Discussiones Mathematicae
Graph Theory 36 (2016) 565–575
doi:10.7151/dmgt.1875

ON LONGEST CYCLES IN ESSENTIALLY 4-CONNECTED PLANAR GRAPHS

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Abstract

A planar 3-connected graph G is essentially 4-connected if, for any 3-separator S of G, one component of the graph obtained from G by removing S is a single vertex. Jackson and Wormald proved that an essentially 4-connected planar graph on n vertices contains a cycle C such that $|V(C)| \geq \frac{2n+4}{5}$. For a cubic essentially 4-connected planar graph G, Grünbaum with Malkevitch, and Zhang showed that G has a cycle on at least $\frac{3}{4}n$ vertices. In the present paper the result of Jackson and Wormald is improved. Moreover, new lower bounds on the length of a longest cycle of G are presented if G is an essentially 4-connected planar graph of maximum degree 4 or G is an essentially 4-connected maximal planar graph.

Keywords: planar graph, longest cycle.

2010 Mathematics Subject Classification: 05C10, 05C38.

1. Introduction and Results

We use standard notation and terminology of graph theory ([1]) and consider a finite simple 3-connected planar graph G with vertex set $V(G)$ and edge set $E(G)$. Let $N(x)$, $d(x) = |N(x)|$, and $\Delta(G)$ denote the *neighborhood*, the *degree* of

[*]Supported in part by Research and Development Operating Program for the project "University Science Park Technicom for innovative applications with support of knowledge technologies", code ITMS: 26220220182, co-financed from European funds.

$x \in V(G)$ in G, and the *maximum degree* of G, respectively. A subset $S \subset V(G)$ is an *s-separator* of G if $|S| = s$ and $G - S$ is disconnected. It is well-known that $G - S$ has exactly two components if G is a 3-connected planar graph and S is a 3-separator of G. If S is a 3-separator of a 3-connected planar graph G and one component of $G - S$ is a single vertex, then S is a *trivial* 3*-separator* of G. If G is planar, 3-connected, and each 3-separator S of G is trivial, then G is *essentially* 4*-connected*. In the present paper we are interested in the length of longest cycles of an essentially 4-connected planar graph.

Jackson and Wormald [4] proved that every essentially 4-connected planar graph on n vertices contains a cycle C such that $|V(C)| \geq \frac{2n+4}{5}$. For a cubic essentially 4-connected planar graph G, Grünbaum and Malkevitch [3], and Zhang [8] showed that G has a cycle on at least $\frac{3}{4}n$ vertices. Given a real constant $c > \frac{2}{3}$, Jackson and Wormald [4] presented an infinite family of essentially 4-connected planar graphs G such that G does not contain a cycle on more than $c \cdot n$ vertices. This observation is even true for essentially 4-connected maximal planar graphs. To see this, let G' be a 4-connected maximal planar graph on $n' \geq 6$ vertices embedded into the plane and let G be obtained by inserting a new vertex into each face of G' and connecting it with all three vertices of that face by an edge. Obviously, G is an essentially 4-connected maximal planar graph on $n = n' + (2n' - 4)$ vertices and the $2n' - 4$ vertices in $V(G) \setminus V(G')$ are pairwise independent. Hence each cycle of G contains at most $2n' = \frac{2}{3}(n + 4)$ vertices. At the end of Section 2 we will show that G contains a cycle on exactly $2n' = \frac{2}{3}(n+4)$ vertices.

It is well-known that a 3-connected planar graph on $4 \leq n \leq 10$ vertices is Hamiltonian. It remains open whether a maximal planar (or even an arbitrary planar) essentially 4-connected graph on $n \geq 11$ vertices contains a cycle C such that $|V(C)| \geq \frac{2}{3}(n + 4)$.

Our results are presented in the following Theorem 1.

Theorem 1. *Let G be an essentially* 4*-connected planar graph on $n \geq 11$ vertices and C be a longest cycle of G. Then $|V(C)| \geq \frac{1}{2}(n+4)$, $|V(C)| \geq \frac{3}{5}n$ if $\Delta(G) = 4$, and $|V(C)| \geq \frac{13}{21}(n + 4)$ if G is maximal planar.*

2. Proofs

In the remainder of the paper we assume that G is embedded into the plane. The two open sets into which a cycle C of G partitions the plane are the *interior* $\mathrm{int}(C)$ and the *exterior* $\mathrm{ext}(C)$ of C. Furthermore, let B be a component of $G - V(C)$. A vertex $x \in V(C)$ is a *touch vertex* of B if x is adjacent to a vertex of $V(B)$. Note that B has at least 3 touch vertices, if G is a 3-connected planar graph. In [7], Tutte proved a remarkable and famous result on cycles in 2-connected planar graphs implying that a 4-connected planar graph is Hamiltonian. This result has been extended several times ([5, 6]). We will use the following Lemma 2 of Sanders ([5]) as a version of Tutte's result for 3-connected planar graphs.

Lemma 2. *Every* 3*-connected planar graph* G *with two prescribed edges* a *and* b *contains a cycle* C *through* a *and* b *such that each component of* $G - V(C)$ *has exactly* 3 *touch vertices.*

A cycle C of G is an *outer-independent-*3*-cycle* (OI3*-cycle*), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.

Lemma 3. *Let* G *be an essentially* 4*-connected planar graph, and let* a *and* b *be non-adjacent edges of* G*. If* a *and* b *belong to a common face of* G *or all end vertices of* a *and* b *have degree at least* 4 *in* G*, then* G *contains an* OI3*-cycle* C *through* a *and* b*.*

Proof. By Lemma 2, let C be a cycle of G through a and b such that each component of $G - V(C)$ has exactly three touch vertices. Since a and b are non-adjacent, $|V(C)| \geq 4$. We will show that C is an OI3-cycle of G. Suppose to the contrary that $G - V(C)$ has a component B with at least two inner vertices (w.l.o.g. let $V(B) \subset \text{int}(C)$). Since G is essentially 4-connected and $|V(C)| \geq 4$, the three touch vertices y, z, u of B separate G, hence they form the neighborhood of a vertex x of degree 3.

First assume that $x \in V(C)$ as shown in Figure 1 (C is the fat-drawn cycle).

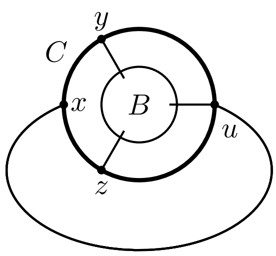

Figure 1

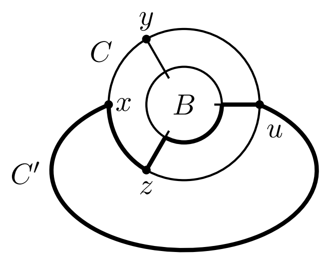

Figure 2

Let α be the face of G containing z, u and at least one vertex of $V(B)$ and let P be the boundary path of α connecting u and z and containing some vertex of $V(B)$. Furthermore, let C' be the (fat-drawn) cycle with $V(C') = V(P) \cup \{x\}$ as shown in Figure 2. It is clear that z and u are the only vertices of C' which possibly have a neighbor in $\text{int}(C') \cap V(G)$. It follows that $\text{int}(C') \cap V(G) = \emptyset$, because otherwise $\{z, u\}$ forms a 2-separator of G contradicting the 3-connectedness of G. Thus z and u are neighbors on C and, by symmetry, y and u are also neighbors on C. Consequently, $|V(C)| = 4$, the edges a and b cannot belong to a common face, and one of them is incident with the vertex x of degree 3 contradicting the choice of a and b.

If $x \notin V(C)$ as shown in Figure 3, then, considering the (fat-drawn) cycles C'' in Figure 4 and C''' in Figure 5, it follows that $\mathrm{int}(C'') \cap V(G) = \emptyset$ and $\mathrm{int}(C''') \cap V(G) = \emptyset$ with similar arguments, hence $|V(C)| = 3$, also a contradiction.

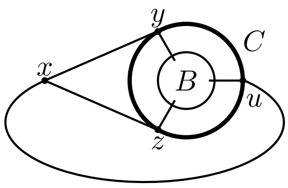

Figure 3

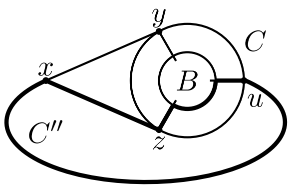

Figure 4

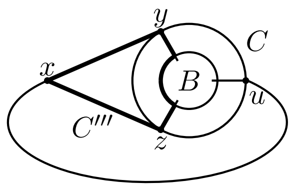

Figure 5

Consequently, C is an OI3-cycle through a and b. ■

Note that a Hamiltonian cycle of a graph is an OI3-cycle. Let $a = yz$ be an edge of an OI3-cycle C of a graph G and assume that y and z have a common neighbor $x \in V(G) \setminus V(C)$. Then let C' be the cycle of G obtained from C by replacing the edge a with the path (y, x, z). In this case, a is an *extendable edge of* C. Note that C' is again an OI3-cycle of G, $|V(C')| = |V(C)| + 1$, and that C' has less extendable edges than C. Obviously, a longest OI3-cycle of G does not contain an extendable edge.

For the proof of Theorem 1 it suffices to show the following lemma.

Lemma 4. *Let G be an essentially 4-connected planar graph on $n \geq 11$ vertices.*

(i) *G contains an* OI3*-cycle.*

(ii) *If C is an* OI3*-cycle of G without extendable edges, then $|V(C)| \geq \frac{1}{2}(n+4)$.*

(iii) *If $\Delta(G) = 4$ and C is an* OI3*-cycle of G, then $|V(C)| \geq \frac{3}{5}n$.*

(iv) *If G is maximal planar and C is a longest* OI3*-cycle of G, then $|V(C)| \geq \frac{13}{21}(n+4)$.*

Proof. If G is an essentially 4-connected plane graph without vertices of degree 3, then G is even 4-connected, hence, G contains a Hamiltonian cycle (Lemma 2). Since every Hamiltonian cycle is an OI3-cycle, Lemma 4(i) is true in this case. If G is not maximal planar, then there exist two non-adjacent edges a and b of G belonging to a common face, hence, by Lemma 3, Lemma 4(i) follows.

Thus, for the proof of Lemma 4(i), it remains to deal with the case that G is maximal planar and contains a vertex of degree 3. Let $a = yz$ be an edge connecting two neighbors y and z of a vertex x of degree 3 in G. In this case we will show that $d(y) \geq 4$, $d(z) \geq 4$, and that there is an edge b being non-adjacent with a, and with both end vertices of degree at least 4. Consequently, the existence of an OI3-cycle in G follows by Lemma 3, and Lemma 4(i) is true also in this case. Let u be the third neighbor of x. The vertices y, z, u form a separating 3-cycle, hence because G is 3-connected, all of them have degree at least 4. Let $w \in N(u) \setminus \{x, y, z\}$ be a fourth neighbor of u. If $d(u) = 4$, then $\{y, z, w\}$ is a 3-separator and both components of $G - \{y, z, w\}$ contain at least two vertices, a contradiction to the essentially 4-connectedness of G. It follows that $d(u) \geq 5$. Let $v \in N(u) \setminus \{x, y, z, w\}$ such that $v \in N(w)$. Since $G \not\simeq K_4$, vertices of degree three are not adjacent in G, thus one of the vertices w and v has degree at least four. We are done with $b = uw$ or $b = uv$, respectively, and Lemma 4(i) is completely proved.

The following Lemma 5 is proved in [2]. For completeness, we present its short proof here.

Lemma 5. *If C is a cycle of a plane graph G on at least 4 vertices such that $\operatorname{int}(C) \cap V(G)$ is an independent set of vertices of degree 3 in G and, for each edge xy of C, x and y do not have a common neighbor in $\operatorname{int}(C) \cap V(G)$, then $|\operatorname{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$.*

Proof. We proceed by induction on $c = |V(C)|$. If $c \leq 5$, then, obviously, $|\operatorname{int}(C) \cap V(G)| = 0$. Now let $c \geq 6$, $d = |\operatorname{int}(C) \cap V(G)| > 0$, and ϕ be an orientation of C. Consider a fixed vertex $x \in \operatorname{int}(C) \cap V(G)$ and let x_1, x_2, and x_3 be the neighbours of x on C met in this order following ϕ. For $i = 1, 2, 3$, let C_i be the cycle obtained by the union of the path on C from x_i to x_{i+1} following ϕ and the two edges xx_i and xx_{i+1} (where $x_4 = x_1$), $c_i = |V(C_i)|$, and $d_i = |\operatorname{int}(C_i) \cap V(G)|$. Obviously, $c > c_i \geq 4$ and for each edge xy of C_i, x and y do not have a common neighbor in $\operatorname{int}(C_i) \cap V(G)$ $(i = 1, 2, 3)$. We have $c_1 + c_2 + c_3 = c + 6$, $d_1 + d_2 + d_3 = d - 1$, and, by induction hypothesis, $d_i \leq \frac{c_i}{2} - 2$ for $i = 1, 2, 3$. This implies $d \leq \frac{c}{2} - 2$. □

To prove Lemma 4(ii), consider an OI3-cycle C of G without an extendable edge. Obviously, $|V(C)| \geq 4$ because $n \geq 4$. Moreover, for each edge xy of C, x and y do not have a common neighbor in $(\operatorname{int}(C) \cup \operatorname{ext}(C)) \cap V(G)$. By Lemma 5, $|\operatorname{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$ and, by symmetry, $|\operatorname{ext}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4)$. Thus $n = |V(C)| + |\operatorname{int}(C) \cap V(G)| + |\operatorname{ext}(C) \cap V(G)| \leq 2|V(C)| - 4$ and Lemma 4(ii) is proved.

For the proof of Lemma 4(iii) consider an arbitrary OI3-cycle C of G. Since $V(G) \setminus V(C)$ is an independent set and $d(x) = 3$ for every $x \in V(G) \setminus V(C)$, $3(n - |V(C)|)$ equals the number e of edges between $V(C)$ and $V(G) \setminus V(C)$. If $y \in V(C)$, then, because $d(y) \leq 4$, y has at most two neighbors in $V(G) \setminus V(C)$. It follows $e \leq 2|V(C)|$ and Lemma 4(iii) is proved.

It remains to prove Lemma 4(iv).

Let C be a longest OI3-cycle of G. By Lemma 4(ii) and $n \geq 11$, we have $|V(C)| \geq 8$. Moreover, let $H = G[V(C)]$ be the graph obtained from G by removing all vertices of degree 3 which do not lie on C. Obviously, H is maximal planar and C is a Hamiltonian cycle of H. A face α of H is an *empty face* of H if α is also a face of G, otherwise α is a *non-empty face* of H. Denote by $\mathcal{F}$ the set of empty faces of H. Note that every face of G has at least two (of three) vertices on C. The three neighbors of a vertex of $V(G) \setminus V(C)$ induce a separating 3-cycle of G creating the boundary of a non-empty face of H.

Lemma 6. *Let $t = |\mathcal{F}|$ be the number of empty faces of H. For a positive real a, the inequalities $|V(C)| \leq at$ and $|V(C)| \geq \frac{a}{3a-1}(n+4)$ are equivalent.*

Proof. Since every face of G which is not an empty face of H has exactly one vertex in $V(G) \setminus V(C)$, calculating the number of faces of G leads to $2n - 4 = t + 3(n - |V(C)|)$. It follows $t = 3|V(C)| - n - 4$ and directly the equivalence of $|V(C)| \leq at$ and $|V(C)| \geq \frac{a}{3a-1}(n+4)$. □

Using Lemma 6, it suffices to prove $|V(C)| \leq \frac{13}{18}t$.

Let H_1 and H_2 be the spanning subgraphs of H consisting of the cycle C and of its chords lying in the interior and in the exterior of C, respectively. Note that $E(H_1) \cap E(H_2) = E(C)$ and H_1 and H_2 are maximal outerplanar graphs.

An empty face φ of H is a *j-face* if exactly j of its three incident edges belong to $E(C)$. Since $|V(C)| \geq 8$, it follows $j \in \{0, 1, 2\}$ for any j-face φ of H. Note that C and a non-empty face of H do not have an edge in common because otherwise such an edge would be an extendable edge of C in G.

Since C does not contain extendable edges, every face of H incident with an edge of C is an empty face. An edge e of C incident with the faces φ and ψ is a *(j,k)-edge* for $1 \leq j, k \leq 2$, if φ is a j-face and ψ is a k-face.

For every edge $e \in E(C)$ we define the weight $w_0(e) = 1$. Obviously, $\sum_{e \in E(C)} w_0(e) = |V(C)|$.

First redistribution of weights

If x, y, and z are the vertices incident with a face φ of H, then we write $\varphi = [x, y, z]$. Let (u, x, y, v) be a subpath of C, xy be a $(2,2)$-edge of C, and $\alpha = [u, x, y]$ and $\sigma = [x, y, v]$ be two adjacent 2-faces of H. Moreover, let β and τ be the faces of H incident with uy and xv and distinct from α and σ, respectively (see Figure 6). The cycle $\widetilde{C}$ obtained from C by replacing the path (u, x, y, v) by the path (u, y, x, v) is also a longest OI3-cycle of G, hence both uy and xv are not extendable edges of $\widetilde{C}$ and therefore β and τ are also empty faces of H.

The weight of all edges of C will be completely redistributed to empty faces of H by the following rules.

Rule R1. A $(2,2)$-edge xy of C (Figure 6) sends weight $\frac{1}{3}$ to both incident 2-faces α and σ and weight $\frac{1}{6}$ to β (through the edge uy) and to τ (through the edge xv).

Rule R2. A $(1,2)$-edge of C sends weight $\frac{2}{3}$ to the incident 1-face and weight $\frac{1}{3}$ to the incident 2-face.

Rule R3. A $(1,1)$-edge of C sends weight $\frac{1}{2}$ to both incident 1-faces.

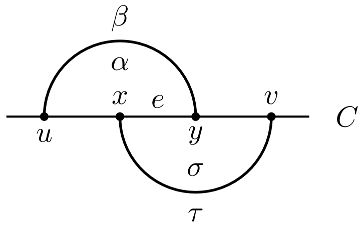

Figure 6

For an empty face φ, let $w_1(\varphi)$ be the total weight obtained by φ (in first redistribution). Obviously, $\sum_{\varphi\in\mathcal{F}} w_1(\varphi) = |V(C)|$.

Every empty face gets weight from (or through) at most two of its three incident edges (otherwise $|V(C)| \le 6$, a contradiction). An empty face φ of H is *good* if $w_1(\varphi) \le \frac{2}{3}$, otherwise it is *bad*.

Every 2-face φ gets weight only by rules R1 or R2, thus $w_1(\varphi) \le \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ and φ is good.

A 0-face φ can get weight only by rule R1. It can get weight $\frac{1}{6}$ from two distinct edges of C through the same incident edge, thus $w_1(\varphi) \le \left(\frac{1}{6}+\frac{1}{6}\right) + \left(\frac{1}{6}+\frac{1}{6}\right) = \frac{2}{3}$ and φ is good.

Every 1-face φ gets weight $\frac{2}{3}$ (by R2) or weight $\frac{1}{2}$ (by R3) from the incident edge lying on C. Furthermore, φ can get weight also through one of the remaining two incident edges (by R1). Thus $w_1(\varphi) \le \frac{2}{3} + \left(\frac{1}{6}+\frac{1}{6}\right) = 1$. Moreover, if φ is bad, then $w_1(\varphi) = \frac{5}{6}$ or $w_1(\varphi) = 1$.

Now we describe all possible neighborhoods of bad faces.

Lemma 7. *Let $\beta \in F(H_i)$, $i \in \{1,2\}$, be a bad face of H and let α and γ be the two faces of H_i adjacent to β, where α is a 2-face of H. The face β is of one of the following four types* (*Figure* 7)*:*

(B1) $w_1(\beta) = \frac{5}{6}$ *and γ is an empty face,*

(B2) $w_1(\beta) = 1$ *and γ is an empty 0-face,*

(B3) $w_1(\beta) = 1$ *and* $w_1(\gamma) = \frac{1}{2}$,

(B4) *there is a 2-face σ of H_{3-i} adjacent* (*in H*) *to α, β, and τ, where τ is an empty 0-face of H.*

Proof. If $\beta \in F(H_i)$, $i \in \{1,2\}$, is a bad face of H, then there is a 2-face α of H_i adjacent to β. Let γ ($\gamma \ne \alpha$) be the second face of H_i adjacent to β (Figure 8).

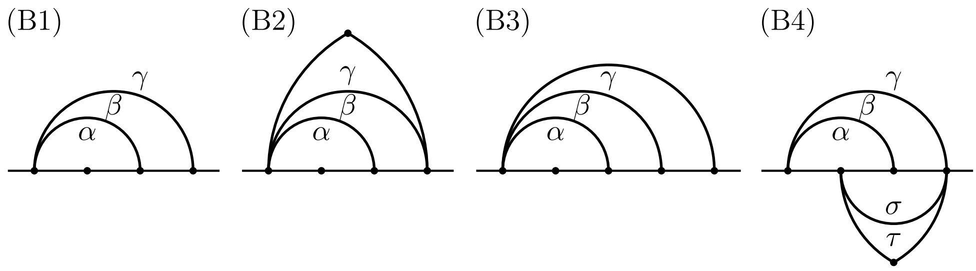

Figure 7

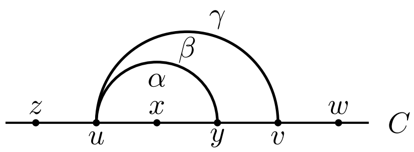

Figure 8

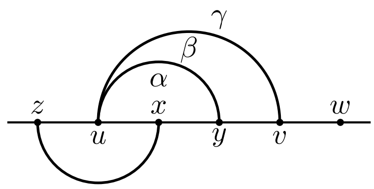

Figure 9

Case 1. Let $w_1(\beta) = \frac{5}{6}$ and ux be a $(2,2)$-edge (i.e., $zx \in E(H_{3-i})$, see Figure 9). The cycle $\widetilde{C}$ obtained from C by replacing the path (z, u, x, y, v) by the path (z, x, y, u, v) is a longest OI3-cycle of G and contains the edge uv, thus γ is an empty face of H (and β is of type B1).

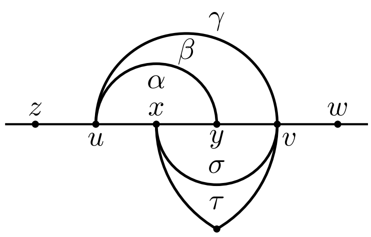

Figure 10

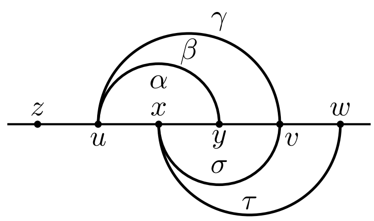

Figure 11

Case 2. Let $w_1(\beta) = \frac{5}{6}$ and xy be a $(2,2)$-edge (i.e., $xv \in E(H_{3-i})$). The face $\sigma = [x, y, v]$ is a 2-face of H_{3-i}. Let τ $(\tau \neq \sigma)$ be the second face of H_{3-i} incident with xv. Since $|V(C)| \geq 8$, it follows $u \neq w$, hence τ cannot be a 2-face of H_{3-i}.

Case 2.1. If τ is a 0-face (Figure 10), then the cycle $\widetilde{C}$ obtained from C by replacing the path (u, x, y, v) by the path (u, y, x, v) is a longest OI3-cycle of G and contains the edge xv, thus τ is an empty face of H (and β is of type B4).

Case 2.2. If τ is a 1-face (Figure 11), then $\tau = [x, v, w]$ (since $uv \in E(H_i) \setminus E(C)$, uv is not an edge of H_{3-i}). The cycle $\widetilde{C}$ obtained from C by replacing the path (u, x, y, v, w) by the path (u, v, y, x, w) is a longest OI3-cycle of G and contains the edge uv, thus γ is an empty face of H (and β is of type B1).

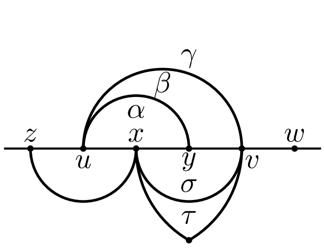

Figure 12

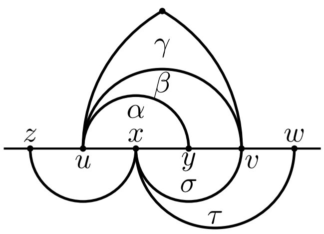

Figure 13

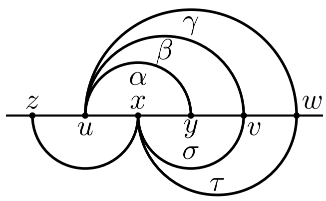

Figure 14

Case 3. Let $w_1(\beta) = 1$. Now both ux and xy are $(2,2)$-edges (i.e., $zx, xv \in E(H_{3-i})$). The face $\sigma = [x, y, v]$ is a 2-face of H_{3-i}. Let τ ($\tau \neq \sigma$) be the second face of H_{3-i} incident with xv. Again, τ cannot be a 2-face of H_{3-i} and we consider two subcases.

Case 3.1. If τ is a 0-face (see Figure 12, possibly $\tau = [z, x, v]$), then, for a similar reason as in Case 2.1, τ is an empty face of H (and β is of type B4).

Case 3.2. If τ is a 1-face, then $\tau = [x, v, w]$. Since $|V(C)| \geq 8$, it follows $z \neq w$, hence γ is not a 2-face of H_i. We consider the last two subcases.

Case 3.2.1. If γ is a 0-face (see Figure 13), then, for a similar reason as in Case 1, γ is an empty face of H (and β is of type B2).

Case 3.2.2. If γ is a 1-face, then $\gamma \neq [z, u, v]$ (otherwise $\{z, x, v\}$ is a non-trivial 3-separator, a contradiction). Thus $\gamma = [u, v, w]$ (see Figure 14) and vw is a $(1,1)$-edge (and β is of type B3). □

For a better overview, we list the current weights of all faces considered in Lemma 7:

(B1) $w_1(\alpha) = \frac{2}{3}$, $w_1(\beta) = \frac{5}{6}$, and $w_1(\gamma) \leq \frac{2}{3}$;

(B2) $w_1(\alpha) = \frac{2}{3}$, $w_1(\beta) = 1$, and $w_1(\gamma) \leq \frac{1}{3}$, because γ obtains no weight through its common edge with β and at most $\frac{1}{6} + \frac{1}{6}$ through at most one of its remaining two edges;

(B3) $w_1(\alpha) = \frac{2}{3}$, $w_1(\beta) = 1$, and $w_1(\gamma) = \frac{1}{2}$;

(B4) $w_1(\alpha) = \frac{2}{3}$, $\frac{5}{6} \leq w_1(\beta) \leq 1$, $w_1(\sigma) = \frac{2}{3}$, and $w_1(\tau) \leq \frac{1}{2}$, because τ obtains weight $\frac{1}{6}$ through its common edge with σ and at most $\frac{1}{6} + \frac{1}{6}$ through at most one of its remaining two edges.

Second redistribution of weights

The weight of all bad faces exceeded $\frac{13}{18}$ will be redistributed to good faces in their neighborhoods.

Rule R4. A bad face β of type B1 sends weight $\frac{1}{18}$ to α and to γ (through the common edge).

Rule R5. A bad face β of type B2 or B3 sends weight $\frac{1}{18}$ to α and weight $\frac{2}{9}$ to γ (through the common edge).

Rule R6. A bad face β of type B4 sends weight $\frac{1}{18}$ to α and to σ (through the common edge) and the weight $\frac{1}{6}$ to τ (through the edge xv, see Figure 10).

For an empty face φ, let $w_2(\varphi)$ be the total weight of φ (after second redistribution). Obviously, $\sum_{\varphi\in\mathcal{F}} w_2(\varphi) = \sum_{\varphi\in\mathcal{F}} w_1(\varphi) = |V(C)|$.

A bad face φ of type B1 sends weight $2 \times \frac{1}{18}$ to good faces, thus $w_2(\varphi) = \frac{5}{6} - 2 \times \frac{1}{18} = \frac{13}{18}$. A bad face φ of type B2 or B3 sends weight $\frac{1}{18} + \frac{2}{9}$ to good faces, thus $w_2(\varphi) = 1 - \frac{1}{18} - \frac{2}{9} = \frac{13}{18}$. Finally, a bad face φ of type B4 sends weight $2 \times \frac{1}{18} + \frac{1}{6}$ to good faces, thus $w_2(\varphi) \leq 1 - 2 \times \frac{1}{18} - \frac{1}{6} = \frac{13}{18}$.

If a 2-face φ gets weight by the rules R4, R5, or R6, then either by exactly one of the rules R4 and R5 ($\varphi = \alpha$ is adjacent to a 1-face β in this case) or by R6 ($\varphi = \sigma$ is adjacent to a 0-face τ in this case). Thus $w_2(\varphi) \leq \frac{2}{3} + \frac{1}{18} = \frac{13}{18}$.

A good 1-face φ has at most one adjacent bad face (otherwise $|V(C)| \leq 7$ by Lemma 7, a contradiction). If $w_1(\varphi) = \frac{1}{2}$, then $w_2(\varphi) \leq \frac{1}{2} + \frac{2}{9} = \frac{13}{18}$ (by R5). If $w_1(\varphi) = \frac{2}{3}$, then $w_2(\varphi) \leq \frac{2}{3} + \frac{1}{18} = \frac{13}{18}$ (by R4).

A 0-face φ gets through at least one of its incident edges no weight in first redistribution (1RD) and also in second redistribution (2RD). Let e be an edge incident with φ. If φ gets weight $\frac{2}{9}$ through e (by R5) in 2RD, then φ obtained no weight through e in 1RD. If φ gets weight $\frac{1}{6}$ through e (by R6) in 2RD, then φ has already obtained weight $\frac{1}{6}$ through e in 1RD. Finally, if φ gets no weight through e in 2RD, then φ has obtained weight at most $\frac{1}{3}$ through e in 1RD. Thus φ obtain through e weight at most $\frac{1}{3}$ (in 1RD and 2RD in total) and $w_2(\varphi) \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ follows. Thus, Lemma 4 is completely proved. ■

It remains to show that the essentially 4-connected maximal planar graph G on $n = n' + (2n' - 4)$ vertices constructed in Section 1 from the 4-connected maximal planar graph G' on $n' \geq 6$ vertices contains a cycle on exactly $2n'$ vertices. To see this, let a and b be two adjacent edges of G' which do not belong to a common face of G'. Note that a and b exist since $n \geq 6$ implies that each vertex of G' has degree at least 4. Consider a Hamiltonian cycle C' of G' through a and b (apply Lemma 2). Let $a = e_1, e_2, \ldots, e_{n'-1}, e_{n'} = b$ be the edges of C' met in this order along C'. For $j = 1, \ldots, n'$, consider the common neighbors $x_j \in (V(G)\setminus V(G'))\cap\text{int}(C')$ and $y_j \in (V(G)\setminus V(G'))\cap\text{ext}(C')$ of the end vertices u_j and v_j of e_j. It is easy to see that the vertices in $\{x_1, \ldots, x_{n'}, y_1, \ldots, y_{n'}\}$ are pairwise distinct (if n' is odd, then note that a and b do not belong to a common face of G'). Eventually, let the cycle C of G be obtained by replacing e_j in C' with the path (u_j, x_j, v_j) if j is odd and (u_j, y_j, v_j) if j is even ($j = 1, \ldots, n'$).

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Received 16 June 2015
Revised 23 September 2015
Accepted 23 September 2015