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A CHARACTERIZATION OF HYPERGRAPHS WITH LARGE DOMINATION NUMBER

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Abstract

Let $H = (V, E)$ be a hypergraph with vertex set V and edge set E. A dominating set in H is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. The domination number $\gamma(H)$ is the minimum cardinality of a dominating set in H. It is known [Cs. Bujtás, M.A. Henning and Zs. Tuza, *Transversals and domination in uniform hypergraphs*, European J. Combin. **33** (2012) 62–71] that for $k \geq 5$, if H is a hypergraph of order n and size m with all edges of size at least k and with no isolated vertex, then $\gamma(H) \leq (n + \lfloor (k-3)/2 \rfloor m)/(\lfloor 3(k-1)/2 \rfloor)$. In this paper, we apply a recent result of the authors on hypergraphs with large transversal number [M.A. Henning and C. Löwenstein, *A characterization of hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem*, Discrete Math. **323** (2014) 69–75] to characterize the hypergraphs achieving equality in this bound.

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1. Introduction

In this paper we continue the study of domination in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*.

We shall use the notation $n_H = |V|$ and $m_H = |E|$, and sometimes simply n and m without subscript if the actual H need not be emphasized, to denote the order and size of H, respectively. A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$.

The *degree* of a vertex v in H, denoted by $d_H(v)$ or simply by $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The minimum degree among the vertices of H is denoted by $\delta(H)$. We define a hypergraph H to be *edge-minimal* if every edge of H contains at least one vertex of degree 1 in H.

If H' is a hypergraph such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$, then H' is called a *subhypergraph* of H. Possibly, $H' = H$. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2, \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components. A component of a hypergraph H that is isomorphic to a hypergraph F we call an *F-component* of H.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a *$\tau(H)$-set*.

A *dominating set* in H is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. Equivalently, every vertex $v \in V \setminus D$ is adjacent to a vertex in D. The *domination number* $\gamma(H)$ is the minimum cardinality of a dominating set in H. A dominating set of H of cardinality $\gamma(H)$ is called a *$\gamma(H)$-set*.

If $X, Y \subseteq V$, then we say that Y *dominates* X if every vertex in X is in Y or is adjacent to some vertex of Y. If X is a nonempty subset of vertices in H, then we define an *X-dominating set* in H as a set Y that dominates X and we define the *X-domination number*, denoted $\gamma(X; H)$ as the minimum cardinality of an X-dominating set in H. In particular, we note that $\gamma(H) = \gamma(V; H)$.

Domination in graphs is still a very active area inside graph theory; see, for example, the recent papers [6, 11, 14]. Domination in hypergraphs, however, was introduced recently by Acharya [1] and studied further, for example, in [2, 3, 4, 8, 12].

Bujtás, Henning, Tuza [4] proved the following upper bounds on the domination number of a hypergraph.

Theorem A [4]**.** *For $k \ge 3$, if H is a hypergraph of order n and size m with all edges of size at least k and with $\delta(H) \ge 1$, then*

$$\gamma(H) \le \frac{n + \left\lfloor \frac{k-3}{2} \right\rfloor m}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}.$$

Theorem B [4]**.** *If H is a hypergraph of order n with all edges of size at least* 5 *and with $\delta(H) \ge 1$, then $\gamma(H) \le 2n/9$.*

We have two immediate aims in this paper. Our first aim is to characterize the extremal hypergraphs in Theorem A for each $k \ge 5$. Our second aim is to characterize the extremal hypergraphs in Theorem B. For this purpose, we define three special families of hypergraphs.

1.1. Special families of hypergraphs

For $k \ge 4$, let E_k denote the k-uniform hypergraph on k vertices with exactly one edge. The hypergraph E_4 is illustrated in Figure 1.

For $k \ge 4$, we define the k-uniform hypergraph T_k as follows. Let A, B, C and D be vertex-disjoint sets of vertices with $|A| = \lceil k/2 \rceil$, $|B| = |C| = \lfloor k/2 \rfloor$ and $|D| = \lceil k/2 \rceil - \lfloor k/2 \rfloor$. In particular, if k is even, the set $D = \emptyset$, while if k is odd, the set D consists of a singleton vertex. Let T_k denote the k-uniform hypergraph with $V(T_k) = A \cup B \cup C \cup D$ and with $E(T_k) = \{e_1, e_2, e_3\}$, where $V(e_1) = A \cup B$, $V(e_2) = A \cup C$, and $V(e_3) = B \cup C \cup D$. The hypergraphs T_4 and T_5 are illustrated in Figure 1.

For odd $k \ge 5$, we define the hypergraph T_k^* as follows. Let A, B and C be vertex-disjoint sets of vertices with $|A| = |B| = (k+1)/2$ and $|C| = (k-1)/2$. Let T_k^* denote the hypergraph with $V(T_k^*) = A \cup B \cup C$ and with $E(T_k^*) = \{e_1, e_2, e_3\}$, where $V(e_1) = A \cup B$, $V(e_2) = A \cup C$, and $V(e_3) = B \cup C$. The hypergraph T_5^* is illustrated in Figure 1. Note that every edge in T_k^* has size at least k.

Given a hypergraph $H = (V, E)$, let H' be obtained from H by expanding every edge in H by adding to it one new vertex, where all added vertices have degree 1 in H'. We call $H' = (V', E')$ the *expanded hypergraph* $\mathrm{expa}(H)$ of H. That is for every edge $e \in E$, if v_e denotes the new vertex added to e, where $v_e \ne v_f$ for edges $e \ne f$ in H, then $E' = \{V(e) \cup \{v_e\} \mid e \in E\}$ and $V' = V \cup \bigcup_{e \in E} \{v_e\}$.

For $k \geq 5$, let $D_k = \mathrm{expa}(T_{k-1})$. The hypergraphs D_5 and D_6 are illustrated in Figure 2. Note that D_k is k-uniform.

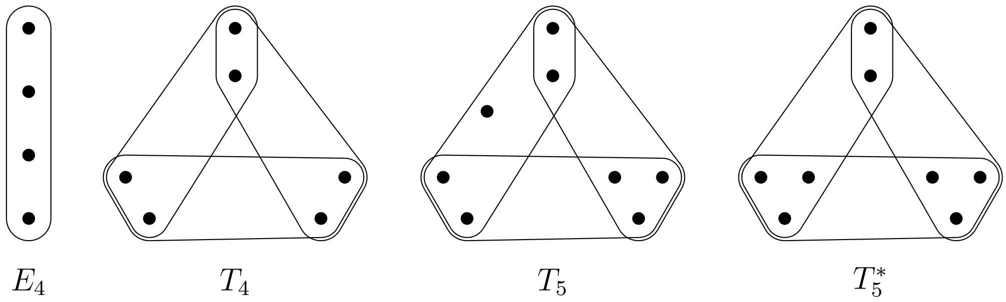

Figure 1. The hypergraphs E_4, T_4, T_5 and T_5^*.

For even $k \geq 6$, let $D_k^* = \mathrm{expa}(T_{k-1}^*)$. The hypergraph D_6^* is illustrated in Figure 2. Note that D_k^* is of edge size at least k.

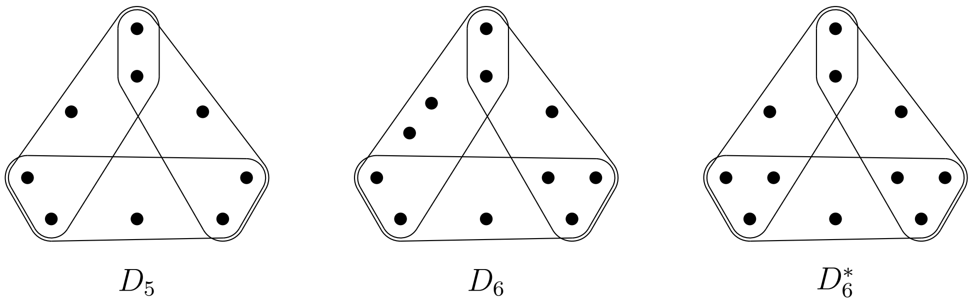

Figure 2. The hypergraphs D_5, D_6 and D_6^*.

We are now in a position to define our three special families of hypergraphs.

The Family $\mathcal{T}_k$. For even $k \geq 4$, we define $\mathcal{T}_k = \{E_k, T_k\}$ and for odd $k \geq 5$, we define $\mathcal{T}_k = \{E_k, T_k, T_k^*\}$.

The Family $\mathcal{D}_k$. For odd $k \geq 5$, we define $\mathcal{D}_k = \{E_k, D_k\}$ and for even $k \geq 6$, we define $\mathcal{D}_k = \{E_k, D_k, D_k^*\}$.

The Family $\mathcal{H}$. Let D_5 be the hypergraph shown in Figure 2. Let H_{under} be a hypergraph every component of which is isomorphic to D_5. Let H be a hypergraph obtained from H_{under} by adding edges of size at least five, called *link edges*, in such a way that every added edge contains only vertices of degree 2 in H_{under}. Possibly, H is disconnected or $H = D_5$. We call the hypergraph H_{under} an *underlying hypergraph* of H. Let $\mathcal{H}$ denote the family of all such hypergraphs H.

2. Main Result

Our aim in this paper is to characterize the extremal hypergraphs in Theorem A and Theorem B. We shall prove:

Theorem 1. *For $k \geq 5$, let H be a connected hypergraph on n vertices and m edges with edge size at least k and with $\delta(H) \geq 1$. Then,*

$$\gamma(H) \leq \frac{n + \left\lfloor \frac{k-3}{2} \right\rfloor m}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}$$

with equality if and only if $H \in \mathcal{D}_k$.

Theorem 2. *Let H be a connected hypergraph on n vertices with edge size at least 5 and with $\delta(H) \geq 1$. Then, $\gamma(H) \leq 2n/9$, with equality if and only if $H \in \mathcal{H}$.*

A proof of Theorem 1 is presented in Section 5, while a proof of Theorem 2 is given in Section 6.

3. Known Results on Transversals in Hypergraphs

Chvátal and McDiarmid [5] established the following upper bound on the transversal number of a uniform hypergraphs in terms of its order and size.

Chvátal-McDiarmid Theorem. *For $k \geq 2$, if H is a k-uniform hypergraph on n vertices with m edges, then $\tau(H) \leq (n + \left\lfloor \frac{k}{2} \right\rfloor m)/(\left\lfloor \frac{3k}{2} \right\rfloor)$.*

The extremal connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem when $k = 3$ were characterized by Henning and Yeo [7]. For $k = 2$ and $k \geq 4$, the extremal hypergraphs were characterized by the authors in [10].

Theorem C [10]**.** *For $k = 2$ and $k \geq 4$, let H be a connected k-uniform hypergraph on n vertices and m edges. Then, $\tau(H) \leq (n + \left\lfloor \frac{k}{2} \right\rfloor m)/(\left\lfloor \frac{3k}{2} \right\rfloor)$ with equality if and only if $H \in \{E_k, T_k\}$.*

In the special case when $k = 4$, Lai and Chang [13] established the following upper bound on the transversal number of a k-uniform hypergraph.

Lai-Chang Theorem. *If H is a 4-uniform hypergraph on n vertices and m edges, then $\tau(H) \leq 2(n + m)/9$.*

The extremal hypergraphs achieving equality in the Lai-Chang Theorem were characterized by the authors in [9].

Theorem D [9]**.** *Let H be a connected hypergraph on n vertices and m edges with edge size at least 4. If $\tau(H) = 2(n+m)/9$, then $H = T_4$.*

4. Preliminary Results

In this section, we present some preliminary observations, lemmas and theorems that we will need in proving our main results. Our first result is a generalization of Theorem C where we relax the k-uniformity condition to edge sizes at least k.

Theorem 3. *For $k \geq 4$, let H be a connected hypergraph on n vertices and m edges with edge size at least k and with $\delta(H) \geq 1$. Then,*

$$\tau(H) \leq \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}$$

with equality if and only if $H \in \mathcal{T}_k$.

Proof. For $k \geq 4$, let H be a hypergraph of order n and size m with all edges of size at least k. If H is a k-uniform hypergraph, then the result follows from Theorem C. Hence, we may assume that H is not k-uniform, for otherwise the desired result follows. Let H' be a k-uniform hypergraph obtained from H by shrinking every edge of size at least $k+1$ to an edge of size k (by removing vertices from a large edge until the edge size is reduced to size k) and deleting resulting isolated vertices, if any. Let H' have order n' and size m'. Then, $n' \leq n$ and $m' = m$. Every transversal in H' is a transversal in H, and so $\tau(H) \leq \tau(H')$. Thus, by the Chvátal-McDiarmid Theorem,

$$\tau(H) \leq \tau(H') \leq \frac{n' + \lfloor \frac{k}{2} \rfloor m'}{\lfloor \frac{3k}{2} \rfloor} \leq \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}.$$

This establishes the upper bound. To prove the necessity, suppose that $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m)/\lfloor \frac{3k}{2} \rfloor$. Then,

$$\frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor} = \tau(H) \leq \tau(H') \leq \frac{n' + \lfloor \frac{k}{2} \rfloor m'}{\lfloor \frac{3k}{2} \rfloor} \leq \frac{n + \lfloor \frac{k}{2} \rfloor m}{\lfloor \frac{3k}{2} \rfloor}.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $n' = n$ and H' is a k-uniform hypergraph, possibly disconnected, satisfying $\tau(H') = (n + \lfloor \frac{k}{2} \rfloor m)/\lfloor \frac{3k}{2} \rfloor$. By Theorem C, $\tau(C) = (n_C + \lfloor \frac{k}{2} \rfloor m_C)/\lfloor \frac{3k}{2} \rfloor$

for every component C of H' on n_C vertices and m_C edges. Applying Theorem C to each component of H', we note that every component of H' is an E_k-component or a T_k-component. Let T' be a $\tau(H')$-set. Note that T' contains one vertex from every E_k-component of H' and two vertices from every T_k-component of H'.

Claim A. *H' is connected.*

Proof. Suppose, to the contrary, that there is an edge $e \in E(H)$ whose vertex set is not entirely contained in a component of H'. Since $n' = n$, we note that the edge e has a nonempty intersection with at least two components of H'. Since $e \notin E(H')$, let $e' \in E(H')$ be the edge that resulted from shrinking the edge e. Let F_e denote the component of H' containing e' and let F_v denote a component of H' different from F_e that has a nonempty intersection with e. Let $v \in V(e) \cap V(F_v)$. If F_v is a T_k-component, then let v' be a vertex in F_v that is contained in all edges of F_v that do not contain v. If F_e is a T_k-component, then let u be a vertex in F_e that is contained in both edges of F_e different from e'.

Suppose that F_e is an E_k-component of H'. If F_v is an E_k-component, let $T = (T' \setminus (V(F_v) \cup V(F_e))) \cup \{v\}$. If F_v is a T_k-component, let $T = (T' \setminus (V(F_v) \cup V(F_e))) \cup \{v, v'\}$. In both cases, T is a transversal in H of size $|T'| - 1$, implying that $\tau(H) \le \tau(H') - 1 < (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$, a contradiction. Hence, F_e is a T_k-component. If F_v is an E_k-component, then $(T' \setminus (V(F_v) \cup V(F_e))) \cup \{u, v\}$ is a transversal in H of size $|T'| - 1$. If F_v is a T_k-component, then $(T' \setminus (V(F_v) \cup V(F_e))) \cup \{u, v, v'\}$ is a transversal in H of size $|T'| - 1$. Both cases produce a contradiction. Hence there is no edge $e \in E(H)$ that has a nonempty intersection with at least two components of H', which implies that H' is connected. □

By Claim A, H' is connected and therefore, by Theorem C, $H' \in \{E_k, T_k\}$. Hence we may assume that $H \neq H'$, for otherwise $H \in \mathcal{T}_k$, as desired. Therefore there is an edge e of H that is of size at least $k + 1$, implying that $H' = T_k$. If there is a vertex w of degree 3 in H, then $\{w\}$ is a transversal in H of size $|T'| - 1$, a contradiction. Hence every vertex in H has degree at most 2. This implies that if k is even, then $H = H'$, contradicting our earlier assumption that $H \neq H'$. Hence, k is odd, and so H has order $n(T_k) = (3k + 1)/2$. It follows that H consists of two edges of size k and one edge, namely the edge e, of size $k + 1$. Further every vertex of H is of degree 2. Hence, $H = T_k^*$, and so $H \in \mathcal{T}_k$. This establishes the necessity. If $H \in \mathcal{T}_k$, then it is straightforward to check that $\tau(H) = (n + \lfloor \frac{k}{2} \rfloor m) / \lfloor \frac{3k}{2} \rfloor$. This establishes the sufficiency and completes the proof of the Theorem 3. ■

Since every transversal in a hypergraph with no isolated vertex is a dominating set in the hypergraph, we have the following observation.

Observation 4. *If H is a hypergraph with $\delta(H) \ge 1$, then $\gamma(H) \le \tau(H)$.*

We shall need the following properties of the hypergraphs in the family $\mathcal{D}_k$ and the hypergraph D_5.

Observation 5. *For $k \geq 5$, let $F \in \mathcal{D}_k$ have order n_F and size m_F. Let $H = D_5$ have order n_H and size m_H, and let $X \subseteq V(H)$. Then the following holds.*

(a) $\gamma(F) = (n_F + \lfloor \frac{k-3}{2} \rfloor m_F)/ \left\lfloor \frac{3(k-1)}{2} \right\rfloor$.

(b) $\gamma(H) = 2 = 2n_H/9$.

(c) *Every vertex in H belongs to some $\gamma(H)$-set.*

(d) *If some vertex of degree 1 in H is not in X, then $\gamma(X; H) = 1 < 2n_H/9$.*

Lemma 6. *If $H \in \mathcal{H}$ has order n_H, then $\gamma(H) = 2n_H/9$.*

Proof. Let H_{under} be an underlying hypergraph of H and let $\mathcal{C}(H_{\text{under}})$ be the set of all components of H_{under}. Further, let $F \in \mathcal{C}(H_{\text{under}})$ and let X_F be the set of all vertices of degree 1 in F. Then, F is isomorphic to D_5, and $|X_F| = 3$. By construction of H, every vertex in X_F has degree 1 in H, and is adjacent in H only to vertices in $V(F)$. Now let D be a minimum dominating set of H. In order to dominate the vertices in X_F, the dominating set D must contain at least two vertices of $V(F)$. This is true for every component, F, in H_{under}, implying that

$$|D| \geq 2|\mathcal{C}(H_{\text{under}})| = \frac{2}{9} n_H,$$

and so $\gamma(H) \geq \frac{2}{9} n_H$. By Theorem B, $\gamma(H) \leq 2n_H/9$. Consequently, $\gamma(H) = 2n_H/9$. ■

Recall that a hypergraph is edge-minimal if every edge of the hypergraph contains at least one vertex of degree 1.

Lemma 7. *For $k \geq 5$, let H be a connected edge-minimal hypergraph of order n_H and size m_H with all edges of size at least k and with $\delta(H) \geq 1$. Then the following holds.*

(a) $\gamma(H) = (n_H + \lfloor \frac{k-3}{2} \rfloor m_H)/ \left\lfloor \frac{3(k-1)}{2} \right\rfloor$ *if and only if $H \in \mathcal{D}_k$.*

(b) *For $k = 5$, we have $\gamma(H) = 2n_H/9$ if and only if $H = D_5$.*

Proof. By the edge minimality of H, every edge of H contains a vertex of degree 1. Hence every dominating set of H is also a transversal in H, and so $\tau(H) \leq \gamma(H)$. Consequently by Observation 4, $\tau(H) = \gamma(H)$. Let H' be the hypergraph obtained from H by deleting exactly one vertex of degree 1 from each edge of H. Note that $H = \text{expa}(H')$. Since H is connected, so too is H'. We note that all edges of H' have size at least $k-1$ and that H' may have multiple edges. Let n'_H and m'_H denote the order and size of H', respectively. By construction,

$n'_{H} = n_{H} - m_{H}$ and $m'_{H} = m_{H}$. The transversal number of H and H' remains unchanged, and so $\tau(H) = \tau(H')$. We now consider part (a) and part (b) in turn.

(a) The sufficiency follows from Observation 5(a). To prove the necessity, suppose that $\gamma(H) = (n_{H} + \lfloor \frac{k-3}{2} \rfloor m_{H})/\left\lfloor \frac{3(k-1)}{2} \right\rfloor$. Then,

$$\frac{n'_{H} + \lfloor \frac{k-1}{2} \rfloor m'_{H}}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor} = \frac{n_{H} + \lfloor \frac{k-3}{2} \rfloor m_{H}}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor} = \gamma(H) = \tau(H) = \tau(H').$$

By Theorem 3, $H' \in \mathcal{T}_{k-1}$, implying that $H \in \mathcal{D}_k$.

(b) The sufficiency follows from Observation 5(b). To prove the necessity, suppose that $\gamma(H) = 2n_{H}/9$. Then, $2(n'_{H} + m'_{H})/9 = 2n_{H}/9 = \gamma(H) = \tau(H) = \tau(H')$. By Theorem D, $H' = T_4$, implying that $H = D_5$. ■

We remark that with the technique used in the proof of Lemma 7(a) and the result of [7], we can characterize the connected edge-minimal hypergraphs H of order n_{H} with edge size at least 4 and with $\delta(H) \geq 1$ achieving $\gamma(H) = n_{H}/4$. These are exactly the expanded hypergraphs of the hypergraphs mentioned in [7] that achieve equality in Theorem A for $k = 3$.

5. Proof of Theorem 1

Recall the statement of Theorem 1.

Theorem 1. *For $k \geq 5$, let H be a connected hypergraph on n vertices and m edges with edge size at least k and with $\delta(H) \geq 1$. Then,*

$$\gamma(H) \leq \frac{n + \lfloor \frac{k-3}{2} \rfloor m}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}$$

with equality if and only if $H \in \mathcal{D}_k$.

Proof. Let H be a hypergraph of order n and size m with all edges of size at least k and with $\delta(H) \geq 1$. By Theorem A, $\gamma(H) \leq (n + \lfloor \frac{k-3}{2} \rfloor m)/\lfloor 3(k-1)/2 \rfloor$. If $H \in \mathcal{D}_k$, then, by Observation 5(a), $\gamma(H) = (n + \lfloor \frac{k-3}{2} \rfloor m)/\lfloor 3(k-1)/2 \rfloor$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma(H) = (n + \lfloor \frac{k-3}{2} \rfloor m)/\lfloor 3(k-1)/2 \rfloor$. Let H' be a hypergraph obtained from H by successively deleting edges of H that do not contain any vertices of degree 1 in the resulting hypergraph at each stage. We note that H' is a hypergraph with all edges of size at least k and with $V(H') = V(H)$ and $E(H') \subseteq E(H)$. In particular, H' has $n' = n$ vertices and

$m' \leq m$ edges. When H is transformed to H', isolated vertices cannot arise and the domination number cannot decrease. Therefore, $\gamma(H) \leq \gamma(H')$, and so, by Theorem A applied to the hypergraph H', we see that

$$\frac{n + \left\lfloor \frac{k-3}{2} \right\rfloor m}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor} = \gamma(H) \leq \gamma(H') \leq \frac{n' + \left\lfloor \frac{k-3}{2} \right\rfloor m'}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor} \leq \frac{n + \left\lfloor \frac{k-3}{2} \right\rfloor m}{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}.$$

Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(H') = (n' + \left\lfloor \frac{k-3}{2} \right\rfloor m')/\lfloor 3(k-1)/2 \rfloor$ and $m' = m$, implying that $H' = H$. However, every edge of H' contains at least one vertex of degree 1 and hence H' is an edge-minimal connected hypergraph with all edges of size at least k. Therefore, H is an edge-minimal hypergraph with all edges of size at least k, and so, by Lemma 7(a), $H \in \mathcal{D}_k$. This proves the necessity. ■

6. Proof of Theorem 2

Recall the statement of Theorem 2.

Theorem 2. *Let H be a connected hypergraph on n vertices with edge size at least 5 and with $\delta(H) \geq 1$. Then, $\gamma(H) \leq 2n/9$, with equality if and only if $H \in \mathcal{H}$.*

Proof. By Theorem B, $\gamma(H) \leq 2n/9$. If $H \in \mathcal{H}$, then, by Lemma 6, $\gamma(H) = 2n/9$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma(H) = 2n/9$. Let H' be a hypergraph obtained from H by successively deleting edges of H that do not contain any vertices of degree 1 in the resulting hypergraph at each stage. We note that H' is a hypergraph with all edges of size at least 5 and with $V(H') = V(H)$ and $E(H') \subseteq E(H)$. In particular, H' has $n' = n$ vertices and $m' \leq m$ edges. When H is transformed to H', isolated vertices cannot arise and the domination number cannot decrease. Therefore, $\gamma(H) \leq \gamma(H')$, and so, by Theorem B applied to the hypergraph H', we see that

$$\frac{2n}{9} = \gamma(H) \leq \gamma(H') \leq \frac{2n'}{9} = \frac{2n}{9}.$$

Consequently, $\gamma(H') = 2n/9$. Moreover, every edge of H' contains at least one vertex of degree 1 and hence H' is an edge-minimal hypergraph with all edges of size at least 5. By Theorem B, $\gamma(C) = 2n_C/9$ for every component C of H' on n_C vertices. Applying Lemma 7(b) to each component of H', every component of H' is isomorphic to D_5. We proceed further with two claims.

Claim I. *Every vertex v of degree 1 in H' is not adjacent in H to any vertex from a component of H' not containing v.*

Proof. Let v be a vertex of degree 1 in H' and let H_v denote the component of H' containing v. By Observation 5(d), there is a vertex v' that dominates $V(H_v) \setminus \{v\}$. Let F be a component of H' different from H_v. Suppose that v is adjacent in H to a vertex $w \in V(F)$. By Observation 5(c), there exists a $\gamma(F)$-set S_w that contains w. The set $S_w \cup \{v'\}$ can be extended to a dominating set S of H by adding to it a minimum dominating set from every component of H' different from F and H_v. Since every component of H' has domination number 2, we note that $\gamma(H' - V(F_v)) = \gamma(H') - \gamma(F_v) = \gamma(H') - 2 = 2n/9 - 2$. Hence, $\gamma(H) \leq |S| = \gamma(H' - V(F_v)) + 1 = 2n/9 - 1 = \gamma(H) - 1$, a contradiction. Therefore, v and w are not adjacent in H. $\square$

Claim II. *A vertex of degree 1 in H' is of degree 1 in H.*

Proof. Let v_1 be a vertex of degree 1 in H' and let F be the component of v_1 in H'. Further, let $V(F) = \{v_1, v_2, \ldots, v_9\}$ and $E(F) = \{e_1, e_2, e_3\}$, where $e_1 = \{v_1, v_2, v_3, v_4, v_5\}$, $e_2 = \{v_4, v_5, v_6, v_7, v_8\}$ and $e_3 = \{v_7, v_8, v_9, v_2, v_3\}$. Let S be a $\gamma(H' - V(F))$-set. We note that $\gamma(H' - V(F)) = \gamma(H') - \gamma(F) = \gamma(H') - 2 = 2n/9 - 2$. Suppose v_1 is of degree at least 2 in H. Then there is an edge $e \in E(H) \setminus \{e_1\}$ that contains v_1 in H. By Claim I, $V(e) \subseteq V(F)$. If $v' \in V(e)$ where $v' \in \{v_7, v_8\}$, then $S \cup \{v'\}$ is a dominating set of H, and so $\gamma(H) \leq |S| + 1 = (2n/9 - 2) + 1 < 2n/9$, a contradiction. Hence, $\{v_7, v_8\} \cap V(e) = \emptyset$.

Since $e \neq e_1$, we may therefore assume, renaming v_6 and v_9, if necessary, that $v_6 \in V(e)$. If $v' \in V(e)$ where $v' \in \{v_2, v_3\}$, then $S \cup \{v'\}$ is a dominating set of H, and so $\gamma(H) \leq |S| + 1 < 2n/9$, a contradiction. Hence, $e = \{v_1, v_4, v_5, v_6, v_9\}$. But then $S \cup \{v_4\}$ is a dominating set of H, and so $\gamma(H) \leq |S| + 1 < 2n/9$, a contradiction. Hence, $d_H(v_1) = 1$. $\square$

We now return to the proof of the necessity of Theorem 2. By Claim II, the edges of $E' = E(H) \setminus E(H')$ contain only vertices of degree 2 in H'. Therefore since H can be obtained from H' by adding to it the edges in E', we see that $H \in \mathcal{H}$ where H' is an underlying hypergraph of H and where the edges of E' are the link edges. $\blacksquare$

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