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EXTREMAL MATCHING ENERGY OF COMPLEMENTS OF TREES

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Abstract

Gutman and Wagner proposed the concept of the matching energy which is defined as the sum of the absolute values of the zeros of the matching polynomial of a graph. And they pointed out that the chemical applications of matching energy go back to the 1970s. Let T be a tree with n vertices. In this paper, we characterize the trees whose complements have the maximal, second-maximal and minimal matching energy. Furthermore, we determine the trees with edge-independence number p whose complements have the minimum matching energy for $p = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. When we restrict our consideration to all trees with a perfect matching, we determine the trees whose complements have the second-maximal matching energy.

Keywords: matching polynomial, matching energy, Hosoya index, energy.

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1. Introduction

All graphs considered in this paper are undirected simple graphs. For notation and terminologies not defined here, see [7, 18].

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For any a vertex $v \in V(G)$ (or an edge $e \in E(G)$), let $G - v$ (or $G - e$) denote the subgraph obtained from G by deleting v (or e). Denote by $\overline{G}$ the complement of G. The path, star and complete graph with n vertices are denoted by P_n, $K_{1,n-1}$ and K_n, respectively. Let $T_{n,2}$ be a tree obtained from the star $K_{1,3}$ by attaching a path P_{n-3} to one of the pendent vertices of $K_{1,3}$, and let $T^1_{n,2}$ be a tree obtained from the star $K_{1,3}$ by attaching two paths P_2 and P_{n-4} to two different pendent vertices of $K_{1,3}$, respectively. Let T^p_n be a tree with n vertices obtained from the star $K_{1,n-p}$ by attaching a pendent edge to each of $p-1$ pendent vertices in $K_{1,n-p}$ for $p = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

A k-matching in G is a set of k pairwise non-incident edges. The number of k-matchings in G is denoted by $m(G, k)$. Specifically, $m(G, 0) = 1$, $m(G, 1) = m$ and $m(G, k) = 0$ for $k > \frac{n}{2}$ or $k < 0$. For a k-matching M in G, if G has no k'-matching such that $k' > k$, then M is called a *maximum matching* of G. The number $\nu(G)$ of edges in a maximum matching M is called the *edge-independence number* of G. We use $\mathcal{T}_{n,p}$ to denote the set of trees with n vertices and the edge-independence number at least p for $p = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. The *Hosoya index* $Z(G)$ is defined as the total number of matchings of G, that is

$$Z(G) = \sum\nolimits_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(G, k).$$

Recall that for a graph G on n vertices, the *matching polynomial* $\mu(G, x)$ of G is given by

$$\mu(G, x) = \sum\nolimits_{k \geq 0} (-1)^k m(G, k) x^{n-2k}. \tag{1}$$

Its theory is well elaborated [4, 6, 7, 8, 9]. Gutman and Wagner [10] gave the definition of the *quasi-order* $\succeq$ as follows. If G and H have the matching polynomials in the form (1), then the quasi-order $\succeq$ is defined by

$$G \succeq H \Longleftrightarrow m(G, k) \geq m(H, k) \quad for\ all \quad k = 0, 1, \ldots, \lfloor n/2 \rfloor. \tag{2}$$

Particularly, if $G \succeq H$ and there exists some k such that $m(G, k) > m(H, k)$, then we write $G \succ H$.

Gutman and Wagner in [10] first proposed the concept of the *matching energy* of a graph, denoted by $ME(G)$, and defined as

$$ME = ME(G) = \frac{2}{\pi} \int_0^\infty x^{-2} \ln \left[\sum\nolimits_{k \geq 0} m(G, k) x^{2k} \right] dx. \tag{3}$$

Meanwhile, they gave also another form of the definition of matching energy of a graph. That is,

$$ME(G) = \sum\nolimits_{i=1}^{n} |\mu_i|,$$

where μ_i denotes the root of matching polynomial of G. Additionally, they found some relations between the matching energy and energy (or reference energy). By (2) and (3), we easily obtain the fact as follows.

(4) $\quad G \succeq H \Longrightarrow ME(G) \geq ME(H) \quad$ *and* $\quad G \succ H \Longrightarrow ME(G) > ME(H).$

This property is an important technique to determine extremal graphs with the matching energy.

Note that the energy (or reference energy) of graphs are extensively examined (see [1, 4, 5, 11, 12, 16]). However, the literature on the matching energy is far less than that on the energy and reference energy. Up to now, we find only a few papers about the matching energy published. Gutman and Wagner [10] gave some properties and asymptotic results of the matching energy. Li and Yan [15] characterized the connected graph with the fixed connectivity (resp. chromatic number) which has the maximum matching energy. Ji *et al.* in [13] determined the graphs with the extremal matching energy among all bicyclic graphs. Li *et al.* [14] characterized the unicyclic graphs with fixed girth (resp. clique number) which has the maximum and minimum matching energy, respectively. Chen and Shi [2] characterized the graphs with the maximal value of matching energy among all tricyclic graphs. Chen *et al.* in [3] characterized the graphs with minimal matching energy among all unicyclic and bicyclic graphs with a given diameter d. Xu *et al.* [20] determined the extremal graphs from $\mathcal{T}(n)$ with minimal and maximal matching energies, respectively, where $\mathcal{T}(n)$ is a set of t-apex trees of order n. And they also determined the extremal graphs from $\mathcal{G}_{n,m}$ minimizing the matching energy [21], where $\mathcal{G}_{n,m}$ is a set of connected graphs of order n and with m edges. Additionally, the present author [19] characterized completely the graphs which has i-th maximal matching energy, where $i = 2, 3, \ldots, 16$.

In this paper, inspired by the idea given in [22], we investigate the problem of the matching energy of the complements of trees, and obtain the following main theorems.

Theorem 1.1. *Let T be a tree with n vertices. If $T \ncong T_{n,2}$ and $T \ncong P_n$, then*

$$ME\left(\overline{T}\right) < ME\left(\overline{T_{n,2}}\right) < ME\left(\overline{P_n}\right).$$

Theorem 1.2. *Let $\mathcal{T}_{n,p}$ denote the set of trees with n vertices and the edge-independence number at least p for $p = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. For a tree $T \in \mathcal{T}_{n,p}$ it holds*

$$ME\left(\overline{T}\right) \geq ME\left(\overline{T_n^p}\right)$$

with equality if and only if $T \cong T_n^p$.

By Theorems 1.1 and 1.2, we obtain directly the following corollary.

Corollary 1.3. *The complements of P_n and $K_{1,n-1}$ have the maximum and minimum matching energy in all complements of trees, respectively.*

Theorem 1.4. *Let $\mathscr{T}_{n,\frac{n}{2}}$ be a proper subset of $\mathcal{T}_{n,p}$ containing all trees with a perfect matching. Suppose that $T \in \mathscr{T}_{n,\frac{n}{2}}$, $T \ncong T_n^{\frac{n}{2}}$ and $T \ncong P_n$. If $n \geq 6$, then*

$$ME\left(\overline{T_n^{\frac{n}{2}}}\right) < ME\left(\overline{T}\right) \leq ME\left(\overline{T_{n,2}^1}\right) < ME\left(\overline{P_n}\right),$$

where the equality holds if and only if $T \cong T_{n,2}^1$.

2. Some Lemmas

There exists a well-known formula which characterizes the relation between $m(G,r)$ and $m(\overline{G},i)$ (see Lovász [17]), which will play a key role in the proofs of the main theorems.

Lemma 2.1 [17]**.** *Let G be a simple graph with n vertices and $\overline{G}$ the complement of G. Then*

$$m(G,r) = \sum\nolimits_{i=0}^{r} (-1)^i \binom{n-2i}{2r-2i} (2r-2i-1)!!m(\overline{G},i), \tag{5}$$

where $s!! = s \times (s-2)!!$, and $(-1)!! = 0!! = 1$.

The following results about the matching polynomial of G can be found in Godsil [7].

Lemma 2.2 [7]**.** *The matching polynomial satisfies the following identities:*

(i) $\mu(G \cup H, x) = \mu(G,x)\mu(H,x)$,

(ii) $\mu(G,x) = \mu(G \setminus e, x) - \mu(G-u-v,x)$ *if $e = \{u,v\}$ is an edge of G,*

(iii) $\mu(G,x) = x\mu(G \setminus u, x) - \sum_{v \sim u} \mu(G-u-v,x)$ *if $u \in V(G)$.*

Lemma 2.3 [7]**.** *Let m and n be two positive integers. Then*

$$\mu(P_{m+n}) = \mu(P_m)\mu(P_n) - \mu(P_{m-1})\mu(P_{n-1}). \tag{6}$$

Lemma 2.4 [22]**.** *If T is a tree with n vertices and edge-independence number $\nu(T) = p$, then T has at most $n-p$ vertices of degree one. In particular, if T has exactly $n-p$ vertices of degree one, then every vertex of degree at least two in T is adjacent to at least one vertex of degree one.*

3. ORDERING COMPLEMENTS OF TREES WITH RESPECT TO THEIR MATCHINGS

For convenience, we use the same definitions of trees which are given in [22].

Definition 3.1. Let T_1 be a tree with $n+m+k$ vertices shown in Figure 1, where T_0 is a tree with k vertices ($k \geq 2$) and u a vertex of T_0, $n \geq 1$ and $m \geq 1$. Suppose T_2 is a tree with $n+m+k$ vertices obtained from T_0 by attaching a path P_{m+n} to u in T_0 (see Figure 1). We designate the transformation from T_1 to T_2 as of type **1** and denote it by $\mathcal{F}_1$: $T_1 \hookrightarrow T_2$ or $\mathcal{F}_1(T_1) = T_2$.

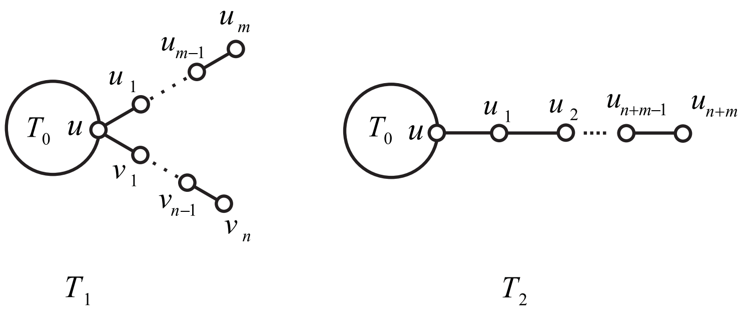

Figure 1. Two trees T_1 and T_2.

Theorem 3.1. *Let T_1 and T_2 be the trees with $m+n+k$ vertices defined in Definition 3.1. Then $\overline{T_2} \succ \overline{T_1}$.*

Proof. By Lemma 2.2,

$$\begin{aligned}\mu(T_1) &= x\mu(T_0-u)\mu(P_m)\mu(P_n) - \mu(T_0-u)\mu(P_{m-1})\mu(P_n)\\ &- \mu(T_0-u)\mu(P_m)\mu(P_{n-1}) - \sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}} \mu(T_0-u-v)\mu(P_m)\mu(P_n),\\ \mu(T_2) &= x\mu(T_0-u)\mu(P_{m+n}) - \mu(T_0-u)\mu(P_{m+n-1})\\ &- \sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}} \mu(T_0-u-v)\mu(P_{m+n}),\end{aligned}$$

where the above sums range over all vertices of T_0 adjacent to u. Hence

$$\mu(T_1)-\mu(T_2) = x\mu(T_0-u)[\mu(P_m)\mu(P_n)-\mu(P_{m+n})] - \mu(T_0-u)[\mu(P_{m-1})\mu(P_n)$$
$$-\mu(P_{m+n-1}) + \mu(P_m)\mu(P_{n-1})] - [\mu(P_m)\mu(P_n)-\mu(P_{m+n})] \sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}} \mu(T_0-u-v).$$

By (6) and a routine calculation,

$$\mu(T_1) - \mu(T_2) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v)\mu(P_{m-1})\mu(P_{n-1}). \tag{7}$$

For an arbitrary vertex v adjacent to u in T_0, let T_v^* be the forest $(T_0 - u - v) \cup P_{m-1} \cup P_{n-1}$, which has $n + m + k - 4$ vertices. By (5), we obtain

$$\begin{aligned} & m\left(\overline{T_1}, r\right) - m\left(\overline{T_2}, r\right) \\ & = \sum_{i=0}^{r} (-1)^i \binom{n+m+k-2i}{2r-2i} (2r-2i-1)!![m(T_1, i) - m(T_2, i)]. \end{aligned} \tag{8}$$

Note that $m(T_1, 0) = m(T_2, 0)$ and $m(T_1, 1) = m(T_2, 1)$. Hence

$$\begin{aligned} & m\left(\overline{T_1}, r\right) - m\left(\overline{T_2}, r\right) \\ & = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \sum_{i=2}^{r} (-1)^i \binom{n+m+k-2i}{2r-2i} (2r-2i-1)!! m(T_v^*, i-2). \end{aligned} \tag{9}$$

Note that T_v^* has $n + m + k - 4$ vertices. So

$$\begin{aligned} m\left(\overline{T_v^*}, r-2\right) &= \sum_{j=0}^{r-2} (-1)^j \binom{n+m+k-4-2j}{2(r-2)-2j} (2(r-2)-2j-1)!! m(T_v^*, j) \\ &= \sum_{i=2}^{r} (-1)^i \binom{n+m+k-2i}{2r-2i} (2r-2i-1)!! m(T_v^*, i-2). \end{aligned}$$

Hence

$$m\left(\overline{T_1}, r\right) - m\left(\overline{T_2}, r\right) = - \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} m\left(\overline{T_v^*}, r-2\right). \tag{10}$$

By the definition of $m(G, r)$ and (10), we have $m(\overline{T_v^*}, r-2) \geq 0$, which implies $m(\overline{T_1}, r) \leq m(\overline{T_2}, r)$. Particularly, if $r = 2$, then $m(\overline{T_1}, r) - m(\overline{T_2}, r) \leq -1$. By (2), $\overline{T_2} \succ \overline{T_1}$. ■

Remark 3.2. By Theorem 3.1 and (4), we obtain immediately a result as follows: If T_1 and T_2 are the two trees defined in Definition 3.1, then $ME(\overline{T_2}) > ME(\overline{T_1})$. Additionally, by the definition of the Hosoya index and Theorem 3.1, it is not difficult to see that $Z(\overline{T_2}) > Z(\overline{T_1})$.

Definition 3.2. Let T_3 and T_4 be two trees with $m + n + s + 1$ vertices shown in Figure 2, where $s \geq m \geq 2, n \geq 1$. We designate the transformation from T_3 to T_4 in Figure 2 as of type **2** and denote it by $\mathcal{F}_2$: $T_3 \mapsto T_4$ or $\mathcal{F}_2(T_3) = T_4$.

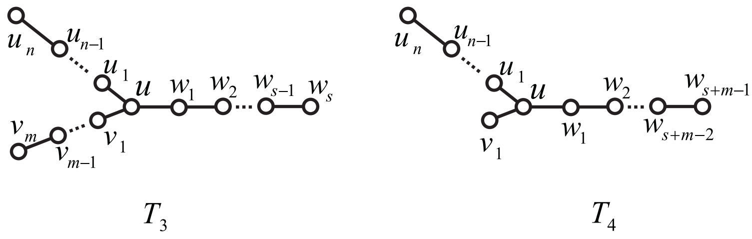

Figure 2. Two trees T_3 and T_4.

Theorem 3.3. *Let T_3 and T_4 be two trees with $m+n+s+1$ vertices defined in Definition* 3.2. *Then $\overline{T_4} \succ \overline{T_3}$.*

Proof. Similarly to the proof of Theorem 3.1, we can obtain that

$$\mu(T_3) - \mu(T_4) = -\mu(P_{m-2})\mu(P_{n-1})\mu(P_{s-2}).$$

Furthermore, we also have

$$m\left(\overline{T_3}, r\right) - m\left(\overline{T_4}, r\right) = -m\left(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}, r-3\right). \tag{11}$$

By the definition of $m(G,r)$ and (11), we have $m\left(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}, r-3\right) \geq 0$, which implies $m\left(\overline{T_3}, r\right) \leq m\left(\overline{T_4}, r\right)$. Especially, if $r = 3$ then

$$m\left(\overline{P_{m-2} \cup P_{n-1} \cup P_{s-2}}, r-3\right) = 1.$$

This means, by (2), that $\overline{T_4} \succ \overline{T_3}$. The proof is completed. ■

Definition 3.3. Let T_5 and T_6 be two trees with $m+n+2$ vertices shown in Figure 3, where $m \geq n \geq 2$. We designate the transformation from T_5 to T_6 in Figure 3 as of type **3** and denote it by $\mathcal{F}_3$: $T_5 \to T_6$ or $\mathcal{F}_3(T_5) = T_6$.

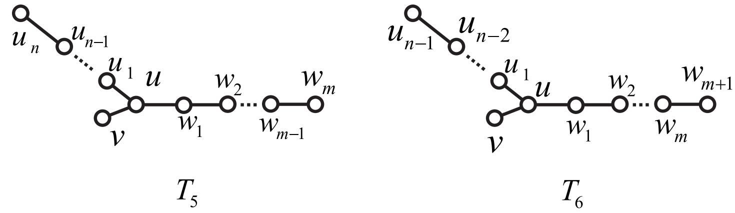

Figure 3. Two trees T_5 and T_6.

Theorem 3.4. *Let T_5 and T_6 be two trees with $m+n+2$ vertices defined in Definition* 3.3. *Then* $\overline{T_6} \succ \overline{T_5}$.

Proof. Similarly to the proof of Theorem 3.1, we have

$$\mu(T_5) - \mu(T_6) = -\mu(P_{m-n})$$

and

$$m\left(\overline{T_5}, r\right) - m\left(\overline{T_6}, r\right) = -m\left(\overline{P_{m-n}}, r-n-1\right). \tag{12}$$

By the definition of $m(G,r)$ and (12), we have $m\left(\overline{P_{m-n}}, r-n-1\right) \geq 0$, which indicates $m\left(\overline{T_5}, r\right) \leq m\left(\overline{T_6}, r\right)$. Especially, when $r = n+1$, then $m\left(\overline{P_{m-n}}, r-n-1\right) = 1$. By (2), we get that $\overline{T_4} \succ \overline{T_3}$. ■

Definition 3.4. Suppose that T_1' and T_2' are two trees with m ($m > 1$) vertices and with n ($n > 1$) vertices, respectively. Take one vertex u of T_1' and one v of T_2'. Construct two trees T_7 and T_8 with $m+n$ vertices as follows. The vertex set $V(T_7)$ of T_7 is $V(T_1') \cup V(T_2')$ and the edge set of T_7 is $E(T_1') \cup E(T_2') \cup uv$. T_8 is the tree obtained from T_1' and T_2' by identifying the vertex u of T_1' and the vertex v of T_2' and adding a pendent edge $uw = vw$ to this new vertex u $(= v)$. The resulting graphs are presented in Figure 4. We designate the transformation from T_7 to T_8 as of type **4** and denote it by $\mathcal{F}_4$: $T_7 \looparrowright T_8$ or $\mathcal{F}_4(T_7) = T_8$.

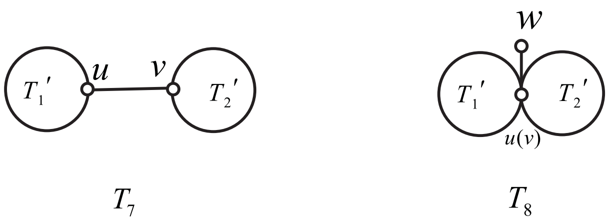

Figure 4. Two trees T_7 and T_8.

Theorem 3.5. *Let T_7 and T_8 be two trees with $m+n$ vertices defined in Definition* 3.4. *Then* $\overline{T_7} \succ \overline{T_8}$.

Proof. By Lemma 2.2,

$$\mu(T_7) = \mu(T_1')\mu(T_2') - \mu(T_1' - u)\mu(T_2' - v), \tag{13}$$

$$\mu(T_8) = x\mu(T_8 - w) - \mu(T_1' - u)\mu(T_2' - v), \tag{14}$$

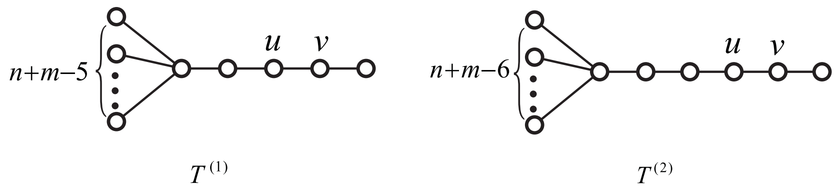

Figure 5. Two trees $T^{(1)}$ and $T^{(2)}$.

$$\mu(T_1') = x\mu(T_1' - u) - \sum_{i=1}^{s} \mu(T_1' - u - u_i) \tag{15}$$

and

$$\mu(T_2') = x\mu(T_2' - v) - \sum_{j=1}^{s} \mu(T_2' - v - v_j), \tag{16}$$

where the first sum ranges over all vertices u_i $(1 \le i \le s)$ of T_1' adjacent to u and the second sum ranges over all v_j $(1 \le j \le t)$ of T_2' adjacent to v. By (15) and (16), we have

$$\begin{aligned} x\mu(T_8 - w) = {} & x^2\mu(T_1' - u)\mu(T_2' - v) - x\sum_{j=1}^{t} \mu(T_1' - u)\mu(T_2' - v - v_j) \\ & - x\sum_{i=1}^{s} \mu(T_2' - v)\mu(T_1' - u - u_i) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \mu(T_1')\mu(T_2') = {} & x^2\mu(T_1' - u)\mu(T_2' - v) - x\sum_{j=1}^{t} \mu(T_1' - u)\mu(T_2' - v - v_j) \\ & - x\sum_{i=1}^{s} \mu(T_2' - v)\mu(T_1' - u - u_i) + \sum_{\substack{1 \le i \le s \\ 1 \le j \le t}} \mu(T_1' - u - u_i)\mu(T_2' - v - v_j). \end{aligned} \tag{18}$$

Combining (13), (14), (17) and (18),

$$\mu(T_7) - \mu(T_8) = \sum_{\substack{1 \le i \le s \\ 1 \le j \le t}} \mu(T_1' - u - u_i)\mu(T_2' - v - v_j). \tag{19}$$

As in the proof of Theorem 3.1, we can show that

$$m(\overline{T_7}, r) - m(\overline{T_8}, r) = \sum_{\substack{1 \le i \le s \\ 1 \le j \le t}} m(\overline{\mu(T_1' - u - u_i) \cup \mu(T_2' - v - v_j)}, r - 2),$$

which implies that

$$m(\overline{T_7}, r) \geq m(\overline{T_8}, r).$$

Note that $m(\overline{T_7}, r) - m(\overline{T_8}, r) \geq 1$ when $r = 2$. So, by (2), the theorem holds. ■

Remark 3.6. For the trees $T^{(1)}$ and $T^{(2)}$ (see Figure 5), we note that neither tree $T^{(1)}$ nor tree $T^{(2)}$ can be transformed into T^p_{m+n} by a single transformation **4**. Hence if T_8 in Theorem 3.5 is T^p_{m+n}, then $\overline{T_7} \succ \overline{T_8} = \overline{T^p_{m+n}}$. Particularly, $\overline{T^p_n} \succ \overline{T^{p-1}_n}$ for $n \geq 5$. Similarly, as in earlier proofs, one can show that this statement holds.

Definition 3.5. Suppose that T_9 is a tree with n vertices and with the edge-independence number p (shown in Figure 6) which has exactly $n - p$ pendent vertices, where $|V(T_0)| \geq 2$ and $r \geq 2$. Let T_{10} be the tree with n vertices shown in Figure 6, which is obtained from T_9. We designate the transformation from T_9 to T_{10} as of type **5** and denote it by $\mathcal{F}_5$: $T_9 \dashrightarrow T_{10}$ or $\mathcal{F}_5(T_9) = T_{10}$.

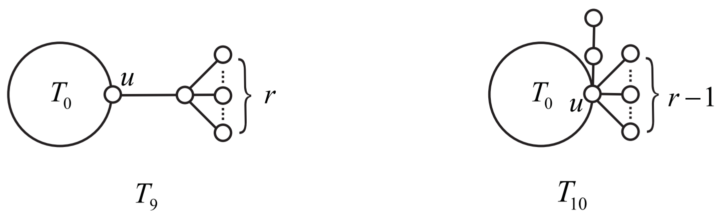

Figure 6. Two trees T_9 and T_{10}.

Theorem 3.7. *Let T_9 and T_{10} be two trees with n vertices defined in Definition 3.5. Then $\overline{T_9} \succ \overline{T_{10}}$.*

Proof. By Lemma 2.2,

$$\begin{aligned}
\mu(T_9) &= x\mu(T_0 - u)\mu(K_{1,r}) - \mu(K_{1,r}) \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) - \mu(T_0 - u)\mu(P_1)^r \\
&= x^2\mu(T_0 - u)\mu(P_1)^r - rx\mu(T_0 - u)\mu(P_1)^{r-1} - x\mu(P_1)^r \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v)
\end{aligned}$$

$$+ r\mu(P_1)^{r-1} \sum_{\substack{v \in V(T_0) \\ uv \in E(T_0)}} \mu(T_0 - u - v) - \mu(T_0 - u)\mu(P_1)^r \tag{20}$$

and

$$
\begin{aligned}
\mu(T_{10}) &= x\mu(T_0-u)\mu(P_2)\mu(P_1)^{r-1} - \mu(P_2)\mu(P_1)^{r-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v)\\
&- \mu(T_0-u)\mu(P_1)^r - (r-1)\mu(T_0-u)\mu(P_2)\mu(P_1)^{r-2}\\
&= x^2\mu(T_0-u)\mu(P_1)^r - x\mu(T_0-u)\mu(P_1)^{r-1}\\
&- x\mu(P_1)^r\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v) + \mu(P_1)^{r-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v) \qquad (21)\\
&- \mu(T_0-u)\mu(P_1)^r - (r-1)x\mu(T_0-u)\mu(P_1)^{r-1}\\
&+ (r-1)\mu(T_0-u)\mu(P_1)^{r-2},
\end{aligned}
$$

where the sum ranges over all vertices of T_0 incident with u.

By (20) and (21), we have

$$\mu(T_9)-\mu(T_{10}) = -(r-1)\mu(T_0-u)\mu(P_1)^{r-2}+(r-1)\mu(P_1)^{r-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v).$$

By Lemma 2.4, there exists at least one pendent vertex v' in T_0 joining vertex u of T_0. Hence, $\mu(T_0-u) = x\mu(T_0-u-v')$, which implies that

$$\mu(T_9)-\mu(T_{10}) = (r-1)\sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}}\mu(P_1)^{r-1}\mu(T_0-u-v).$$

Similarly to the proof of Theorem 3.1,

$$m\left(\overline{T_9},k\right) - m\left(\overline{T_{10}},k\right) = (r-1)\sum_{\substack{v\in V(T_0),v\neq v'\\ uv\in E(T_0)}} m\left(\overline{(r-1)P_1\cup(T_0-u-v)},k-2\right), \qquad (22)$$

for every vertex v ($\neq v'$) of T_0 incident with u. Hence $m\left(\overline{T_9},k\right) \geq m\left(\overline{T_{10}},k\right)$. Furthermore, if $k=2$, then $m\left(\overline{T_9},k\right) - m\left(\overline{T_{10}},k\right) \geq 1$. So $\overline{T_9} \succ \overline{T_{10}}$. ■

Definition 3.6. Suppose that T_{11} is a tree with n vertices and with the edge-independence number p (shown in Figure 7), which has exactly $n-p$ pendent vertices, where $|V(T_0)| \geq 2$, $s \geq 1$ and $t \geq 1$. Let T_{12} be the tree with n vertices shown in Figure 7, which is obtained from T_{11}. We designate the transformation from T_{11} to T_{12} as of type **6** and denote it by $\mathcal{F}_6$: $T_{11} \rightarrowtail T_{12}$ or $\mathcal{F}_6(T_{11}) = T_{12}$.

Theorem 3.8. *Let T_{11} and T_{12} be two trees with n vertices defined in Definition 3.6. Then $\overline{T_{11}} \succ \overline{T_{12}}$.*

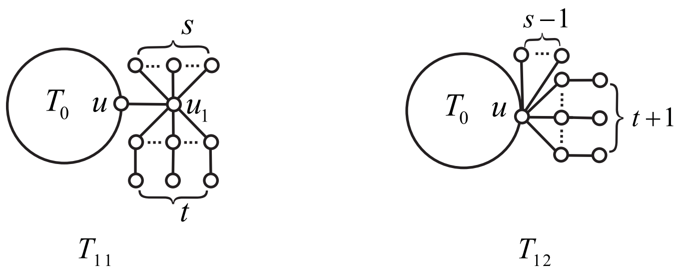

Figure 7. Two trees T_{11} and T_{12}.

Proof. Suppose that $s > 2$. By Lemma 2.2,

$$\begin{aligned}\mu(T_{11}) &= [x^2\mu(P_1)\mu(P_2) - sx\mu(P_2) - tx\mu(P_1)^2 - \mu(P_1)\mu(P_2)] \\ &\quad\times \mu(P_1)^{s-1}\mu(P_2)^{t-1}\mu(T_0-u) - [x\mu(P_1)\mu(P_2) - s\mu(P_2) - t\mu(P_1)^2] \\ &\quad\times \mu(P_1)^{s-1}\mu(P_2)^{t-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v)\end{aligned}$$

and

$$\begin{aligned}\mu(T_{12}) &= [x^2\mu(P_1)^2\mu(P_2) - x\mu(P_1)\mu(P_2) - sx\mu(P_1)\mu(P_2) + s\mu(P_2) - \mu(P_2) \\ &\quad+ x\mu(P_1)\mu(P_2) - (t+1)\mu(P_1)^2\mu(P_2)]\mu(P_1)^{s-2}\mu(P_2)^{t-1}\mu(T_0-u) \\ &\quad- [x\mu(P_1)\mu(P_2) - \mu(P_2)]\mu(P_1)^{s-1}\mu(P_2)^{t-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v),\end{aligned}$$

where the sum ranges over every vertex v of T_0 adjacent to u.

Combining the above two equations, we obtain

$$\begin{aligned}\mu(T_{11}) - \mu(T_{12}) &= -\,[(s+t-1)x^2 - (s-1)]\mu(P_1)^{s-2}\mu(P_2)^{t-1}\mu(T_0-u) \\ &\quad+ [(s+t-1)x^2 - (s-1)]\mu(P_1)^{s-1}\mu(P_2)^{t-1}\sum_{\substack{v\in V(T_0)\\ uv\in E(T_0)}}\mu(T_0-u-v).\end{aligned}$$

By Lemma 2.4, there exists at least one pendent vertex v' of T_0 adjacent to u. Hence, $\mu(T_0-u) = \mu(T_0-u-v')$. Thus, simplifying the above equation, we have

$$\begin{aligned}
&\mu(T_{11}) - \mu(T_{12}) \\
&= [(s+t-1)\mu(P_1)^{s-1}\mu(P_2)^t + t\mu(P_1)^{s-1}\mu(P_2)^{t-1}] \sum_{\substack{v\in V(T_0), v\neq v' \\ uv\in E(T_0)}} \mu(T_0 - u - v).
\end{aligned} \tag{23}$$

As in the proof of Theorem 3.1, we can show that if $s \geq 1$, then

$$\begin{aligned}
& m\left(\overline{T_{11}}, r\right) - m\left(\overline{T_{12}}, r\right) \\
&= (s+t-1) \sum_{\substack{v\in V(T_0), v\neq v' \\ uv\in E(T_0)}} m\left(\overline{(s-1)P_1 \cup tP_2 \cup (T_0 - u - v)}, r-2\right) \\
&+ t \sum_{\substack{v\in V(T_0), v\neq v' \\ uv\in E(T_0)}} m\left(\overline{(s-1)P_1 \cup (t-1)P_2 \cup (T_0 - u - v)}, r-3\right),
\end{aligned}$$

which implies that $m\left(\overline{T_{11}}, r\right) \geq m\left(\overline{T_{12}}, r\right)$. By the above equation, if $r = 2$, then $m\left(\overline{T_{11}}, r\right) - m\left(\overline{T_{12}}, r\right) \geq 1$. By (2) and the properties as above, we have $\overline{T_{11}} \succ \overline{T_{12}}$. ■

4. PROOFS OF THEOREMS 1.1, 1.2 AND 1.4

***Proof of Theorem* 1.1.** We prove that if $T \ncong P_n$ then $ME(\overline{T}) < ME(\overline{P_n})$. By repeated applications of transformation **1** presented in Definition 3.1, we can transform T into P_n, that is, there exist trees $T^{(i)}$ for $0 \leq i \leq l$ such that

$$T = T^{(0)} \hookrightarrow T^{(1)} \hookrightarrow T^{(2)} \hookrightarrow \cdots \hookrightarrow T^{(l-1)} \hookrightarrow T^{(l)} = P_n, \tag{24}$$

where $T^{(l-1)} \neq P_n$. By Theorem 3.1, we have

$$\overline{P_n} = \overline{T^{(l)}} \succ \overline{T^{(l-1)}} \succ \cdots \succ \overline{T^{(2)}} \succ \overline{T^{(1)}} \succ \overline{T}.$$

By (4), we obtain immediately the result as follows:

$$\begin{aligned}
ME\left(\overline{P_n}\right) &= ME\left(\overline{T^{(l)}}\right) > ME\left(\overline{T^{(l-1)}}\right) > \cdots > ME\left(\overline{T^{(2)}}\right) \\
&> ME\left(\overline{T^{(1)}}\right) > ME\left(\overline{T}\right).
\end{aligned}$$

By the transformation **1** presented in Definition 3.1, Theorem 3.1 and (4), it is clear that

$$ME\left(\overline{P_n}\right) > ME\left(\overline{T_{n,2}}\right).$$

Now we show that $ME\left(\overline{T_{n,2}}\right) > ME\left(\overline{T}\right)$. Suppose $T \neq T_{n,2}$. From (24), we know that if $T^{(l-1)} = T_{n,2}$, then $\overline{T_{n,2}} \succ \overline{T}$, which implies $ME\left(\overline{T^{(n,2)}}\right) > ME\left(\overline{T}\right)$. If $T^{(1-1)} \neq T_{n,2}$, then $T^{(1-1)}$ must have the from of T_3 in Figure 2. By repeated applications of the transformations **2** and **3** presented in Definitions 3.2 and 3.3, T_3 can be transformed into $T_{n,2}$. By Theorems 3.3 and 3.4, we have $\overline{T_{n,2}} \succ \overline{T_3} \succ \overline{T}$. By (4), $ME\left(\overline{T_{n,2}}\right) > ME\left(\overline{T}\right)$. This completes the proof of Theorem 1.1. ■

The following two lemmas were proved by Yan *et al.* [22].

Lemma 4.1 [22]**.** *For an arbitrary tree T with n vertices and edge-independence number $\nu(T) = p$, if the number of pendent vertices of T is less than $n - p$, then by repeated applications of the transformation* **4** *presented in Definition* 3.4*, T can be transformed into a tree T' with n vertices and with $\nu(T') = p$, the number of pendent vertices of which is exactly $n - p$.*

Lemma 4.2 [22]**.** *For an arbitrary tree T with n vertices and with $\nu(T) > p$, repeated applications of the transformation* **4** *presented in Definition* 3.4 *transform T into a tree T'' with n vertices and with $\nu(T'') = p$, the number of pendent vertices of which is exactly $n - p$.*

***Proof of Theorem* 1.2.** Assume $T \ncong T_n^p$. Now we prove $ME\left(\overline{T}\right) > ME\left(\overline{T_n^p}\right)$ and distinguish the following three cases.

Case 1. We assume that the edge-independence number of T is p and it has exactly $n - p$ pendent vertices. By Lemma 2.4, the structure of T is clear. It is not difficult that, with repeated applications of the transformations **5** and **6** from Definitions 3.5 and 3.6, T can be transformed into T_n^p. Furthermore, by Theorems 3.7 and 3.8, we have $\overline{T} \succ \overline{T_n^p}$. This indicates, by (4), that $ME\left(\overline{T}\right) > ME\left(\overline{T_n^p}\right)$.

Case 2. Assume $\nu(T) = p$ and the number of pendent vertices of T is less than $n - p$. By Lemma 4.1, T can be transformed into one tree T' with n vertices, $\nu(T') = p$ and the number of pendent vertices of which is exactly $n - p$. If $T' \neq T_n^p$, then, by Theorem 3.5, we have $\overline{T} \succ \overline{T'}$. By Case 1, we note that $\overline{T} \succ \overline{T_n^p}$. If $T' = T_n^p$, then, by Remark 2, we have $\overline{T} \succ \overline{T'}$. Similarly, by Case 1, we have $\overline{T} \succ \overline{T'} \succ \overline{T_n^p}$, which implies $\overline{T} \succ \overline{T_n^p}$. These mean, by (4), that $ME\left(\overline{T}\right) > ME\left(\overline{T_n^p}\right)$.

Case 3. Suppose $\nu(T) > p$. By Lemma 4.2, we know that T can be transformed into one tree T'' with n vertices, $\nu(T'') = p$ and the number of pendent vertices of which is exactly $n - p$. Similarly to Case 2, we can show that $ME\left(\overline{T}\right) > ME\left(\overline{T_n^p}\right)$.

Combining Cases 1–3, the theorem holds. ■

***Proof of Theorem* 1.4.** By Theorems 1.1 and 1.2, it can be seen that $ME\left(\overline{T_n^{\frac{n}{2}}}\right) < ME\left(\overline{T}\right) < ME\left(\overline{P_n}\right)$ and $ME\left(\overline{T_{n,2}^1}\right) < ME\left(\overline{P_n}\right)$. The following we prove that $ME\left(\overline{T}\right) \leq ME\left(\overline{T_{n,2}^1}\right)$ when $T \in \mathscr{T}_{n,\frac{n}{2}}$ and $T \ncong P_n$.

Assume $T \ncong P_n$. Similarly to the proof of Theorem 1.1, there exist trees $T^{(i)}$ for $0 \leq i \leq l$ such that

$$\overline{P_n} = \overline{T^{(l)}} \succ \overline{T^{(l-1)}} \succ \overline{T^{(l-2)}} \succ \overline{T^{(l-3)}} \succ \cdots \succ \overline{T^{(2)}} \succ \overline{T^{(1)}} \succ \overline{T}.$$

Obviously, $T^{(l-2)} = T_{n,2}^1$ or $T^{(l-2)}$ has the from of T_3 in Figure 2. By (4), we know that if $T^{(l-2)} = T_{n,2}^1$, then $ME\left(\overline{T}\right) < ME\left(\overline{T_{n,2}^1}\right)$. If $T^{(l-2)} \neq T_{n,2}^1$, then by repeated applications of the transformations **2** and **3** from Definitions 3.2 and 3.3, T_3 can be transformed into $T_{n,2}^1$. By Theorems 3.3 and 3.4, we have $\overline{T_{n,2}^1} \succ \overline{T}$. By (4), $ME\left(\overline{T_{n,2}^1}\right) > ME\left(\overline{T}\right)$. ■

Remark 4.3. Denote by $\mathscr{T}_{n,p}$ the proper subset of $\mathcal{T}_{n,p}$ containing all trees with edge-independence number p. Examining Theorem 1.4, we see that if $p = \frac{n}{2}$ in $\mathscr{T}_{n,p}$, then $\overline{P_n}$ and $\overline{T_{n,2}^1}$ have the maximal and second-maximal matching energy, respectively. A natural question is how to characterize the trees with edge-independence number p whose complements have the maximum matching energy in complements of all trees with edge-independence number p.

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