charge from $v$ is to be transmitters- 1 , which can happen with at most one of them due to Lemma 11. So we can assume that $d\left(v_{3}\right)=3$.

Since $z$ is bad, there is a 4 -face $z v_{2} x u$ with $d(x) \leq 4$. If $d(x)=3$, then the only way for $f_{2}$, being incident with three 3 -vertices and a 5 -vertex, to conduct a positive charge from $v$ is to be a transmitter-2 for a poor 11 -vertex. However,
the 3 -vertex $x$, which is the only "suspicious" 3 -vertex lying between a 4 -vertex and a 3 -vertex $v_{2}$ in the boundary of $f_{2}$, in fact cannot see an 11-vertex since $x$ is in a common 4 -face with the bad vertex $z$. Therefore, R11 is not applicable to $f_{2}$, and so we can assume from now on that $d(x)=4$.

Now we are done unless $f_{2}=v v_{2} x y v_{3}$, and $f_{2}$ sees a poor 11-vertex $z^{\prime}$ through the 3 -vertex $y$. In this case, we have a 4 -face $z^{\prime} y x w$ with $d(w)=3$.

On the other hand, the bad 11-vertex $z$ has a 4 -face $z v_{2} x u$ with $d(u)=3$ and another 4 -face zuts with $t \neq x$. We note that $d(t) \leq 4$, since $z$ cannot belong to two 5 -sharp faces according to (B4). Thus there is a $5^{+}$-face $f^{*}=\cdots$ tuxw lying at $z$.

We see that $f^{*}$ cannot be strong due to the property (B2) in the definition of $z$. If $d(t)=4$, then $f^{*}$ can have only two 3 -vertices, $u$ and $w$, on its boundary, as $w\left(f^{*}\right) \geq 18$. Thus $f^{*}$ is helpful for $z$, which violates (B3), and so we can assume that $d(t)=3$. Hence $f^{*}$ is a transmitter-2 for $z$, but this contradicts the property (B4) for $z$.

We are now ready to complete the proof of Subcase 2.3. If $v$ gives $\frac{1}{3}$ through $f_{3}$ by R12a, then it gives at most $\frac{1}{3}$ through $f_{1}$ and $f_{2}$ together due to Lemma 13. By symmetry, Lemma 13 is applied also to $f_{4}$ and $f_{5}$. This implies $\mu^{\prime}(v) \geq$ $5-4-3 \times \frac{1}{3}=0$.

It remains to assume that R12a is not applied to $v$. If $v$ gives $\frac{1}{3}$ by R10 and R11a at most once, then we have $\mu^{\prime}(v) \geq 5-4-\frac{1}{3}-4 \times \frac{1}{6}=0$. If these rules are applied to two consecutive faces, say $f_{2}$ and $f_{3}$, then $f_{1}$ and $f_{4}$ conduct nothing from $v$ due to Lemmas 11 and 12 , which yields $\mu^{\prime}(v) \geq 5-43 \times \frac{1}{3}=0$.

Otherwise, each of at most two faces takes $\frac{1}{3}$ from $v$, and there is a face taking nothing from $v$ by the two lemmas, so we have $\mu^{\prime}(v) \geq 5-4-2 \times \frac{1}{3}-2 \times \frac{1}{6}=0$, as desired.

Subcase 2.4. $6 \leq d(v) \leq 9$. Now $v$ gives $\frac{1}{3}$ to each incident face according to R3a and does not participate in the other rules, whence $\mu^{\prime}(v) \geq d(v)-4-d(v) \times$ $\frac{1}{3}=\frac{2(d(v)-6)}{3} \geq 0$.

Subcase 2.5. $d(v)=10$. This time, $v$ gives $\frac{3}{5}$ to each incident face by R3b, so we have $\mu^{\prime}(v) \geq 10-4-10 \times \frac{3}{5}=0$.

Subcase 2.6. $d(v)=11$. Note that $v$ gives either $\frac{1}{3}$ or $\frac{2}{3}$ to each incident face by R4. If $v$ gives $\frac{1}{3}$ at least once, then we have $\mu^{\prime}(v) \geq 11-4-\frac{1}{3}-10 \times \frac{2}{3}=0$. So suppose that $v$ gives $\frac{2}{3}$ to each incident face, which means due to R4 that each incident face is a 4 -face incident with two 3 -vertices and one vertex of degree 4 or 5. Hence, our $v$ from now on is poor.

If $v$ has a 5 -neighbor, then $\mu^{\prime}(v) \geq 11-4-12 \times \frac{2}{3}+2 \times \frac{1}{6}=0$ by R 6 , so suppose otherwise in what follows. Thus $v$ satisfies (B1) in the definition of a bad 11-vertex.

If $v$ belongs to at least two 5 -sharp faces, then $v$ is not bad due to (B4), hence $\mu^{\prime}(v) \geq 11-4-11 \times \frac{2}{3}+2 \times \frac{1}{6}=0$ by R12b.

Next suppose $v$ is incident with precisely one 5 -sharp face. If $v$ is bad, then $\mu^{\prime}(v) \geq 11-7-11 \times \frac{2}{3}+\frac{1}{3}=0$ by R12a. Otherwise, $v$ must violate at least one of the properties (B2)-(B4), which implies that $v$ receives $\frac{1}{6}$ by R12b and at least $\frac{1}{6}$ by one of the rules R7-R11 due to Remark 9 , so again $\mu^{\prime}(v) \geq 0$.

Finally, suppose $v$ does not belong to 5 -sharp faces. Due to Remark 7 , there is a 4 -sharp face $f_{1}=v v_{1} w_{1} v_{2}$ with $d\left(v_{1}\right)=d\left(v_{2}\right)=3$ and $d\left(w_{1}\right)=4$. For further notation, we return to Figure 2.

It is not hard to check that each lying $5^{+}$-face $f_{1}^{*}=\cdots x_{1} w_{1} v_{1} w_{11}$ brings $v$ at least $\frac{1}{6}$ by R8-R11 since $d\left(w_{11}\right) \leq 4$ due to the absence of 5 -sharp faces. This is obvious if $f_{1}^{*}$ is strong, so suppose $f_{1}$ is weak and hence $d\left(f_{1}^{*}\right)=5$. If $d\left(w_{11}\right)=4$, then $f_{1}^{*}$ is helpful and hence participates in R9. Otherwise, $f_{1}^{*}$ gives $v$ at least $\frac{1}{6}$ by R10 or R11.

Therefore, from now on we can assume that $f_{1}^{*}$ is the only one lying $5^{+}$-face at $v$. Since $d\left(f_{2}^{*}\right)=4$, we have $d\left(x_{2}\right) \geq 21-3-4-4=10$, so R8a is not applicable to $w_{1}$. This means that the violation of (B2) by $v$ implies $\mu^{\prime}(v) \geq 0$, and we have nothing to prove. So from now on we can assume that (B2) is satisfied; in particular, $f_{1}^{*}$ is a weak 5 -face.

Note that $f_{1}^{*}$ is either helpful or a transmitter since it is incident with a 4 -vertex $w_{1}$ and satisfies $w\left(f_{1}^{*}\right) \geq 18$. Recall that still $d\left(w_{11}\right) \leq 4$ since $v$ is poor.

If $d\left(w_{11}\right)=4$, then $f_{1}^{*}$ is helpful. This implies by Remark 7 that there is a 4-sharp face whose summit $w_{i}$ differs from $w_{1}$ and $w_{11}$. However, then due to Remark 9 there is a lying $5^{+}$-face at $v$ other than $f_{1}^{*}$, which contradicts the assumptions made.

So suppose $d\left(w_{11}\right)=3$. If $f_{1}^{*}$ is helpful or transmitter- 1 , then $v$ receives $\frac{1}{3}$ by R9a or R10, respectively, and we are done. So we can assume that (B3) is also satisfied by $v$. This means that $f_{1}^{*}$ is a transmitter- 2 for $v$, so ( B 4 ) is also true for $v$. Thus $v$ is bad, and it remains to observe that $v$ receives $\frac{1}{3}$ by R11a, which yields $\mu^{\prime}(v) \geq 0$, as desired.

Subcase 2.7. $d(v) \geq 12$. Since $v$ gives at most $\frac{2}{3}$ to each incident face by R5, we have $\mu^{\prime}(v) \geq d(v)-4-d(v) \times \frac{2}{3}=\frac{d(v)-12}{3} \geq 0$.

Thus we have proved that $\mu^{\prime}(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and thus completes the proof of Theorem 6.

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