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END SIMPLICIAL VERTICES IN PATH GRAPHS

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Abstract

A graph is a path graph if there is a tree, called UV-model, whose vertices are the maximal cliques of the graph and for each vertex x of the graph the set of maximal cliques that contains it induces a path in the tree. A graph is an interval graph if there is a UV-model that is a path, called an interval model. Gimbel [3] characterized those vertices in interval graphs for which there is some interval model where the interval corresponding to those vertices is an end interval. In this work, we give a characterization of those simplicial vertices x in path graphs for which there is some UV-model where the maximal clique containing x is a leaf in this UV-model.

Keywords: chordal graphs, clique trees, path graphs.

2010 Mathematics Subject Classification: 05C75.

1. Introduction

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. Various characterizations of chordal graphs have appeared in the literature [2, 6, 7]. Here we use the classical result of Gavril that states that a graph G is chordal if and only if there is a tree T, called *clique tree*, whose vertices are the maximal cliques of the graph and for every vertex x of G the maximal cliques that contain x, called C_x, induce in the tree a subtree which we will denote by T_x. Clique trees are also called *models* of the graph. A model of a chordal graph often reduces the size of the data structure needed to store the graph, permitting the use of efficient algorithms that take advantage the compactness of the representation [1]. Since some chordal graphs have many distinct models, it is interesting to consider which one is most desirable under various circumstances.

Natural subclasses of chordal graphs are path graphs and interval graphs. In [5], Monma and Wei introduced the notation UV to refer to the class of path graphs. They also proved that a graph G is a *path graph*, or a UV graph, if it admits a *UV-model*, i.e., a clique tree T such that T_x is a subpath of T for every $x \in V(G)$. It is clear that a graph is an *interval graph* if it admits a clique tree which is a path. By definition we have the following inclusions between the different considered classes (and these inclusions are strict): interval $\subset$ path graphs $\subset$ chordal, see Figure 1.

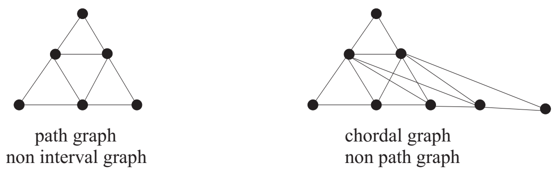

Figure 1.

Given an interval graph G, a vertex is an *end vertex* if the maximal clique that contains it is a leaf in some interval model, see Figure 2. Gimbel [3] characterized those vertices that are end vertices in interval graphs.

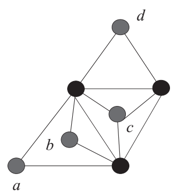

Figure 2. a, b, d are end vertices, c is not an end vertex.

Theorem 1 (Gimbel)**.** *A vertex a in an interval graph G is an end vertex if and only if G contains, as induced subgraphs, none of the graphs presented in Figure* 3, *where a is the designated vertex.*

Simplicial vertices play an important role in chordal graphs; some of them occupy special positions in the models. Blair and Peyton [1] gave a characterization of maximal cliques that are leaves of some model; the simplicial vertices that are in these maximal cliques are called *end vertices of chordal graphs*. We will study the *end vertices of UV graphs*, i.e., simplicial vertices that are in maximal cliques which are leaves in some UV-model. It is clear that if a is an end vertex of G, then there do not exist two non-simplicial vertices $u \neq v$ that are neighbors of a in G such that $C_u \cap C_v = \{Q^a\}$, where Q^a is the unique maximal clique of G that contains a. On the other hand, we observe that among the family given by Gimbel, the graphs in Figure 4 have a UV-model where a is an end vertex.

In this paper, we obtain a characterization of end vertices of UV graphs similar to Gimbel's characterization, stated in the following result.

Theorem 2. *Let G be a UV graph. A simplicial vertex a is an end vertex of G if and only if there do not exist two non-simplicial vertices $u \neq v$ that are neighbors of a in G such that $C_u \cap C_v = \{Q^a\}$ and G contains as induced subgraphs none of the graphs presented in Figure* 5*, where a is the designated vertex.*

Observe that in Figure 5, the graphs G_1, G_2, G_3 and the family G_4 are obtained from Gimbel's graph by adding an universal vertex.

The paper is organized as follows: in Section 2, we give some definitions and background. In Section 3, we prove some lemmas that allow us to restrict the study to UV graphs with certain conditions. Finally, in Section 4, we give a proof of Theorem 2.

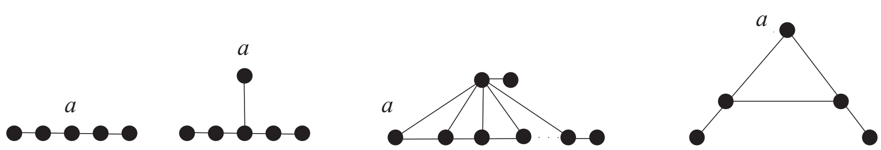

Figure 3. Gimbel's graphs.

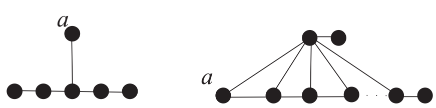

Figure 4. Gimbel's graphs with UV-models where a is an end vertex.

2. Definitions and Background

A *clique* in a graph G is a set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all maximal cliques* of G. We denote by C_x the set of the maximal cliques that contain x.

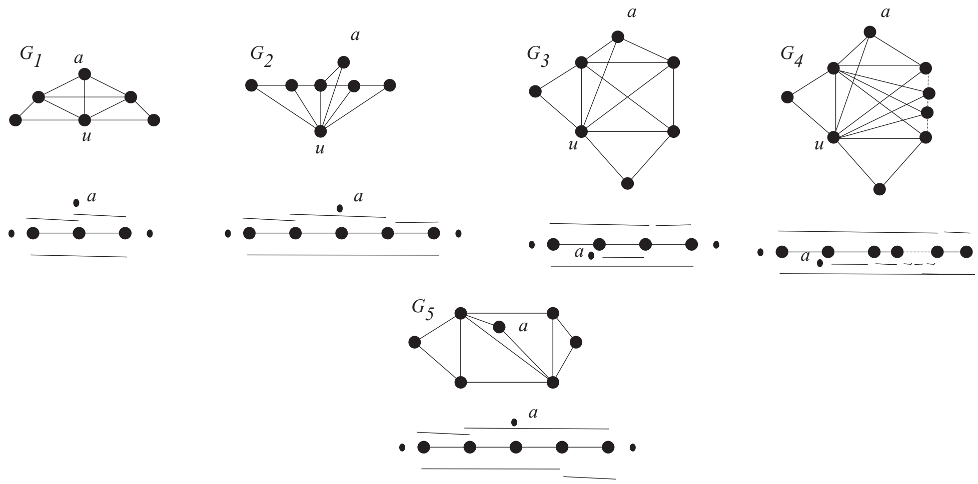

Figure 5. Graphs and its unique UV-models.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x, and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$. A vertex a is *simplicial* if its (closed) neighborhood is a maximal clique, which we will denote Q^a instead of $N[a]$. Two adjacent vertices x and y are *twins* if $N[x] = N[y]$.

A *clique tree* T of a graph G is a tree whose vertices are the elements of $\boldsymbol{C}(G)$ and such that for each vertex x of G, C_x induces a subtree of T, which we will denote by T_x. When T_x is not a subpath of T, we will say that T has a *claw*.

Let F be a finite family of non-empty sets. A graph G is the *intersection graph* for F if there is a one-to-one correspondence between the vertices of G and the sets in F such that two vertices in G are adjacent if and only if the corresponding sets have non-empty intersection.

Note that if T is a clique tree of G, G is the intersection graph of the subtrees $(T_x)_{x \in V(G)}$. Gavril [2] proved that a graph is chordal if and only if it has a clique tree. Clique trees are called *models* of the graph.

A graph is a *path graph* or a *UV graph* (see [5]) if it admits a *UV-model*, i.e., a clique tree T such that T_x is a subpath of T for every $x \in V(G)$.

Let T be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to maximal cliques of G. In order to simplify the notation, we often write $Q \in T$ instead of $Q \in V(T)$, and $e \in T$ instead of $e \in E(T)$. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\bigcup_{Q \in V(T')} Q$.

In a clique tree T, the *label* of an edge QQ' of T is defined as $lab(QQ') = Q \cap Q'$. For each edge e of a clique tree, in every clique tree, there is $\widetilde{e}$ such that $lab(e) = lab(\widetilde{e})$; we will say that e and $\widetilde{e}$ are *equivalent*, see Figure 7. We will say that e, e' in the same clique tree T are *twin edges* if $lab(e) = lab(e')$, see Figure 6.

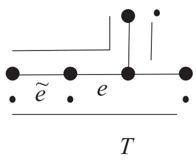

Figure 6. Twin edges.

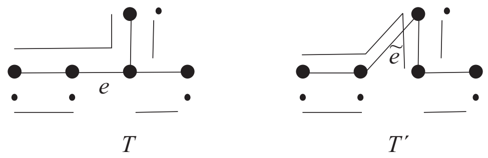

Figure 7. Equivalent edges.

Let T be a UV-model of G, let Q be a vertex of T, and let e be an edge of T. Let T_1 and T_2 be the two connected components of $T - e$ where Q is in T_1. We say that vertices in $lab(e)$ have *the same end* with respect to Q if there exists a vertex Q' in T_1, possibly $Q' = Q$, such that for each $x \in lab(e)$, one endpoint of T_x is Q' (recall that T_x is a subpath of T).

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G + G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and the edges are $E(G) \cup E(G')$. Note that if T, T' are two trees such that $|V(T) \cap V(T')| = 0$, then $T + T'$ is a forest.

Let T be a tree. For $X, Y \in V(T)$, $T[X, Y]$ is the subpath of T between X and Y. Note that the path may be reduced to a single vertex when X and Y are equal. For $e, e' \in E(T)$ and $X \in V(T)$, we denote $Q \in T[e, e']$ if e, Q, e' appear in this order in T.

A simplicial vertex a in a UV graph G is an *end vertex* if there is a UV-model T of G where Q^a is a leaf of T.

3. About Reduced Conditions

The following lemmas allow us to reduce the study to UV graphs with certain conditions.

Lemma 3. *Let G be a UV graph, let T be a UV-model of G, and let a be a simplicial vertex of G that is not an end vertex of G, $u \in N(a)$ is a cut vertex of G. Suppose that $e_i \in T$ for $i = 1, \ldots, n$ are edges whose label is $\{u\}$, and T_1 is the connected component of $T - \{e_1, \ldots, e_n\}$ with $Q^a \in T_1$. Then a is not an end vertex of G_{T_1} or for all $v \in N(a) - \{u\}$, $C_v \cap C_u = \{Q^a\}$.*

Proof. Suppose that a is an end vertex of G_{T_1}, and there is $v \in N(a) - \{u\}$ such that $C_v \cap C_u \neq \{Q^a\}$. As a is an end vertex of G_{T_1}, then there is a UV-model T_1' of G_{T_1} such that Q^a is a leaf. Let $T_2, \ldots, T_k$ be the connected components of $T - \{e_1, \ldots, e_n\}$ different from T_1. Let $Q_1 \in T_1'$ be such that $u \in Q_1$ and the distance in T_1' between Q^a and Q_1 is largest possible. Observe that Q_1 may be Q^a. As there is $v \in N(a) - \{u\}$ such that $C_v \cap C_u \neq \{Q^a\}$, then $Q_1 \neq Q^a$ (recall that Q^a is a leaf of T_1'). Let $Q_i, Q_i' \in T_i$ for $i = 2, \ldots, k$ be such that $(T_i)_u = T_i[Q_i, Q_i']$. It is clear that $T' = T_1' + Q_1Q_2 + T_2 + \sum_{i=2}^{k-1}(Q_i'Q_{i+1} + T_{i+1})$ is a model of G which has Q^a as a leaf, see Figure 8. As $Q_1 \cap Q_2 = Q_i' \cap Q_{i+1} = \{u\}$ and Q_1, Q_i', Q_{i+1} are leaves of $(T_1')_u$, $(T_i)_u$ and $(T_{i+1})_u$ respectively, then T' does not have claws. Therefore, T' is a UV-model of G which has Q^a as a leaf since $Q_1 \neq Q^a$, contradicting the fact that a is not an end vertex.

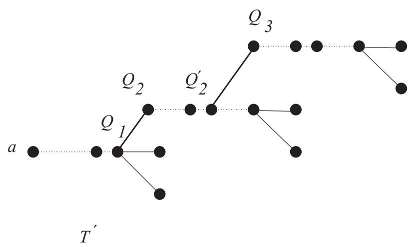

Figure 8. T' UV-model for Lemma 3.

■

Lemma 4. *Let G be a UV graph, let T be a UV-model of G, let a be a simplicial vertex of G that is not an end vertex, let $e \in T$ be an edge that is not incident to Q^a such that for all $x \in lab(e)$, T_x in direction to Q^a in T have the same end, and let T_1 be the connected component of $T - e$ with $Q^a \in T_1$. Then a is not an end vertex of G_{T_1}.*

Proof. Let $e = QQ'$ with $Q' \in T[Q^a, Q]$. Observe that $Q' \neq Q^a$ because e is not incident to Q^a. Suppose that a is an end vertex of G_{T_1}. Then there is a UV-model T_1' of G_{T_1} such that Q^a is a leaf. By our choice of e, the elements of $lab(e)$ are twins in $G_{T_1} = G_{T_1'}$, and e is not incident to Q^a. It follows that, for $x \in lab(e)$, $(T_1')_x = T_1'[Q_1, Q'']$ for some $Q_1, Q'' \in C(G)$ with $Q'' \neq Q^a$ (observe that Q_1 may be Q^a). Let T_2 be the connected component of $T - e$ different from T_1. It is clear that $T' = T_1' + Q''Q + T_2$ is a model of G which has Q^a as a leaf, see Figure 9. As $Q'' \neq Q^a$ is a leaf of $(T_1')_x$ where $x \in lab(e)$, Q is a leaf of $(T_2)_x$ and $lab(e) = lab(Q''Q)$, we conclude that T' does not have claws. Therefore T' is a UV-model of G which has Q^a as a leaf, a contradiction. ■

Lemma 5. *Let G be a UV graph, let T be a UV-model of G, let a be a simplicial vertex of G that is not an end vertex, let $e, e' \in T$ be twin edges where $e = QQ'$,*

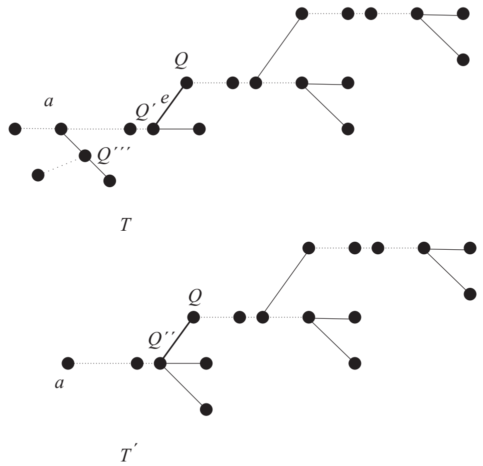

Figure 9. T and T' UV-model for Lemma 4.

$e' = Q_1Q'_1$ with $Q', Q_1 \in T[Q, Q'_1]$, and let T_1, T_2, T_3 be the connected components of $T - \{e, e'\}$ with $Q', Q_1 \in T_2$, $Q^a \notin T_2$, and $T'_1 = T_1 + QQ'_1 + T_3$. Then a is not an end vertex of $G_{T'_1}$.

Proof. Suppose that a is an end vertex of $G_{T'_1}$. Then there is a UV-model T''_1 of $G_{T'_1}$ such that Q^a is a leaf. Let $\overline{QQ'_1}$ be an edge of T''_1 equivalent to QQ'_1 with ends $\overline{Q}$, $\overline{Q'_1}$. It is clear that $T' = T''_1 - \overline{QQ'_1} + \overline{Q}Q' + Q_1\overline{Q'_1} + T_2$ is a model of G which has Q^a as a leaf, see Figure 10. Let L_1 and L_2 be the connected components of $T''_1 - \overline{QQ'_1}$ with $Q^a, \overline{Q} \in L_1$ and $\overline{Q'_1} \in L_2$. For all $x \in lab(e) = lab(e') = lab(\overline{QQ'_1})$, $(T_2)_x = T_2[Q', Q_1]$ and $\overline{Q}$, $\overline{Q'_1}$ are leaves of $(L_1)_x$, $(L_2)_x$ respectively, so T' does not have claws. Therefore T' is a UV-model of G which has Q^a as a leaf. ■

4. Proof of the Main Theorem

Proof. ($\Longrightarrow$) Suppose that G has as induced subgraph G_i, one of the graphs presentaed in Figure 5 for $i \in \{1, 2, 3, 4, 5\}$, and there is a UV-model T of G which has Q^a as a leaf of T. It is possible to build T_i, a UV-model of G_i, from T considering T and $(T_x)_{x \in V(G_i)}$ which has Q^a as a leaf. This is a contradiction because G_i does not have such a UV-model (this can be readily verified, see Figure 5). Hence, in every T, UV-model of G, Q^a is not a leaf of T.

($\Longleftarrow$) Suppose that G is the smallest graph such that a is not an end vertex of G, and there do not exist two non-simplicial vertices $u \neq v$ that are neighbors of a in G such that $C_u \cap C_v = \{Q^a\}$. We will prove that G contains some G_i where $i \in \{1, 2, 3, 4, 5\}$ as induced subgraph. By our assumption, G is a connected

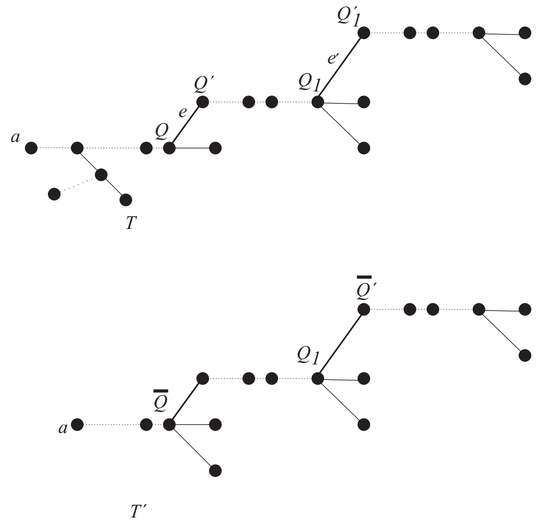

Figure 10. T and T' UV-model for Lemma 5.

graph, there are no two twin simplicial vertices b, c and as a consequence of Lemmas 3, 4, 5, G does not have cut vertices in the neighborhood of a. Also, in each UV-model T of G there are no twin edges in the same connected component of $T - Q^a$, and the elements of the edges whose labels are contained in $N(a)$ do not have the same end to Q^a with the exception of edges incident to Q^a.

Since a is not an end vertex, in each UV-model Q^a is an internal vertex. As G is a connected graph, it follows that $|Q^a| > 1$. By the assumption, G does not have neighbors of a which are cut vertices, so $|Q^a| \geq 3$.

Let T be a UV-model, $e, e' \in T$ with $Q^a \in T[e, e']$ and $lab(e) \subset N(a)$, $lab(e') \subset N(a)$ maximizing the distance between e and e'. By the assumption, $|lab(e)| > 1$ and $|lab(e')| > 1$ since there are no cut vertices that are neighbors of a.

Case 1. If $lab(e) = lab(e')$, then neither e nor e' are incident in leaves of T since the elements in these labels do not have the same end to Q^a. In particular, there are vertices in $lab(e)$ with different ends to Q^a. Namely, we may assume that e, Q^a, e' appear in this order in T, and there are two vertices x and y in $lab(e)$ such that $T_x = T[Q, Q_1]$, $T_y = T[Q_2, Q_3]$, $Q \neq Q_2$, $Q_3 \neq Q_1$. Observe that Q, Q_1, Q_2, Q_3 may be on the same path in T.

Case 1.1. In case that Q, Q_1, Q_2, Q_3 appear on the same path in T we have that Q, Q_2, Q_3, Q_1 (Q_2, Q, Q_3, Q_1) or Q, Q_2, Q_1, Q_3 (Q_2, Q, Q_1, Q_3) appear in this order in T. As $x, y \in lab(e)$ and $lab(e) \subset N(a)$, $\{x, y, a\}$ is a clique.

Case 1.1.1. Suppose that Q, Q_2, Q_3, Q_1 appear in this order in T. So $x, y \in Q_2$ and $x, y \in Q_3$. Let Q'_2, Q'_3 be vertices adjacent to Q_2 and Q_3 respectively with $Q_2, Q_3 \in T[Q'_2, Q'_3]$ and $Q'_2, Q'_3 \in T[Q, Q_1]$. By the choice of e and e', there are

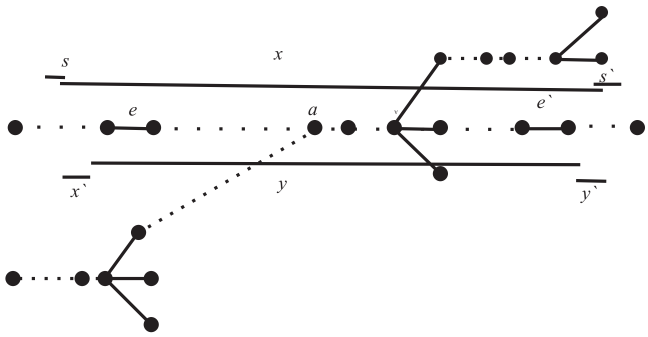

Figure 11. T UV-model, $lab(e) = lab(e')$.

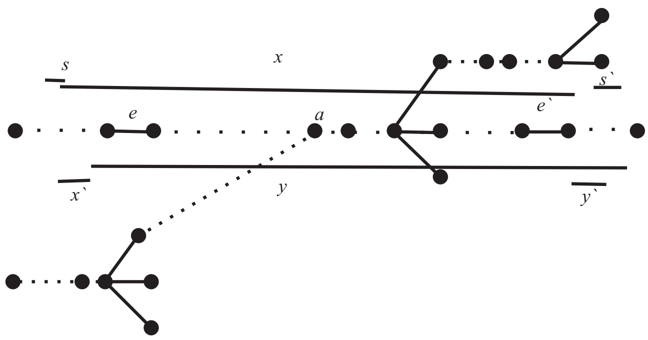

Figure 12. T UV-model, $lab(e) = lab(e')$.

vertices $x' \in Q'_2 \cap Q_2 - N(a)$, $y' \in Q'_3 \cap Q_3 - N(a)$, $s \in Q'_2 - Q_2$ and $s' \in Q'_3 - Q_3$. Observe that $s, s' \notin N(a)$ since $Q'_2, Q_2, Q^a, Q_3, Q'_3$ appear in this order in T, see Figure 11. It is clear that $x'y'$ is not an edge of G by the choice of x', y' and since $x', y' \notin N(a)$. Observe that $\{s, x, x'\}$, $\{x, x', y\}$, $\{x, y, a\}$, $\{x, y, y'\}$, $\{x, y', s'\}$ are cliques. Clearly, $G[s, s', x, x', y, y', a] = G_2$.

Case 1.1.2. Suppose that Q, Q_2, Q_1, Q_3 appear in this order in T. So $x, y \in Q_2 \cap Q_1$. Let Q'_2, Q'_1 be vertices adjacent to Q_2 and Q_1, respectively, with $Q_2, Q_1 \in T[Q'_2, Q'_1]$ and $Q'_2, Q'_1 \in T[Q, Q_3]$. By the choice of e and e', there are vertices $x' \in Q'_2 \cap Q_2 - N(a)$, $y' \in Q'_1 \cap Q_1 - N(a)$, $s \in Q'_2 - Q_2$ and $s' \in Q'_1 - Q_1$. It is clear that $s, s' \notin N(a)$ since $Q'_2, Q_2, Q^a, Q_1, Q'_1$ appear in this order in T, see Figure 12. Observe that $\{s, x, x'\}$, $\{x, x', y\}$, $\{x, y, a\}$, $\{x, y, y'\}$, $\{y, y', s'\}$ are cliques. Clearly, $G[s, s', x, x', y, y', a] = G_5$.

Case 1.2. In case that all four maximal cliques Q, Q_1, Q_2, Q_3 are not on the same path in T, by symmetry we can assume that Q, Q_1, Q_3 are not on the same path in T, see Figure 13. So suppose that Q_1, Q_3 are not. Let Q^*, Q^{**} be vertices of T such that $T_x \cap T_y = T[Q^*, Q^{**}]$. By the assumption, $Q^{**} \neq Q_1, Q_3$.

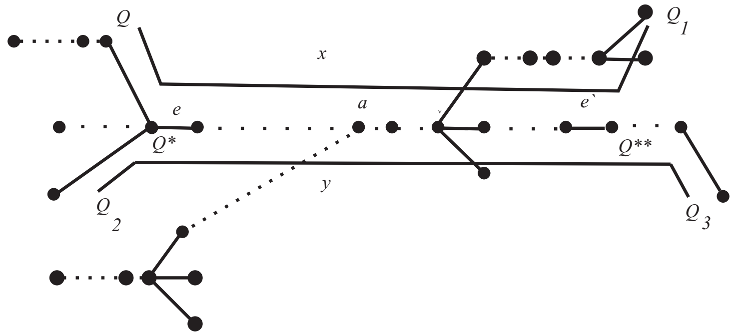

Figure 13. T UV-model, $lab(e) = lab(e')$.

Observe that Q^* may be Q or Q_2 ($Q \neq Q_2$). By symmetry, we may assume that $Q^* \neq Q$. Let $Q_0^* \in T[Q, Q_1]$ be a vertex adjacent to Q^* with $Q^* \in T[Q_0^*, Q^a]$, and $Q_1^* \in T[Q, Q_1]$ be a vertex adjacent to Q^{**} with $Q^{**} \in T[Q^a, Q_1^*]$. Observe that Q_1^* may be Q_1 and Q_0^* may be Q. By the choice of e and e', there are vertices $x' \in Q_0^* \cap Q^* - N(a)$ and $y' \in Q_1^* \cap Q^{**} - N(a)$. Clearly, x', y' are not adjacent, but x', y' are both adjacent to x and y. Let $s' \in Q_1^* - Q^{**}$ and $s \in Q_0^* - Q^*$. Observe that $s', s \notin N(a)$. Since $x, x', s \in Q_0^*$ and $y', x, s' \in Q_1^*$, so $\{x, x', s\}$, $\{y', x, s'\}$ are cliques. Clearly, $G[s', x', x, y, a, y', s] = G_2$.

Case 2. In case that $lab(e) \subset lab(e')$ and $lab(e') \nsubseteq lab(e)$, suppose that e, Q^a, e' appear in this order in T. By our assumption, e' is not incident to a leaf since the elements in $lab(e)$ have different ends to Q^a and there are vertices $x, y \in lab(e) \cap lab(e')$ with different ends to Q^a, $x' \in lab(e') - lab(e)$ such that $T_x = T[Q, Q_1]$, $T_y = T[Q_2, Q_3]$, $T_{x'} = T[Q_5, Q_6]$, and $Q_4 \in T[Q, Q_1] \cap T[Q_5, Q_6]$ is the closest vertex of $T_{x'}$ to Q and Q_2. Clearly, $Q_4 \in T[Q, Q_1] \cap T[Q_2, Q_3]$, $Q_1 \neq Q_3$ and Q, Q_1, Q_3 may be on the same path in T.

Case 2.1. Suppose that Q_1, Q_3, Q_6 are on the same path in T. Then Q_6, Q_1, Q_3 or Q_1, Q_6, Q_3 or Q_1, Q_3, Q_6 or Q_6, Q_3, Q_1 or Q_3, Q_6, Q_1 or Q_3, Q_1, Q_6 appear in this order in T. Observe that Q_6 may be equal to Q_1 or to Q_3. As $x, y, x' \in lab(e')$ and $lab(e') \subset N(a)$, it follows that $\{x, y, x', a\}$ is a clique. Let $e = Q_0Q_0'$ with $Q_0 \in T[Q_0', Q^a]$, and let $s \in Q_0' - Q_0$. Clearly, $s \notin N(a)$. Since $x, y \in lab(e)$, so $x, y \in Q_0'$. Also $x' \notin lab(e)$, so x', s are not adjacent. Hence, $\{x, y, s\}$ is a clique.

Case 2.1.1. Suppose that Q_6, Q_1, Q_3 appear in this order in T. Let Q_1' be a vertex adjacent to Q_1 with $Q_1' \in T[Q_1, Q_3]$. By the choice of e', there are vertices in the labels of edges in $T[Q_6, Q_1']$ that are not neighbors of a. Let $x_1, \ldots, x_k$ be vertices chosen in the labels of these edges such that x_i is only adjacent to x_{i+1} for $i = 1, \ldots, k-1$, $x_1 \in Q_6$, $x_k \in Q_1'$, $s' \in Q_1' - Q_1$. $s' \notin N(a)$ since Q^a,

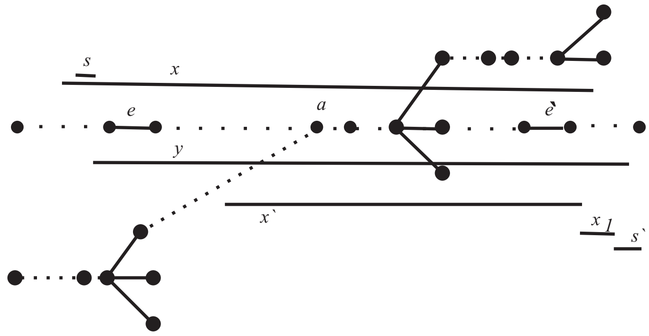

Figure 14. T UV-model, $lab(e) \subset lab(e')$ and $lab(e') \not\subseteq lab(e)$.

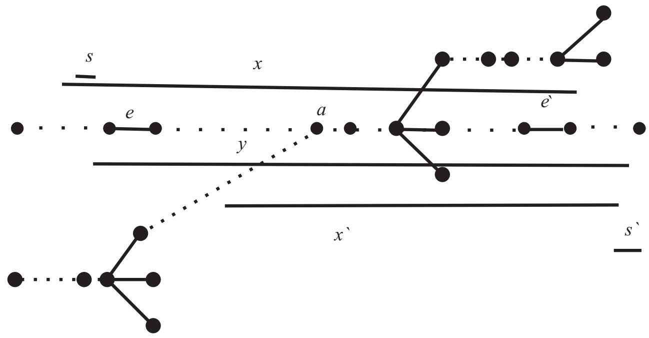

Figure 15. T UV-model, $lab(e) \subset lab(e')$ and $lab(e') \not\subseteq lab(e)$.

Q_1, Q_1' appear in this order in T, see Figure 14. Observe that k may be equal to 1. Moreover, $\{x, y, s\}$, $\{x, y, x', a\}$, $\{x, y, x', x_1\}$, $\{x, y, x_i, x_{i+1}\}_{i=1,\ldots,k-1}$ and $\{x_k, y, s'\}$ are cliques. Clearly, $G[s, s', x, x', x_1, \ldots, x_k, y, a] = G_i$ for $i = 3$ if $k = 1$ or $i = 4$, otherwise. It is easy to see that we obtain the same induced subgraphs if the order is Q_6, Q_3, Q_1.

Case 2.1.2. Suppose that Q_1, Q_6, Q_3 or Q_1, Q_3, Q_6 appear in this order in T. In the first case, we can assume that $Q_1 \neq Q_6$, since otherwise we obtain G_i for $i = 3$ or $i = 4$ following the earlier arguments. Let $s' \in Q_6 - Q_1$ if Q_1, Q_6, Q_3 appear in this order in T or $s' \in Q_3 - Q_1$ if Q_1, Q_3, Q_6 appear in this order in T. Clearly, $s' \notin N(a)$ and s', y are adjacent. As $Q_1 \neq Q_6, Q_3$, x, s' are not adjacent, see Figure 15. Observe that $\{s, x, y\}$, $\{x, y, x', a\}$, $\{x', y, s'\}$ are cliques. Clearly, $G[x, x', y, s, s', a] = G_1$. It is easy to see that we obtain the same induced subgraphs if the order is Q_3, Q_6, Q_1 or Q_3, Q_1, Q_6.

Case 2.2. Now suppose that Q_1, Q_3, Q_6 are not on the same path in T. Let $Q_1^* \in T_x \cap T_y$ be the closest vertex to Q_1 and Q_3, let $Q_6^* \in T_x \cap T_y \cap T_{x'}$ be the closest vertex to Q_1^*. Observe that Q_1^* may be Q_6^*. Clearly, Q_1^*, Q_6^*, Q_3 are on the

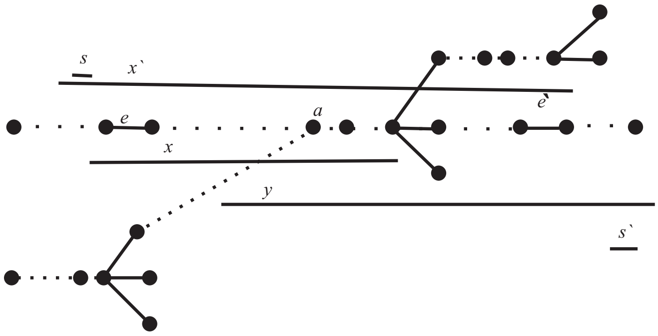

Figure 16. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') \neq \emptyset$.

same path in T and Q_6^*, Q_1^*, Q_3 or Q_1^*, Q_6^*, Q_3 appear in this order in T. Following the earlier arguments we obtain G_i for $i = 1$ or $i = 3$ or $i = 4$.

Case 3. In case that $lab(e) \nsubseteq lab(e')$ and $lab(e') \nsubseteq lab(e)$, there are $x \in lab(e) - lab(e')$ and $y \in lab(e') - lab(e)$. Clearly, $\{x, y, a\}$ is a clique since $lab(e) \subset N(a)$ and $lab(e') \subset N(a)$. Let $e = Q_0Q_0'$ with $Q_0 \in T[Q_0', Q^a]$ and $e' = \overline{Q_0}\overline{Q_0'}$ with $\overline{Q_0} \in T[Q^a, \overline{Q_0'}]$.

Case 3.1. If there is a vertex $x' \in lab(e) \cap lab(e')$, then let $s \in Q_0' - Q_0$ and $s' \in \overline{Q_0'} - \overline{Q_0}$. Clearly, $s, s' \notin N(a)$. As $x, x' \in Q_0'$ and $y \notin Q_0'$, s is adjacent to x, x' but not adjacent to y. Since $y, x' \in \overline{Q_0'}$ and $x \notin \overline{Q_0'}$, s' is adjacent to y, x' but not adjacent to x, see Figure 16. Observe that $\{s, x, x'\}$, $\{x, x', y, a\}$, $\{x', y, s'\}$ are cliques. Clearly, $G[x, x', y, s, s', a] = G_1$.

Case 3.2. If there is not a vertex in $lab(e) \cap lab(e')$, then since there are no cut vertices in G, we get $|lab(e) - lab(e')| > 1$ and $|lab(e') - lab(e)| > 1$. Thus there exist $x, x' \in lab(e) - lab(e')$ with different ends to Q^a and $y, y' \in lab(e') - lab(e)$ with different ends to Q^a. It is clear that $\{x, y, x', y', a\}$ is a clique since $lab(e) \subset N(a)$ and $lab(e') \subset N(a)$. Let $T_x = T[Q, Q_1]$, $T_{x'} = T[Q_2, Q_3]$ with $Q_1 \neq Q_3$ and let Q_4, Q_5 be the ends of T_x and $T_{x'}$, respectively, between e and e' with Q^a, Q_4, Q_5 appearing in this order in T, see Figure 18. Let $Q_1^* \in T_x \cap T_{x'}$ be the closest vertex to Q_1 and Q_3. Observe that Q_1^* may be Q_1 or Q_3 or Q_4. We may assume that $Q_1^* \neq Q_3$ and Q^a, Q_1^*, Q_3 appear in this order in T. Observe that Q_1^* may be Q_4, and Q_3 may be Q_5.

Case 3.2.1. Suppose that $Q_4 \neq Q_5$. Clearly, there is $s' \in Q_5 - Q_4$ and $y \in Q_4 \cap Q_5$; $s' \notin N(a)$ since Q^a, Q_4, Q_5 appear in this order in T. Also s' is not adjacent to x since $Q_4 \neq Q_5$. Observe that $\{s', x', y\}$ is a clique. Let $s \in Q_0' - Q_0$. Clearly $s \notin N(a)$. As $y \notin lab(e)$ but $x, x' \in lab(e)$ it follows that s is adjacent to x and x' but not adjacent to y, see Figure 17. The sets $\{x, x', y, a\}$, $\{x, x', s\}$ and $\{s', x', y\}$ are cliques and $G[x, x', y, s, s', a] = G_1$.

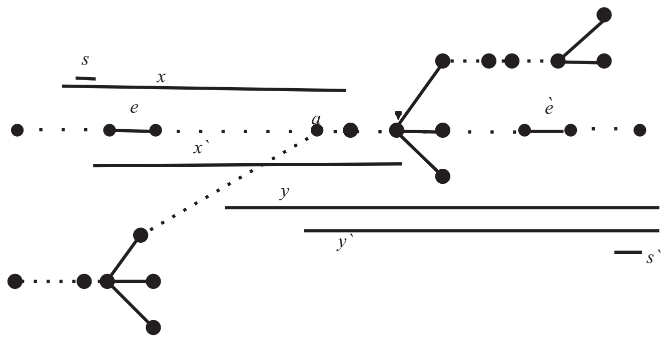

Figure 17. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') = \emptyset$.

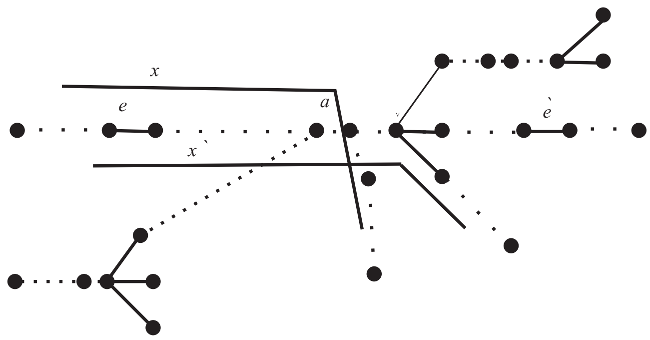

Figure 18. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') = \emptyset$.

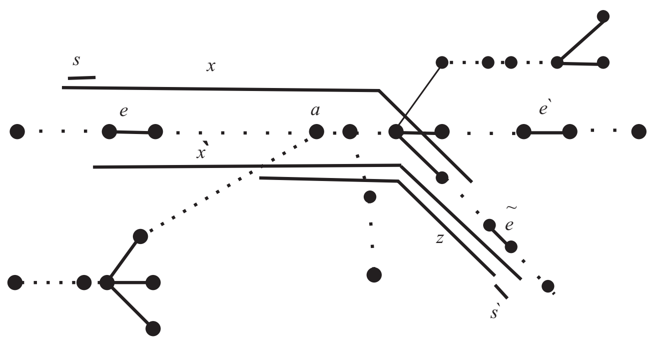

Figure 19. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') = \emptyset$.

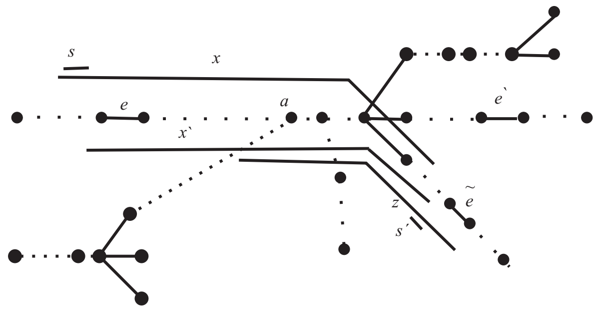

Figure 20. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') = \emptyset$.

Case 3.2.2.i. Suppose that $Q_4 = Q_5$ and there is an edge $\widetilde{e}$ such that $lab(\widetilde{e}) \subset N(a)$; $Q^a, Q_1^*, Q_3, \widetilde{e}$ or $Q^a, Q_1^*, \widetilde{e}, Q_3$ or $Q^a, Q_4, \widetilde{e}, Q_1^*, Q_3$ appear in this order in T. Choose $\widetilde{e}$ maximizing the distance between $\widetilde{e}$ and e. By *Case* 2, we can assume that $lab(\widetilde{e}) \nsubseteq lab(e)$; $lab(e) \nsubseteq lab(\widetilde{e})$ and $lab(\widetilde{e}) \cap lab(e) \cap N(a) = \emptyset$.

• In case that Q_1^*, $\widetilde{e}$, Q_3 appear in this order in T, since $lab(\widetilde{e}) \nsubseteq lab(e)$, there is $z \in lab(\widetilde{e}) - lab(e)$. Let $s \in Q_0' - Q_0$. Clearly $s \notin N(a)$. As $x, x' \in Q_0'$ and $z \notin Q_0'$, so s is adjacent to x and x' but is not adjacent to z. Since Q_1^*, $\widetilde{e}$, Q_3 appear in this order, and x' is in $lab(\widetilde{e})$. Let $\widetilde{e} = \widetilde{Q_0}\widetilde{Q_0'}$ with $\widetilde{Q_0} \in T[Q^a, \widetilde{Q_0'}]$ and $s' \in \widetilde{Q_0'} - \widetilde{Q_0}$. Clearly $s' \notin N(a)$. As $x', z \in lab(\widetilde{e})$ and $x \notin lab(\widetilde{e})$, we get $x', z \in \widetilde{Q_0'}$ and $x \notin \widetilde{Q_0'}$. Hence, s' is adjacent to x' and z but is not adjacent to x, see Figure 19. Observe that $\{x, x', z, a\}$, $\{x, x', s\}$ and $\{s', x', z\}$ are cliques and $G[x, x', y, s, s', a] = G_1$.

• In case that Q_1^*, Q_3, $\widetilde{e}$ appear in this order in T, since $lab(\widetilde{e}) \nsubseteq lab(e)$ there is $z \in lab(\widetilde{e}) - lab(e)$. Let $s \in Q_0' - Q_0$. Clearly $s \notin N(a)$ and is adjacent to x and x' but not adjacent to z. Since Q_1^*, Q_3, $\widetilde{e}$ appear in this order and $z \in N(a)$ so $z \in Q_3 \cap Q_1^*$. As $Q_1^* \neq Q_3$, let $s' \in Q_3 - Q_1^*$; clearly $\{z, x', s\}$ is a clique, see Figure 20. Since Q^a, Q_4, Q_1^*, Q_3 appear in this order in T, $s' \notin N(a)$. The sets $\{x, x', z, a\}$, $\{x, x', s\}$ and $\{s', x', z\}$ are cliques and $G[x, x', z, s, s', a] = G_1$.

• Finally, suppose that Q_4, $\widetilde{e}$, Q_1^* appear in this order in T. Since $lab(\widetilde{e}) \nsubseteq lab(e)$ and $lab(e) \nsubseteq lab(\widetilde{e})$ we conclude that there are $z \in lab(\widetilde{e}) - lab(e)$ and $v \in lab(e) - lab(\widetilde{e})$. Observe that $v \neq x, x'$ since $x, x' \in lab(\widetilde{e})$ (this follows from the fact that Q_4, $\widetilde{e}$, Q_1^* appear in this order in T). Let $s \in Q_0' - Q_0$, let $\widetilde{e} = \widetilde{Q_0}\widetilde{Q_0'}$ with $\widetilde{Q_0} \in T[Q^a, \widetilde{Q_0'}]$ and $s' \in \widetilde{Q_0'} - \widetilde{Q_0}$. Clearly $s, s' \notin N(a)$. Since $v, x' \in lab(e)$ and $z, x' \in lab(\widetilde{e})$, s is adjacent to v and x' but is not adjacent to z and s' is adjacent to x', z but is not adjacent to v, see Figure 21. Observe that $\{v, x', z, a\}$, $\{v, x', s\}$ and $\{s', x', z\}$ are cliques and $G[v, x', z, s, s', a] = G_1$.

Case 3.2.2.ii. Suppose that $Q_4 = Q_5$ and there is not an edge $\widetilde{e}$ such that $lab(\widetilde{e}) \subset N(a)$, and $Q^a, Q_4, Q_3, \widetilde{e}$ or $Q^a, Q_4, \widetilde{e}, Q_3$ or $Q^a, Q_4, \widetilde{e}, Q_1^*, Q_3$ ap-

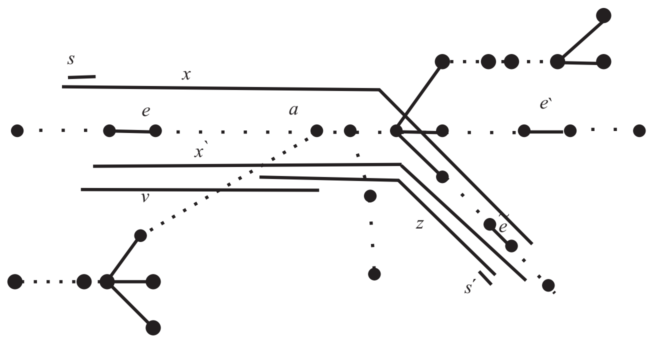

Figure 21. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') = \emptyset$.

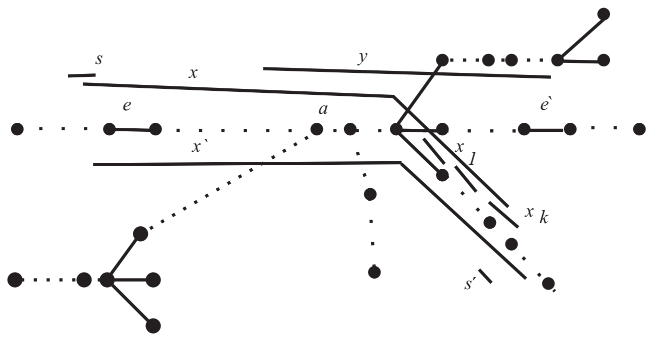

Figure 22. T UV-model; $lab(e) \nsubseteq lab(e')$, $lab(e') \nsubseteq lab(e)$ and $lab(e) \cap lab(e') = \emptyset$.

pear in this order in T. Then $Q_4 \neq Q^a$. Also there are vertices in the labels of edges in $T[Q_4, Q_3]$ that are not neighbors of a. Let Q'_1 be adjacent to Q^*_1 with $Q'_1 \in T[Q^*_1, Q_3]$ and let $x_1, \ldots, x_k$ be vertices chosen in these labels such that x_i is only adjacent to x_{i+1} for $i = 1, \ldots, k-1$, $x_1 \in Q_4$, $x_k \in Q'_1$, $s' \in Q'_1 - Q^*_1$ and $s \in Q'_0 - Q_0$. Clearly, $s \notin N(a)$ and s is adjacent to x, x' but not adjacent to y since $y \in lab(e') - lab(e)$. Observe that k may be equal 1 and $y \in Q_4 - Q'_1$ (Q'_1 is not between e and e'), see Figure 22. Moreover, $\{x, x', s\}$, $\{x, y, x', a\}$, $\{x, y, x', x_1\}$, $\{x, y, x_i, x_{i+1}\}_{i=1,\ldots,k-1}$ and $\{x_k, x', s'\}$ are cliques. Clearly, $G[s, s', x, x', x_1, \ldots, x_k, y, a] = G_i$ for $i = 3$ if $k = 1$ or $i = 4$, otherwise. ∎

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