

EXTREMAL MATCHING ENERGY OF COMPLEMENTS OF TREES

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Abstract

Gutman and Wagner proposed the concept of the matching energy which is defined as the sum of the absolute values of the zeros of the matching polynomial of a graph. And they pointed out that the chemical applications of matching energy go back to the 1970s. Let T be a tree with n vertices. In this paper, we characterize the trees whose complements have the maximal, second-maximal and minimal matching energy. Furthermore, we determine the trees with edge-independence number p whose complements have the minimum matching energy for $p = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. When we restrict our consideration to all trees with a perfect matching, we determine the trees whose complements have the second-maximal matching energy.

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1. INTRODUCTION

All graphs considered in this paper are undirected simple graphs. For notation and terminologies not defined here, see [7, 18].

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. For any a vertex $v \in V(G)$ (or an edge $e \in E(G)$), let $G - v$ (or $G - e$) denote the subgraph obtained from G by deleting v (or e). Denote by $\overline{G}$ the complement of G . The path, star and complete graph with n vertices are denoted by P_n , $K_{1,n-1}$ and K_n , respectively. Let $T_{n,2}$ be a tree obtained from the star $K_{1,3}$ by attaching a path P_{n-3} to one of the pendent vertices of $K_{1,3}$, and let $T_{n,2}^1$ be a tree obtained from the star $K_{1,3}$ by attaching two paths P_2 and P_{n-4} to two different pendent vertices of $K_{1,3}$, respectively. Let T_n^p be a tree with n vertices obtained from the star $K_{1,n-p}$ by attaching a pendent edge to each of $p - 1$ pendent vertices in $K_{1,n-p}$ for $p = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$.

A k -matching in G is a set of k pairwise non-incident edges. The number of k -matchings in G is denoted by $m(G, k)$. Specifically, $m(G, 0) = 1$, $m(G, 1) = m$ and $m(G, k) = 0$ for $k > \frac{n}{2}$ or $k < 0$. For a k -matching M in G , if G has no k' -matching such that $k' > k$, then M is called a *maximum matching* of G . The number $\nu(G)$ of edges in a maximum matching M is called the *edge-independence number* of G . We use $\mathcal{T}_{n,p}$ to denote the set of trees with n vertices and the edge-independence number at least p for $p = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. The *Hosoya index* $Z(G)$ is defined as the total number of matchings of G , that is

$$Z(G) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(G, k).$$

Recall that for a graph G on n vertices, the *matching polynomial* $\mu(G, x)$ of G is given by

$$(1) \quad \mu(G, x) = \sum_{k \geq 0} (-1)^k m(G, k) x^{n-2k}.$$

Its theory is well elaborated [4, 6, 7, 8, 9]. Gutman and Wagner [10] gave the definition of the *quasi-order* $\succeq$ as follows. If G and H have the matching polynomials in the form (1), then the quasi-order $\succeq$ is defined by

$$(2) \quad G \succeq H \iff m(G, k) \geq m(H, k) \text{ for all } k = 0, 1, \dots, \lfloor n/2 \rfloor.$$

Particularly, if $G \succeq H$ and there exists some k such that $m(G, k) > m(H, k)$, then we write $G \succ H$.

Gutman and Wagner in [10] first proposed the concept of the *matching energy* of a graph, denoted by $ME(G)$, and defined as

$$(3) \quad ME = ME(G) = \frac{2}{\pi} \int_0^\infty x^{-2} \ln \left[\sum_{k \geq 0} m(G, k) x^{2k} \right] dx.$$

Meanwhile, they gave also another form of the definition of matching energy of a graph. That is,

$$ME(G) = \sum_{i=1}^n |\mu_i|,$$

where μ_i denotes the root of matching polynomial of G . Additionally, they found some relations between the matching energy and energy (or reference energy). By (2) and (3), we easily obtain the fact as follows.

$$(4) \quad G \succeq H \implies ME(G) \geq ME(H) \quad \text{and} \quad G \succ H \implies ME(G) > ME(H).$$

This property is an important technique to determine extremal graphs with the matching energy.

Note that the energy (or reference energy) of graphs are extensively examined (see [1, 4, 5, 11, 12, 16]). However, the literature on the matching energy is far less than that on the energy and reference energy. Up to now, we find only a few papers about the matching energy published. Gutman and Wagner [10] gave some properties and asymptotic results of the matching energy. Li and Yan [15] characterized the connected graph with the fixed connectivity (resp. chromatic number) which has the maximum matching energy. Ji *et al.* in [13] determined the graphs with the extremal matching energy among all bicyclic graphs. Li *et al.* [14] characterized the unicyclic graphs with fixed girth (resp. clique number) which has the maximum and minimum matching energy, respectively. Chen and Shi [2] characterized the graphs with the maximal value of matching energy among all tricyclic graphs. Chen *et al.* in [3] characterized the graphs with minimal matching energy among all unicyclic and bicyclic graphs with a given diameter d . Xu *et al.* [20] determined the extremal graphs from $\mathcal{T}(n)$ with minimal and maximal matching energies, respectively, where $\mathcal{T}(n)$ is a set of t -apex trees of order n . And they also determined the extremal graphs from $\mathcal{G}_{n,m}$ minimizing the matching energy [21], where $\mathcal{G}_{n,m}$ is a set of connected graphs of order n and with m edges. Additionally, the present author [19] characterized completely the graphs which has i -th maximal matching energy, where $i = 2, 3, \dots, 16$.

In this paper, inspired by the idea given in [22], we investigate the problem of the matching energy of the complements of trees, and obtain the following main theorems.

Theorem 1.1. *Let T be a tree with n vertices. If $T \not\cong T_{n,2}$ and $T \not\cong P_n$, then*

$$ME(\overline{T}) < ME(\overline{T_{n,2}}) < ME(\overline{P_n}).$$

Theorem 1.2. *Let $\mathcal{T}_{n,p}$ denote the set of trees with n vertices and the edge-independence number at least p for $p = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. For a tree $T \in \mathcal{T}_{n,p}$ it holds*

$$ME(\overline{T}) \geq ME(\overline{T_n^p})$$

with equality if and only if $T \cong T_n^p$.

By Theorems 1.1 and 1.2, we obtain directly the following corollary.

Corollary 1.3. *The complements of P_n and $K_{1,n-1}$ have the maximum and minimum matching energy in all complements of trees, respectively.*

Theorem 1.4. *Let $\mathcal{T}_{n, \frac{n}{2}}$ be a proper subset of $\mathcal{T}_{n,p}$ containing all trees with a perfect matching. Suppose that $T \in \mathcal{T}_{n, \frac{n}{2}}$, $T \not\cong T_n^{\frac{n}{2}}$ and $T \not\cong P_n$. If $n \geq 6$, then*

$$ME\left(\overline{T_n^{\frac{n}{2}}}\right) < ME(\overline{T}) \leq ME\left(\overline{T_{n,2}^1}\right) < ME(\overline{P_n}),$$

where the equality holds if and only if $T \cong T_{n,2}^1$.

2. SOME LEMMAS

There exists a well-known formula which characterizes the relation between $m(G, r)$ and $m(\overline{G}, i)$ (see Lovász [17]), which will play a key role in the proofs of the main theorems.

Lemma 2.1 [17]. *Let G be a simple graph with n vertices and $\overline{G}$ the complement of G . Then*

$$(5) \quad m(G, r) = \sum_{i=0}^r (-1)^i \binom{n-2i}{2r-2i} (2r-2i-1)!! m(\overline{G}, i),$$

where $s!! = s \times (s-2)!!$, and $(-1)!! = 0!! = 1$.

The following results about the matching polynomial of G can be found in Godsil [7].

Lemma 2.2 [7]. *The matching polynomial satisfies the following identities:*

- (i) $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$,
- (ii) $\mu(G, x) = \mu(G \setminus e, x) - \mu(G - u - v, x)$ if $e = \{u, v\}$ is an edge of G ,
- (iii) $\mu(G, x) = x\mu(G \setminus u, x) - \sum_{v \sim u} \mu(G - u - v, x)$ if $u \in V(G)$.

Lemma 2.3 [7]. *Let m and n be two positive integers. Then*

$$(6) \quad \mu(P_{m+n}) = \mu(P_m)\mu(P_n) - \mu(P_{m-1})\mu(P_{n-1}).$$

Lemma 2.4 [22]. *If T is a tree with n vertices and edge-independence number $\nu(T) = p$, then T has at most $n - p$ vertices of degree one. In particular, if T has exactly $n - p$ vertices of degree one, then every vertex of degree at least two in T is adjacent to at least one vertex of degree one.*

3. ORDERING COMPLEMENTS OF TREES WITH RESPECT TO THEIR MATCHINGS

For convenience, we use the same definitions of trees which are given in [22].

Definition 3.1. Let T_1 be a tree with $n + m + k$ vertices shown in Figure 1, where T_0 is a tree with k vertices ($k \geq 2$) and u a vertex of T_0 , $n \geq 1$ and $m \geq 1$. Suppose T_2 is a tree with $n + m + k$ vertices obtained from T_0 by attaching a path P_{m+n} to u in T_0 (see Figure 1). We designate the transformation from T_1 to T_2 as of type 1 and denote it by $\mathcal{F}_1: T_1 \hookrightarrow T_2$ or $\mathcal{F}_1(T_1) = T_2$.

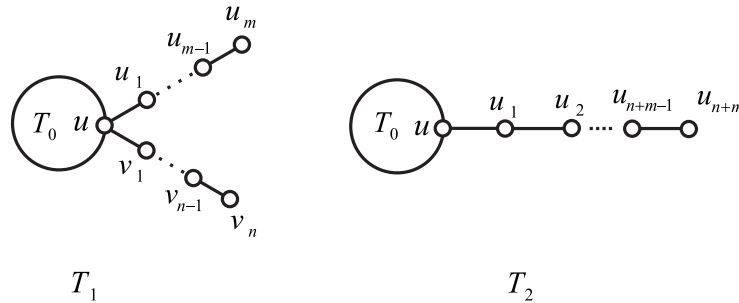


Figure 1. Two trees T_1 and T_2 .

Theorem 3.1. Let T_1 and T_2 be the trees with $m + n + k$ vertices defined in Definition 3.1. Then $\overline{T_2} \succ \overline{T_1}$.

Proof. By Lemma 2.2,

$$\begin{aligned} \mu(T_1) &= x\mu(T_0 - u)\mu(P_m)\mu(P_n) - \mu(T_0 - u)\mu(P_{m-1})\mu(P_n) \\ &\quad - \mu(T_0 - u)\mu(P_m)\mu(P_{n-1}) - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v)\mu(P_m)\mu(P_n), \\ \mu(T_2) &= x\mu(T_0 - u)\mu(P_{m+n}) - \mu(T_0 - u)\mu(P_{m+n-1}) \\ &\quad - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v)\mu(P_{m+n}), \end{aligned}$$

where the above sums range over all vertices of T_0 adjacent to u . Hence

$$\begin{aligned} \mu(T_1) - \mu(T_2) &= x\mu(T_0 - u)[\mu(P_m)\mu(P_n) - \mu(P_{m+n})] - \mu(T_0 - u)[\mu(P_{m-1})\mu(P_n) \\ &\quad - \mu(P_{m+n-1}) + \mu(P_m)\mu(P_{n-1})] - [\mu(P_m)\mu(P_n) - \mu(P_{m+n})] \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v). \end{aligned}$$

By (6) and a routine calculation,

$$(7) \quad \mu(T_1) - \mu(T_2) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) \mu(P_{m-1}) \mu(P_{n-1}).$$

For an arbitrary vertex v adjacent to u in T_0 , let T_v^* be the forest $(T_0 - u - v) \cup P_{m-1} \cup P_{n-1}$, which has $n + m + k - 4$ vertices. By (5), we obtain

$$(8) \quad \begin{aligned} & m(\overline{T_1}, r) - m(\overline{T_2}, r) \\ &= \sum_{i=0}^r (-1)^i \binom{n+m+k-2i}{2r-2i} (2r-2i-1)!! [m(T_1, i) - m(T_2, i)]. \end{aligned}$$

Note that $m(T_1, 0) = m(T_2, 0)$ and $m(T_1, 1) = m(T_2, 1)$. Hence

$$(9) \quad \begin{aligned} & m(\overline{T_1}, r) - m(\overline{T_2}, r) \\ &= - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \sum_{i=2}^r (-1)^i \binom{n+m+k-2i}{2r-2i} (2r-2i-1)!! m(T_v^*, i-2). \end{aligned}$$

Note that T_v^* has $n + m + k - 4$ vertices. So

$$\begin{aligned} m(\overline{T_v^*}, r-2) &= \sum_{j=0}^{r-2} (-1)^j \binom{n+m+k-4-2j}{2(r-2)-2j} (2(r-2)-2j-1)!! m(T_v^*, j) \\ &= \sum_{i=2}^r (-1)^i \binom{n+m+k-2i}{2r-2i} (2r-2i-1)!! m(T_v^*, i-2). \end{aligned}$$

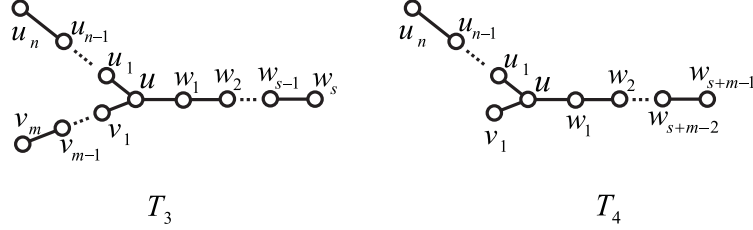
Hence

$$(10) \quad m(\overline{T_1}, r) - m(\overline{T_2}, r) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} m(\overline{T_v^*}, r-2).$$

By the definition of $m(G, r)$ and (10), we have $m(\overline{T_v^*}, r-2) \geq 0$, which implies $m(\overline{T_1}, r) \leq m(\overline{T_2}, r)$. Particularly, if $r = 2$, then $m(\overline{T_1}, r) - m(\overline{T_2}, r) \leq -1$. By (2), $\overline{T_2} \succ \overline{T_1}$. $\blacksquare$

Remark 3.2. By Theorem 3.1 and (4), we obtain immediately a result as follows: If T_1 and T_2 are the two trees defined in Definition 3.1, then $ME(\overline{T_2}) > ME(\overline{T_1})$. Additionally, by the definition of the Hosoya index and Theorem 3.1, it is not difficult to see that $Z(\overline{T_2}) > Z(\overline{T_1})$.

Definition 3.2. Let T_3 and T_4 be two trees with $m + n + s + 1$ vertices shown in Figure 2, where $s \geq m \geq 2, n \geq 1$. We designate the transformation from T_3 to T_4 in Figure 2 as of type **2** and denote it by $\mathcal{F}_2$: $T_3 \mapsto T_4$ or $\mathcal{F}_2(T_3) = T_4$.

Figure 2. Two trees T_3 and T_4 .

Theorem 3.3. Let T_3 and T_4 be two trees with $m + n + s + 1$ vertices defined in Definition 3.2. Then $\overline{T_4} \succ \overline{T_3}$.

Proof. Similarly to the proof of Theorem 3.1, we can obtain that

$$\mu(T_3) - \mu(T_4) = -\mu(P_{m-2})\mu(P_{n-1})\mu(P_{s-2}).$$

Furthermore, we also have

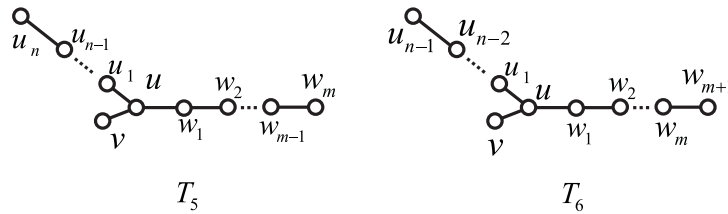
$$(11) \quad m(\overline{T_3}, r) - m(\overline{T_4}, r) = -m(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}, r - 3).$$

By the definition of $m(G, r)$ and (11), we have $m(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}, r - 3) \geq 0$, which implies $m(\overline{T_3}, r) \leq m(\overline{T_4}, r)$. Especially, if $r = 3$ then

$$m(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}, r - 3) = 1.$$

This means, by (2), that $\overline{T_4} \succ \overline{T_3}$. The proof is completed. $\blacksquare$

Definition 3.3. Let T_5 and T_6 be two trees with $m + n + 2$ vertices shown in Figure 3, where $m \geq n \geq 2$. We designate the transformation from T_5 to T_6 in Figure 3 as of type **3** and denote it by $\mathcal{F}_3$: $T_5 \rightarrow T_6$ or $\mathcal{F}_3(T_5) = T_6$.

Figure 3. Two trees T_5 and T_6 .

Theorem 3.4. *Let T_5 and T_6 be two trees with $m + n + 2$ vertices defined in Definition 3.3. Then $\overline{T_6} \succ \overline{T_5}$.*

Proof. Similarly to the proof of Theorem 3.1, we have

$$\mu(T_5) - \mu(T_6) = -\mu(P_{m-n})$$

and

$$(12) \quad m(\overline{T_5}, r) - m(\overline{T_6}, r) = -m(\overline{P_{m-n}}, r - n - 1).$$

By the definition of $m(G, r)$ and (12), we have $m(\overline{P_{m-n}}, r - n - 1) \geq 0$, which indicates $m(\overline{T_5}, r) \leq m(\overline{T_6}, r)$. Especially, when $r = n + 1$, then $m(\overline{P_{m-n}}, r - n - 1) = 1$. By (2), we get that $\overline{T_4} \succ \overline{T_3}$. ■

Definition 3.4. Suppose that T'_1 and T'_2 are two trees with m ($m > 1$) vertices and with n ($n > 1$) vertices, respectively. Take one vertex u of T'_1 and one v of T'_2 . Construct two trees T_7 and T_8 with $m + n$ vertices as follows. The vertex set $V(T_7)$ of T_7 is $V(T'_1) \cup V(T'_2)$ and the edge set of T_7 is $E(T'_1) \cup E(T'_2) \cup uv$. T_8 is the tree obtained from T'_1 and T'_2 by identifying the vertex u of T'_1 and the vertex v of T'_2 and adding a pendent edge $uw = vw$ to this new vertex $u (= v)$. The resulting graphs are presented in Figure 4. We designate the transformation from T_7 to T_8 as of type 4 and denote it by $\mathcal{F}_4$: $T_7 \rightsquigarrow T_8$ or $\mathcal{F}_4(T_7) = T_8$.

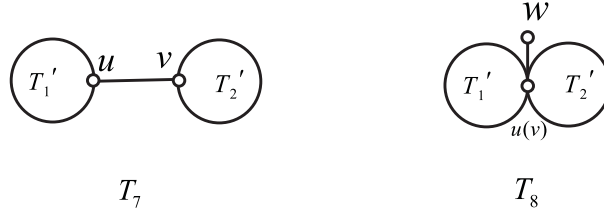


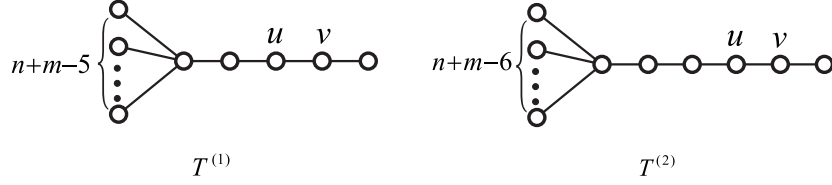
Figure 4. Two trees T_7 and T_8 .

Theorem 3.5. *Let T_7 and T_8 be two trees with $m + n$ vertices defined in Definition 3.4. Then $\overline{T_7} \succ \overline{T_8}$.*

Proof. By Lemma 2.2,

$$(13) \quad \mu(T_7) = \mu(T'_1)\mu(T'_2) - \mu(T'_1 - u)\mu(T'_2 - v),$$

$$(14) \quad \mu(T_8) = x\mu(T_8 - w) - \mu(T'_1 - u)\mu(T'_2 - v),$$

Figure 5. Two trees $T^{(1)}$ and $T^{(2)}$.

$$(15) \quad \mu(T'_1) = x\mu(T'_1 - u) - \sum_{i=1}^s \mu(T'_1 - u - u_i)$$

and

$$(16) \quad \mu(T'_2) = x\mu(T'_2 - v) - \sum_{j=1}^t \mu(T'_2 - v - v_j),$$

where the first sum ranges over all vertices u_i ($1 \leq i \leq s$) of T'_1 adjacent to u and the second sum ranges over all v_j ($1 \leq j \leq t$) of T'_2 adjacent to v . By (15) and (16), we have

$$(17) \quad \begin{aligned} x\mu(T_8 - w) &= x^2\mu(T'_1 - u)\mu(T'_2 - v) - x \sum_{j=1}^t \mu(T'_1 - u)\mu(T'_2 - v - v_j) \\ &\quad - x \sum_{i=1}^s \mu(T'_2 - v)\mu(T'_1 - u - u_i) \end{aligned}$$

and

$$(18) \quad \begin{aligned} \mu(T'_1)\mu(T'_2) &= x^2\mu(T'_1 - u)\mu(T'_2 - v) - x \sum_{j=1}^t \mu(T'_1 - u)\mu(T'_2 - v - v_j) \\ &\quad - x \sum_{i=1}^s \mu(T'_2 - v)\mu(T'_1 - u - u_i) + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} \mu(T'_1 - u - u_i)\mu(T'_2 - v - v_j). \end{aligned}$$

Combining (13), (14), (17) and (18),

$$(19) \quad \mu(T_7) - \mu(T_8) = \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} \mu(T'_1 - u - u_i)\mu(T'_2 - v - v_j).$$

As in the proof of Theorem 3.1, we can show that

$$m(\overline{T_7}, r) - m(\overline{T_8}, r) = \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} m(\overline{\mu(T'_1 - u - u_i) \cup \mu(T'_2 - v - v_j)}, r - 2),$$

which implies that

$$m(\overline{T_7}, r) \geq m(\overline{T_8}, r).$$

Note that $m(\overline{T_7}, r) - m(\overline{T_8}, r) \geq 1$ when $r = 2$. So, by (2), the theorem holds. ■

Remark 3.6. For the trees $T^{(1)}$ and $T^{(2)}$ (see Figure 5), we note that neither tree $T^{(1)}$ nor tree $T^{(2)}$ can be transformed into T_{m+n}^p by a single transformation 4. Hence if T_8 in Theorem 3.5 is T_{m+n}^p , then $\overline{T_7} \succ \overline{T_8} = \overline{T_{m+n}^p}$. Particularly, $\overline{T_n^p} \succ \overline{T_n^{p-1}}$ for $n \geq 5$. Similarly, as in earlier proofs, one can show that this statement holds.

Definition 3.5. Suppose that T_9 is a tree with n vertices and with the edge-independence number p (shown in Figure 6) which has exactly $n - p$ pendent vertices, where $|V(T_0)| \geq 2$ and $r \geq 2$. Let T_{10} be the tree with n vertices shown in Figure 6, which is obtained from T_9 . We designate the transformation from T_9 to T_{10} as of type 5 and denote it by $\mathcal{F}_5$: $T_9 \dashrightarrow T_{10}$ or $\mathcal{F}_5(T_9) = T_{10}$.

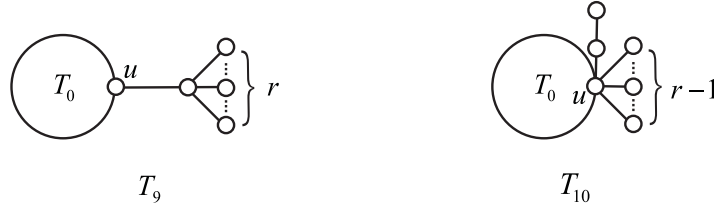


Figure 6. Two trees T_9 and T_{10} .

Theorem 3.7. Let T_9 and T_{10} be two trees with n vertices defined in Definition 3.5. Then $\overline{T_9} \succ \overline{T_{10}}$.

Proof. By Lemma 2.2,

$$\begin{aligned}
 \mu(T_9) &= x\mu(T_0 - u)\mu(K_{1,r}) - \mu(K_{1,r}) \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) - \mu(T_0 - u)\mu(P_1)^r \\
 &= x^2\mu(T_0 - u)\mu(P_1)^r - rx\mu(T_0 - u)\mu(P_1)^{r-1} - x\mu(P_1)^r \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) \\
 (20) \quad &+ r\mu(P_1)^{r-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) - \mu(T_0 - u)\mu(P_1)^r
 \end{aligned}$$

and

$$\begin{aligned}
 \mu(T_{10}) &= x\mu(T_0 - u)\mu(P_2)\mu(P_1)^{r-1} - \mu(P_2)\mu(P_1)^{r-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) \\
 &\quad - \mu(T_0 - u)\mu(P_1)^r - (r-1)\mu(T_0 - u)\mu(P_2)\mu(P_1)^{r-2} \\
 &= x^2\mu(T_0 - u)\mu(P_1)^r - x\mu(T_0 - u)\mu(P_1)^{r-1} \\
 (21) \quad &\quad - x\mu(P_1)^r \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) + \mu(P_1)^{r-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) \\
 &\quad - \mu(T_0 - u)\mu(P_1)^r - (r-1)x\mu(T_0 - u)\mu(P_1)^{r-1} \\
 &\quad + (r-1)\mu(T_0 - u)\mu(P_1)^{r-2},
 \end{aligned}$$

where the sum ranges over all vertices of T_0 incident with u .

By (20) and (21), we have

$$\mu(T_9) - \mu(T_{10}) = -(r-1)\mu(T_0 - u)\mu(P_1)^{r-2} + (r-1)\mu(P_1)^{r-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v).$$

By Lemma 2.4, there exists at least one pendent vertex v' in T_0 joining vertex u of T_0 . Hence, $\mu(T_0 - u) = x\mu(T_0 - u - v')$, which implies that

$$\mu(T_9) - \mu(T_{10}) = (r-1) \sum_{\substack{v \in V(T_0), v \neq v' \\ uv \in E(T_0)}} \mu(P_1)^{r-1} \mu(T_0 - u - v).$$

Similarly to the proof of Theorem 3.1,

$$(22) \quad m(\overline{T_9}, k) - m(\overline{T_{10}}, k) = (r-1) \sum_{\substack{v \in V(T_0), v \neq v' \\ uv \in E(T_0)}} m\left(\overline{(r-1)P_1 \cup (T_0 - u - v)}, k-2\right),$$

for every vertex v ($\neq v'$) of T_0 incident with u . Hence $m(\overline{T_9}, k) \geq m(\overline{T_{10}}, k)$. Furthermore, if $k = 2$, then $m(\overline{T_9}, k) - m(\overline{T_{10}}, k) \geq 1$. So $\overline{T_9} \succ \overline{T_{10}}$. ■

Definition 3.6. Suppose that T_{11} is a tree with n vertices and with the edge-independence number p (shown in Figure 7), which has exactly $n - p$ pendent vertices, where $|V(T_0)| \geq 2$, $s \geq 1$ and $t \geq 1$. Let T_{12} be the tree with n vertices shown in Figure 7, which is obtained from T_{11} . We designate the transformation from T_{11} to T_{12} as of type **6** and denote it by $\mathcal{F}_6$: $T_{11} \rightsquigarrow T_{12}$ or $\mathcal{F}_6(T_{11}) = T_{12}$.

Theorem 3.8. *Let T_{11} and T_{12} be two trees with n vertices defined in Definition 3.6. Then $\overline{T_{11}} \succ \overline{T_{12}}$.*

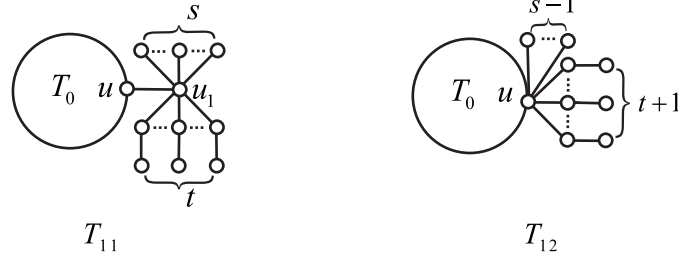


Figure 7. Two trees T_{11} and T_{12} .

Proof. Suppose that $s > 2$. By Lemma 2.2,

$$\begin{aligned}
 \mu(T_{11}) &= [x^2\mu(P_1)\mu(P_2) - sx\mu(P_2) - tx\mu(P_1)^2 - \mu(P_1)\mu(P_2)] \\
 &\quad \times \mu(P_1)^{s-1}\mu(P_2)^{t-1}\mu(T_0 - u) - [x\mu(P_1)\mu(P_2) - s\mu(P_2) - t\mu(P_1)^2] \\
 &\quad \times \mu(P_1)^{s-1}\mu(P_2)^{t-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v)
 \end{aligned}$$

and

$$\begin{aligned}
 \mu(T_{12}) &= [x^2\mu(P_1)^2\mu(P_2) - x\mu(P_1)\mu(P_2) - sx\mu(P_1)\mu(P_2) + s\mu(P_2) - \mu(P_2) \\
 &\quad + x\mu(P_1)\mu(P_2) - (t+1)\mu(P_1)^2\mu(P_2)]\mu(P_1)^{s-2}\mu(P_2)^{t-1}\mu(T_0 - u) \\
 &\quad - [x\mu(P_1)\mu(P_2) - \mu(P_2)]\mu(P_1)^{s-1}\mu(P_2)^{t-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v),
 \end{aligned}$$

where the sum ranges over every vertex v of T_0 adjacent to u .

Combining the above two equations, we obtain

$$\begin{aligned}
 \mu(T_{11}) - \mu(T_{12}) &= -[(s+t-1)x^2 - (s-1)]\mu(P_1)^{s-2}\mu(P_2)^{t-1}\mu(T_0 - u) \\
 &\quad + [(s+t-1)x^2 - (s-1)]\mu(P_1)^{s-1}\mu(P_2)^{t-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v).
 \end{aligned}$$

By Lemma 2.4, there exists at least one pendent vertex v' of T_0 adjacent to u . Hence, $\mu(T_0 - u) = \mu(T_0 - u - v')$. Thus, simplifying the above equation,

we have

$$(23) \quad \begin{aligned} & \mu(T_{11}) - \mu(T_{12}) \\ &= [(s+t-1)\mu(P_1)^{s-1}\mu(P_2)^t + t\mu(P_1)^{s-1}\mu(P_2)^{t-1}] \sum_{\substack{v \in V(T_0), v \neq v' \\ uv \in E(T_0)}} \mu(T_0 - u - v). \end{aligned}$$

As in the proof of Theorem 3.1, we can show that if $s \geq 1$, then

$$\begin{aligned} & m(\overline{T_{11}}, r) - m(\overline{T_{12}}, r) \\ &= (s+t-1) \sum_{\substack{v \in V(T_0), v \neq v' \\ uv \in E(T_0)}} m\left(\overline{(s-1)P_1 \cup tP_2 \cup (T_0 - u - v)}, r-2\right) \\ &+ t \sum_{\substack{v \in V(T_0), v \neq v' \\ uv \in E(T_0)}} m\left(\overline{(s-1)P_1 \cup (t-1)P_2 \cup (T_0 - u - v)}, r-3\right), \end{aligned}$$

which implies that $m(\overline{T_{11}}, r) \geq m(\overline{T_{12}}, r)$. By the above equation, if $r = 2$, then $m(\overline{T_{11}}, r) - m(\overline{T_{12}}, r) \geq 1$. By (2) and the properties as above, we have $\overline{T_{11}} \succ \overline{T_{12}}$. ■

4. PROOFS OF THEOREMS 1.1, 1.2 AND 1.4

Proof of Theorem 1.1. We prove that if $T \not\cong P_n$ then $ME(\overline{T}) < ME(\overline{P_n})$. By repeated applications of transformation 1 presented in Definition 3.1, we can transform T into P_n , that is, there exist trees $T^{(i)}$ for $0 \leq i \leq l$ such that

$$(24) \quad T = T^{(0)} \hookrightarrow T^{(1)} \hookrightarrow T^{(2)} \hookrightarrow \dots \hookrightarrow T^{(l-1)} \hookrightarrow T^{(l)} = P_n,$$

where $T^{(l-1)} \neq P_n$. By Theorem 3.1, we have

$$\overline{P_n} = \overline{T^{(l)}} \succ \overline{T^{(l-1)}} \succ \dots \succ \overline{T^{(2)}} \succ \overline{T^{(1)}} \succ \overline{T}.$$

By (4), we obtain immediately the result as follows:

$$\begin{aligned} ME(\overline{P_n}) &= ME(\overline{T^{(l)}}) > ME(\overline{T^{(l-1)}}) > \dots > ME(\overline{T^{(2)}}) \\ &> ME(\overline{T^{(1)}}) > ME(\overline{T}). \end{aligned}$$

By the transformation 1 presented in Definition 3.1, Theorem 3.1 and (4), it is clear that

$$ME(\overline{P_n}) > ME(\overline{T_{n,2}}).$$

Now we show that $ME(\overline{T_{n,2}}) > ME(\overline{T})$. Suppose $T \neq T_{n,2}$. From (24), we know that if $T^{(l-1)} = T_{n,2}$, then $\overline{T_{n,2}} \succ \overline{T}$, which implies $ME(\overline{T^{(n,2)}}) > ME(\overline{T})$. If $T^{(1-1)} \neq T_{n,2}$, then $T^{(1-1)}$ must have the form of T_3 in Figure 2. By repeated applications of the transformations **2** and **3** presented in Definitions 3.2 and 3.3, T_3 can be transformed into $T_{n,2}$. By Theorems 3.3 and 3.4, we have $\overline{T_{n,2}} \succ \overline{T_3} \succ \overline{T}$. By (4), $ME(\overline{T_{n,2}}) > ME(\overline{T})$. This completes the proof of Theorem 1.1. ■

The following two lemmas were proved by Yan *et al.* [22].

Lemma 4.1 [22]. *For an arbitrary tree T with n vertices and edge-independence number $\nu(T) = p$, if the number of pendent vertices of T is less than $n - p$, then by repeated applications of the transformation **4** presented in Definition 3.4, T can be transformed into a tree T' with n vertices and with $\nu(T') = p$, the number of pendent vertices of which is exactly $n - p$.*

Lemma 4.2 [22]. *For an arbitrary tree T with n vertices and with $\nu(T) > p$, repeated applications of the transformation **4** presented in Definition 3.4 transform T into a tree T'' with n vertices and with $\nu(T'') = p$, the number of pendent vertices of which is exactly $n - p$.*

Proof of Theorem 1.2. Assume $T \not\cong T_n^p$. Now we prove $ME(\overline{T}) > ME(\overline{T_n^p})$ and distinguish the following three cases.

Case 1. We assume that the edge-independence number of T is p and it has exactly $n - p$ pendent vertices. By Lemma 2.4, the structure of T is clear. It is not difficult that, with repeated applications of the transformations **5** and **6** from Definitions 3.5 and 3.6, T can be transformed into T_n^p . Furthermore, by Theorems 3.7 and 3.8, we have $\overline{T} \succ \overline{T_n^p}$. This indicates, by (4), that $ME(\overline{T}) > ME(\overline{T_n^p})$.

Case 2. Assume $\nu(T) = p$ and the number of pendent vertices of T is less than $n - p$. By Lemma 4.1, T can be transformed into one tree T' with n vertices, $\nu(T') = p$ and the number of pendent vertices of which is exactly $n - p$. If $T' \neq T_n^p$, then, by Theorem 3.5, we have $\overline{T} \succ \overline{T'}$. By Case 1, we note that $\overline{T} \succ \overline{T_n^p}$. If $T' = T_n^p$, then, by Remark 2, we have $\overline{T} \succ \overline{T'}$. Similarly, by Case 1, we have $\overline{T} \succ \overline{T'} \succ \overline{T_n^p}$, which implies $\overline{T} \succ \overline{T_n^p}$. These mean, by (4), that $ME(\overline{T}) > ME(\overline{T_n^p})$.

Case 3. Suppose $\nu(T) > p$. By Lemma 4.2, we know that T can be transformed into one tree T'' with n vertices, $\nu(T'') = p$ and the number of pendent vertices of which is exactly $n - p$. Similarly to Case 2, we can show that $ME(\overline{T}) > ME(\overline{T_n^p})$.

Combining Cases 1–3, the theorem holds. ■

Proof of Theorem 1.4. By Theorems 1.1 and 1.2, it can be seen that $ME\left(\overline{T_n^{\frac{n}{2}}}\right) < ME(\overline{T}) < ME(\overline{P_n})$ and $ME\left(\overline{T_{n,2}^1}\right) < ME(\overline{P_n})$. The following we prove that $ME(\overline{T}) \leq ME\left(\overline{T_{n,2}^1}\right)$ when $T \in \mathcal{T}_{n, \frac{n}{2}}$ and $T \not\cong P_n$.

Assume $T \not\cong P_n$. Similarly to the proof of Theorem 1.1, there exist trees $T^{(i)}$ for $0 \leq i \leq l$ such that

$$\overline{P_n} = \overline{T^{(l)}} \succ \overline{T^{(l-1)}} \succ \overline{T^{(l-2)}} \succ \overline{T^{(l-3)}} \succ \dots \succ \overline{T^{(2)}} \succ \overline{T^{(1)}} \succ \overline{T}.$$

Obviously, $T^{(l-2)} = T_{n,2}^1$ or $T^{(l-2)}$ has the form of T_3 in Figure 2. By (4), we know that if $T^{(l-2)} = T_{n,2}^1$, then $ME(\overline{T}) < ME\left(\overline{T_{n,2}^1}\right)$. If $T^{(l-2)} \neq T_{n,2}^1$, then by repeated applications of the transformations **2** and **3** from Definitions 3.2 and 3.3, T_3 can be transformed into $T_{n,2}^1$. By Theorems 3.3 and 3.4, we have $\overline{T_{n,2}^1} \succ \overline{T}$. By (4), $ME\left(\overline{T_{n,2}^1}\right) > ME(\overline{T})$. ■

Remark 4.3. Denote by $\mathcal{T}_{n,p}$ the proper subset of $\mathcal{T}_{n,p}$ containing all trees with edge-independence number p . Examining Theorem 1.4, we see that if $p = \frac{n}{2}$ in $\mathcal{T}_{n,p}$, then $\overline{P_n}$ and $\overline{T_{n,2}^1}$ have the maximal and second-maximal matching energy, respectively. A natural question is how to characterize the trees with edge-independence number p whose complements have the maximum matching energy in complements of all trees with edge-independence number p .

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