# A CHARACTERIZATION OF HYPERGRAPHS WITH LARGE DOMINATION NUMBER 

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#### Abstract

Let $H=(V, E)$ be a hypergraph with vertex set $V$ and edge set $E$. A dominating set in $H$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \backslash D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. The domination number $\gamma(H)$ is the minimum cardinality of a dominating set in $H$. It is known [Cs. Bujtás, M.A. Henning and Zs. Tuza, Transversals and domination in uniform hypergraphs, European J. Combin. 33 (2012) 62-71] that for $k \geq 5$, if $H$ is a hypergraph of order $n$ and size $m$ with all edges of size at least $k$ and with no isolated vertex, then $\gamma(H) \leq(n+$ $\lfloor(k-3) / 2\rfloor m) /(\lfloor 3(k-1) / 2\rfloor)$. In this paper, we apply a recent result of the authors on hypergraphs with large transversal number [M.A. Henning and C. Löwenstein, A characterization of hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem, Discrete Math. 323 (2014) 69-75] to characterize the hypergraphs achieving equality in this bound.


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## 1. Introduction

In this paper we continue the study of domination in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A hypergraph $H=(V, E)$ is a finite set $V=V(H)$ of elements, called vertices, together with a finite multiset $E=E(H)$ of subsets of $V$, called hyperedges or simply edges.

We shall use the notation $n_{H}=|V|$ and $m_{H}=|E|$, and sometimes simply $n$ and $m$ without subscript if the actual $H$ need not be emphasized, to denote the order and size of $H$, respectively. A $k$-edge in $H$ is an edge of size $k$. The hypergraph $H$ is said to be $k$-uniform if every edge of $H$ is a $k$-edge. Every (simple) graph is a 2 -uniform hypergraph. Thus graphs are special hypergraphs. Two vertices $x$ and $y$ of $H$ are adjacent if there is an edge $e$ of $H$ such that $\{x, y\} \subseteq e$.

The degree of a vertex $v$ in $H$, denoted by $d_{H}(v)$ or simply by $d(v)$ if $H$ is clear from the context, is the number of edges of $H$ which contain $v$. The minimum degree among the vertices of $H$ is denoted by $\delta(H)$. We define a hypergraph $H$ to be edge-minimal if every edge of $H$ contains at least one vertex of degree 1 in $H$.

If $H^{\prime}$ is a hypergraph such that $V\left(H^{\prime}\right) \subseteq V(H)$ and $E\left(H^{\prime}\right) \subseteq E(H)$, then $H^{\prime}$ is called a subhypergraph of $H$. Possibly, $H^{\prime}=H$. Two vertices $x$ and $y$ of $H$ are connected if there is a sequence $x=v_{0}, v_{1}, v_{2}, \ldots, v_{k}=y$ of vertices of $H$ in which $v_{i-1}$ is adjacent to $v_{i}$ for $i=1,2, \ldots, k$. A connected hypergraph is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of $H$ is a component of $H$. Thus, no edge in $H$ contains vertices from different components. A component of a hypergraph $H$ that is isomorphic to a hypergraph $F$ we call an $F$-component of $H$.

A subset $T$ of vertices in a hypergraph $H$ is a transversal (also called vertex cover or hitting set in many papers) if $T$ has a nonempty intersection with every edge of $H$. The transversal number $\tau(H)$ of $H$ is the minimum size of a transversal in $H$. A transversal of size $\tau(H)$ is called a $\tau(H)$-set.

A dominating set in $H$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \backslash D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. Equivalently, every vertex $v \in V \backslash D$ is adjacent to a vertex in $D$. The domination number $\gamma(H)$ is the minimum cardinality of a dominating set in $H$. A dominating set of $H$ of cardinality $\gamma(H)$ is called a $\gamma(H)$-set.

If $X, Y \subseteq V$, then we say that $Y$ dominates $X$ if every vertex in $X$ is in $Y$ or is adjacent to some vertex of $Y$. If $X$ is a nonempty subset of vertices in $H$, then we define an $X$-dominating set in $H$ as a set $Y$ that dominates $X$ and we define the $X$-domination number, denoted $\gamma(X ; H)$ as the minimum cardinality of an $X$-dominating set in $H$. In particular, we note that $\gamma(H)=\gamma(V ; H)$.

Domination in graphs is still a very active area inside graph theory; see, for
example, the recent papers $[6,11,14]$. Domination in hypergraphs, however, was introduced recently by Acharya [1] and studied further, for example, in $[2,3,4$, $8,12]$.

Bujtás, Henning, Tuza [4] proved the following upper bounds on the domination number of a hypergraph.

Theorem A [4]. For $k \geq 3$, if $H$ is a hypergraph of order $n$ and size $m$ with all edges of size at least $k$ and with $\delta(H) \geq 1$, then

$$
\gamma(H) \leq \frac{n+\left\lfloor\frac{k-3}{2}\right\rfloor m}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}
$$

Theorem B [4]. If $H$ is a hypergraph of order $n$ with all edges of size at least 5 and with $\delta(H) \geq 1$, then $\gamma(H) \leq 2 n / 9$.

We have two immediate aims in this paper. Our first aim is to characterize the extremal hypergraphs in Theorem A for each $k \geq 5$. Our second aim is to characterize the extremal hypergraphs in Theorem B. For this purpose, we define three special families of hypergraphs.

### 1.1. Special families of hypergraphs

For $k \geq 4$, let $E_{k}$ denote the $k$-uniform hypergraph on $k$ vertices with exactly one edge. The hypergraph $E_{4}$ is illustrated in Figure 1.

For $k \geq 4$, we define the $k$-uniform hypergraph $T_{k}$ as follows. Let $A, B, C$ and $D$ be vertex-disjoint sets of vertices with $|A|=\lceil k / 2\rceil,|B|=|C|=\lfloor k / 2\rfloor$ and $|D|=\lceil k / 2\rceil-\lfloor k / 2\rfloor$. In particular, if $k$ is even, the set $D=\emptyset$, while if $k$ is odd, the set $D$ consists of a singleton vertex. Let $T_{k}$ denote the $k$-uniform hypergraph with $V\left(T_{k}\right)=A \cup B \cup C \cup D$ and with $E\left(T_{k}\right)=\left\{e_{1}, e_{2}, e_{3}\right\}$, where $V\left(e_{1}\right)=A \cup B, V\left(e_{2}\right)=A \cup C$, and $V\left(e_{3}\right)=B \cup C \cup D$. The hypergraphs $T_{4}$ and $T_{5}$ are illustrated in Figure 1.

For odd $k \geq 5$, we define the hypergraph $T_{k}^{*}$ as follows. Let $A, B$ and $C$ be vertex-disjoint sets of vertices with $|A|=|B|=(k+1) / 2$ and $|C|=(k-1) / 2$. Let $T_{k}^{*}$ denote the hypergraph with $V\left(T_{k}^{*}\right)=A \cup B \cup C$ and with $E\left(T_{k}^{*}\right)=\left\{e_{1}, e_{2}, e_{3}\right\}$, where $V\left(e_{1}\right)=A \cup B, V\left(e_{2}\right)=A \cup C$, and $V\left(e_{3}\right)=B \cup C$. The hypergraph $T_{5}^{*}$ is illustrated in Figure 1. Note that every edge in $T_{k}^{*}$ has size at least $k$.

Given a hypergraph $H=(V, E)$, let $H^{\prime}$ be obtained from $H$ by expanding every edge in $H$ by adding to it one new vertex, where all added vertices have degree 1 in $H^{\prime}$. We call $H^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ the expanded hypergraph $\operatorname{expa}(H)$ of $H$. That is for every edge $e \in E$, if $v_{e}$ denotes the new vertex added to $e$, where $v_{e} \neq v_{f}$ for edges $e \neq f$ in $H$, then $E^{\prime}=\left\{V(e) \cup\left\{v_{e}\right\} \mid e \in E\right\}$ and $V^{\prime}=V \cup \bigcup_{e \in E}\left\{v_{e}\right\}$.

For $k \geq 5$, let $D_{k}=\operatorname{expa}\left(T_{k-1}\right)$. The hypergraphs $D_{5}$ and $D_{6}$ are illustrated in Figure 2. Note that $D_{k}$ is $k$-uniform.


Figure 1. The hypergraphs $E_{4}, T_{4}, T_{5}$ and $T_{5}^{*}$.
For even $k \geq 6$, let $D_{k}^{*}=\operatorname{expa}\left(T_{k-1}^{*}\right)$. The hypergraph $D_{6}^{*}$ is illustrated in Figure 2. Note that $D_{k}^{*}$ is of edge size at least $k$.


Figure 2. The hypergraphs $D_{5}, D_{6}$ and $D_{6}^{*}$.
We are now in a position to define our three special families of hypergraphs.

The Family $\mathcal{T}_{\boldsymbol{k}}$. For even $k \geq 4$, we define $\mathcal{T}_{k}=\left\{E_{k}, T_{k}\right\}$ and for odd $k \geq 5$, we define $\mathcal{T}_{k}=\left\{E_{k}, T_{k}, T_{k}^{*}\right\}$.

The Family $\mathcal{D}_{\boldsymbol{k}}$. For odd $k \geq 5$, we define $\mathcal{D}_{k}=\left\{E_{k}, D_{k}\right\}$ and for even $k \geq 6$, we define $\mathcal{D}_{k}=\left\{E_{k}, D_{k}, D_{k}^{*}\right\}$.

The Family $\mathcal{H}$. Let $D_{5}$ be the hypergraph shown in Figure 2. Let $H_{\text {under }}$ be a hypergraph every component of which is isomorphic to $D_{5}$. Let $H$ be a hypergraph obtained from $H_{\text {under }}$ by adding edges of size at least five, called link edges, in such a way that every added edge contains only vertices of degree 2 in $H_{\text {under }}$. Possibly, $H$ is disconnected or $H=D_{5}$. We call the hypergraph $H_{\text {under }}$ an underlying hypergraph of $H$. Let $\mathcal{H}$ denote the family of all such hypergraphs $H$.

## 2. Main Result

Our aim in this paper is to characterize the extremal hypergraphs in Theorem A and Theorem B. We shall prove:

Theorem 1. For $k \geq 5$, let $H$ be a connected hypergraph on $n$ vertices and $m$ edges with edge size at least $k$ and with $\delta(H) \geq 1$. Then,

$$
\gamma(H) \leq \frac{n+\left\lfloor\frac{k-3}{2}\right\rfloor m}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}
$$

with equality if and only if $H \in \mathcal{D}_{k}$.
Theorem 2. Let $H$ be a connected hypergraph on $n$ vertices with edge size at least 5 and with $\delta(H) \geq 1$. Then, $\gamma(H) \leq 2 n / 9$, with equality if and only if $H \in \mathcal{H}$.

A proof of Theorem 1 is presented in Section 5, while a proof of Theorem 2 is given in Section 6.

## 3. Known Results on Transversals in Hypergraphs

Chvátal and McDiarmid [5] established the following upper bound on the transversal number of a uniform hypergraphs in terms of its order and size.

Chvátal-McDiarmid Theorem. For $k \geq 2$, if $H$ is a $k$-uniform hypergraph on $n$ vertices with $m$ edges, then $\tau(H) \leq\left(n+\left\lfloor\frac{k}{2}\right\rfloor m\right) /\left(\left\lfloor\frac{3 k}{2}\right\rfloor\right)$.

The extremal connected hypergraphs that achieve equality in the ChvátalMcDiarmid Theorem when $k=3$ were characterized by Henning and Yeo [7]. For $k=2$ and $k \geq 4$, the extremal hypergraphs were characterized by the authors in [10].

Theorem C [10]. For $k=2$ and $k \geq 4$, let $H$ be a connected $k$-uniform hypergraph on $n$ vertices and $m$ edges. Then, $\tau(H) \leq\left(n+\left\lfloor\frac{k}{2}\right\rfloor m\right) /\left(\left\lfloor\frac{3 k}{2}\right\rfloor\right)$ with equality if and only if $H \in\left\{E_{k}, T_{k}\right\}$.

In the special case when $k=4$, Lai and Chang [13] established the following upper bound on the transversal number of a $k$-uniform hypergraph.

Lai-Chang Theorem. If $H$ is a 4-uniform hypergraph on $n$ vertices and $m$ edges, then $\tau(H) \leq 2(n+m) / 9$.

The extremal hypergraphs achieving equality in the Lai-Chang Theorem were characterized by the authors in [9].

Theorem D [9]. Let $H$ be a connected hypergraph on $n$ vertices and $m$ edges with edge size at least 4. If $\tau(H)=2(n+m) / 9$, then $H=T_{4}$.

## 4. Preliminary Results

In this section, we present some preliminary observations, lemmas and theorems that we will need in proving our main results. Our first result is a generalization of Theorem C where we relax the $k$-uniformity condition to edge sizes at least $k$.

Theorem 3. For $k \geq 4$, let $H$ be a connected hypergraph on $n$ vertices and $m$ edges with edge size at least $k$ and with $\delta(H) \geq 1$. Then,

$$
\tau(H) \leq \frac{n+\left\lfloor\frac{k}{2}\right\rfloor m}{\left\lfloor\frac{3 k}{2}\right\rfloor}
$$

with equality if and only if $H \in \mathcal{T}_{k}$.
Proof. For $k \geq 4$, let $H$ be a hypergraph of order $n$ and size $m$ with all edges of size at least $k$. If $H$ is a $k$-uniform hypergraph, then the result follows from Theorem C. Hence, we may assume that $H$ is not $k$-uniform, for otherwise the desired result follows. Let $H^{\prime}$ be a $k$-uniform hypergraph obtained from $H$ by shrinking every edge of size at least $k+1$ to an edge of size $k$ (by removing vertices from a large edge until the edge size is reduced to size $k$ ) and deleting resulting isolated vertices, if any. Let $H^{\prime}$ have order $n^{\prime}$ and size $m^{\prime}$. Then, $n^{\prime} \leq n$ and $m^{\prime}=m$. Every transversal in $H^{\prime}$ is a transversal in $H$, and so $\tau(H) \leq \tau\left(H^{\prime}\right)$. Thus, by the Chvátal-McDiarmid Theorem,

$$
\tau(H) \leq \tau\left(H^{\prime}\right) \leq \frac{n^{\prime}+\left\lfloor\frac{k}{2}\right\rfloor m^{\prime}}{\left\lfloor\frac{3 k}{2}\right\rfloor} \leq \frac{n+\left\lfloor\frac{k}{2}\right\rfloor m}{\left\lfloor\frac{3 k}{2}\right\rfloor}
$$

This establishes the upper bound. To prove the necessity, suppose that $\tau(H)$ $=\left(n+\left\lfloor\frac{k}{2}\right\rfloor m\right) /\left\lfloor\frac{3 k}{2}\right\rfloor$. Then,

$$
\frac{n+\left\lfloor\frac{k}{2}\right\rfloor m}{\left\lfloor\frac{3 k}{2}\right\rfloor}=\tau(H) \leq \tau\left(H^{\prime}\right) \leq \frac{n^{\prime}+\left\lfloor\frac{k}{2}\right\rfloor m^{\prime}}{\left\lfloor\frac{3 k}{2}\right\rfloor} \leq \frac{n+\left\lfloor\frac{k}{2}\right\rfloor m}{\left\lfloor\frac{3 k}{2}\right\rfloor} .
$$

Consequently, we must have equality throughout the above inequality chain. In particular, $n^{\prime}=n$ and $H^{\prime}$ is a $k$-uniform hypergraph, possibly disconnected, satisfying $\tau\left(H^{\prime}\right)=\left(n+\left\lfloor\frac{k}{2}\right\rfloor m\right) /\left\lfloor\frac{3 k}{2}\right\rfloor$. By Theorem C, $\tau(C)=\left(n_{C}+\left\lfloor\frac{k}{2}\right\rfloor m_{C}\right) /\left\lfloor\frac{3 k}{2}\right\rfloor$
for every component $C$ of $H^{\prime}$ on $n_{C}$ vertices and $m_{C}$ edges. Applying Theorem C to each component of $H^{\prime}$, we note that every component of $H^{\prime}$ is an $E_{k}$-component or a $T_{k}$-component. Let $T^{\prime}$ be a $\tau\left(H^{\prime}\right)$-set. Note that $T^{\prime}$ contains one vertex from every $E_{k}$-component of $H^{\prime}$ and two vertices from every $T_{k}$-component of $H^{\prime}$.

Claim A. $H^{\prime}$ is connected.
Proof. Suppose, to the contrary, that there is an edge $e \in E(H)$ whose vertex set is not entirely contained in a component of $H^{\prime}$. Since $n^{\prime}=n$, we note that the edge $e$ has a nonempty intersection with at least two components of $H^{\prime}$. Since $e \notin E\left(H^{\prime}\right)$, let $e^{\prime} \in E\left(H^{\prime}\right)$ be the edge that resulted from shrinking the edge $e$. Let $F_{e}$ denote the component of $H^{\prime}$ containing $e^{\prime}$ and let $F_{v}$ denote a component of $H^{\prime}$ different from $F_{e}$ that has a nonempty intersection with $e$. Let $v \in V(e) \cap V\left(F_{v}\right)$. If $F_{v}$ is a $T_{k}$-component, then let $v^{\prime}$ be a vertex in $F_{v}$ that is contained in all edges of $F_{v}$ that do not contain $v$. If $F_{e}$ is a $T_{k}$-component, then let $u$ be a vertex in $F_{e}$ that is contained in both edges of $F_{e}$ different from $e^{\prime}$.

Suppose that $F_{e}$ is an $E_{k}$-component of $H^{\prime}$. If $F_{v}$ is an $E_{k}$-component, let $T=\left(T^{\prime} \backslash\left(V\left(F_{v}\right) \cup V\left(F_{e}\right)\right)\right) \cup\{v\}$. If $F_{v}$ is a $T_{k}$-component, let $T=\left(T^{\prime} \backslash\left(V\left(F_{v}\right) \cup\right.\right.$ $\left.\left.V\left(F_{e}\right)\right)\right) \cup\left\{v, v^{\prime}\right\}$. In both cases, $T$ is a transversal in $H$ of size $\left|T^{\prime}\right|-1$, implying that $\tau(H) \leq \tau\left(H^{\prime}\right)-1<\left(n+\left\lfloor\frac{k}{2}\right\rfloor m\right) /\left\lfloor\frac{3 k}{2}\right\rfloor$, a contradiction. Hence, $F_{e}$ is a $T_{k}$-component. If $F_{v}$ is an $E_{k}$-component, then $\left(T^{\prime} \backslash\left(V\left(F_{v}\right) \cup V\left(F_{e}\right)\right)\right) \cup\{u, v\}$ is a transversal in $H$ of size $\left|T^{\prime}\right|-1$. If $F_{v}$ is a $T_{k}$-component, then $\left(T^{\prime} \backslash\left(V\left(F_{v}\right) \cup\right.\right.$ $\left.\left.V\left(F_{e}\right)\right)\right) \cup\left\{u, v, v^{\prime}\right\}$ is a transversal in $H$ of size $\left|T^{\prime}\right|-1$. Both cases produce a contradiction. Hence there is no edge $e \in E(H)$ that has a nonempty intersection with at least two components of $H^{\prime}$, which implies that $H^{\prime}$ is connected.

By Claim A, $H^{\prime}$ is connected and therefore, by Theorem C, $H^{\prime} \in\left\{E_{k}, T_{k}\right\}$. Hence we may assume that $H \neq H^{\prime}$, for otherwise $H \in \mathcal{T}_{k}$, as desired. Therefore there is an edge $e$ of $H$ that is of size at least $k+1$, implying that $H^{\prime}=T_{k}$. If there is a vertex $w$ of degree 3 in $H$, then $\{w\}$ is a transversal in $H$ of size $\left|T^{\prime}\right|-1$, a contradiction. Hence every vertex in $H$ has degree at most 2 . This implies that if $k$ is even, then $H=H^{\prime}$, contradicting our earlier assumption that $H \neq H^{\prime}$. Hence, $k$ is odd, and so $H$ has order $n\left(T_{k}\right)=(3 k+1) / 2$. It follows that $H$ consists of two edges of size $k$ and one edge, namely the edge $e$, of size $k+1$. Further every vertex of $H$ is of degree 2. Hence, $H=T_{k}^{*}$, and so $H \in \mathcal{T}_{k}$. This establishes the necessity. If $H \in \mathcal{T}_{k}$, then it is straightforward to check that $\tau(H)=\left(n+\left\lfloor\frac{k}{2}\right\rfloor m\right) /\left\lfloor\frac{3 k}{2}\right\rfloor$. This establishes the sufficiency and completes the proof of the Theorem 3 .

Since every transversal in a hypergraph with no isolated vertex is a dominating set in the hypergraph, we have the following observation.

Observation 4. If $H$ is a hypergraph with $\delta(H) \geq 1$, then $\gamma(H) \leq \tau(H)$.

We shall need the following properties of the hypergraphs in the family $\mathcal{D}_{k}$ and the hypergraph $D_{5}$.

Observation 5. For $k \geq 5$, let $F \in \mathcal{D}_{k}$ have order $n_{F}$ and size $m_{F}$. Let $H=D_{5}$ have order $n_{H}$ and size $m_{H}$, and let $X \subseteq V(H)$. Then the following holds.
(a) $\gamma(F)=\left(n_{F}+\left\lfloor\frac{k-3}{2}\right\rfloor m_{F}\right) /\left\lfloor\frac{3(k-1)}{2}\right\rfloor$.
(b) $\gamma(H)=2=2 n_{H} / 9$.
(c) Every vertex in $H$ belongs to some $\gamma(H)$-set.
(d) If some vertex of degree 1 in $H$ is not in $X$, then $\gamma(X ; H)=1<2 n_{H} / 9$.

Lemma 6. If $H \in \mathcal{H}$ has order $n_{H}$, then $\gamma(H)=2 n_{H} / 9$.
Proof. Let $H_{\text {under }}$ be an underlying hypergraph of $H$ and let $\mathcal{C}\left(H_{\text {under }}\right)$ be the set of all components of $H_{\text {under }}$. Further, let $F \in \mathcal{C}\left(H_{\text {under }}\right)$ and let $X_{F}$ be the set of all vertices of degree 1 in $F$. Then, $F$ is isomorphic to $D_{5}$, and $\left|X_{F}\right|=3$. By construction of $H$, every vertex in $X_{F}$ has degree 1 in $H$, and is adjacent in $H$ only to vertices in $V(F)$. Now let $D$ be a minimum dominating set of $H$. In order to dominate the vertices in $X_{F}$, the dominating set $D$ must contain at least two vertices of $V(F)$. This is true for every component, $F$, in $H_{\text {under }}$, implying that

$$
|D| \geq 2\left|\mathcal{C}\left(H_{\text {under }}\right)\right|=\frac{2}{9} n_{H}
$$

and so $\gamma(H) \geq \frac{2}{9} n_{H}$. By Theorem B, $\gamma(H) \leq 2 n_{H} / 9$. Consequently, $\gamma(H)=$ $2 n_{H} / 9$.

Recall that a hypergraph is edge-minimal if every edge of the hypergraph contains at least one vertex of degree 1 .

Lemma 7. For $k \geq 5$, let $H$ be a connected edge-minimal hypergraph of order $n_{H}$ and size $m_{H}$ with all edges of size at least $k$ and with $\delta(H) \geq 1$. Then the following holds.
(a) $\gamma(H)=\left(n_{H}+\left\lfloor\frac{k-3}{2}\right\rfloor m_{H}\right) /\left\lfloor\frac{3(k-1)}{2}\right\rfloor$ if and only if $H \in \mathcal{D}_{k}$.
(b) For $k=5$, we have $\gamma(H)=2 n_{H} / 9$ if and only if $H=D_{5}$.

Proof. By the edge minimality of $H$, every edge of $H$ contains a vertex of degree 1. Hence every dominating set of $H$ is also a transversal in $H$, and so $\tau(H) \leq \gamma(H)$. Consequently by Observation $4, \tau(H)=\gamma(H)$. Let $H^{\prime}$ be the hypergraph obtained from $H$ by deleting exactly one vertex of degree 1 from each edge of $H$. Note that $H=\operatorname{expa}\left(H^{\prime}\right)$. Since $H$ is connected, so too is $H^{\prime}$. We note that all edges of $H^{\prime}$ have size at least $k-1$ and that $H^{\prime}$ may have multiple edges. Let $n_{H}^{\prime}$ and $m_{H}^{\prime}$ denote the order and size of $H^{\prime}$, respectively. By construction,
$n_{H}^{\prime}=n_{H}-m_{H}$ and $m_{H}^{\prime}=m_{H}$. The transversal number of $H$ and $H^{\prime}$ remains unchanged, and so $\tau(H)=\tau\left(H^{\prime}\right)$. We now consider part (a) and part (b) in turn.
(a) The sufficiency follows from Observation 5(a). To prove the necessity, suppose that $\gamma(H)=\left(n_{H}+\left\lfloor\frac{k-3}{2}\right\rfloor m_{H}\right) /\left\lfloor\frac{3(k-1)}{2}\right\rfloor$. Then,

$$
\frac{n_{H}^{\prime}+\left\lfloor\frac{k-1}{2}\right\rfloor m_{H}^{\prime}}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}=\frac{n_{H}+\left\lfloor\frac{k-3}{2}\right\rfloor m_{H}}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}=\gamma(H)=\tau(H)=\tau\left(H^{\prime}\right) .
$$

By Theorem 3, $H^{\prime} \in \mathcal{T}_{k-1}$, implying that $H \in \mathcal{D}_{k}$.
(b) The sufficiency follows from Observation 5(b). To prove the necessity, suppose that $\gamma(H)=2 n_{H} / 9$. Then, $2\left(n_{H}^{\prime}+m_{H}^{\prime}\right) / 9=2 n_{H} / 9=\gamma(H)=\tau(H)=\tau\left(H^{\prime}\right)$. By Theorem D, $H^{\prime}=T_{4}$, implying that $H=D_{5}$.

We remark that with the technique used in the proof of Lemma 7(a) and the result of [7], we can characterize the connected edge-minimal hypergraphs $H$ of order $n_{H}$ with edge size at least 4 and with $\delta(H) \geq 1$ achieving $\gamma(H)=n_{H} / 4$. These are exactly the expanded hypergraphs of the hypergraphs mentioned in [7] that achieve equality in Theorem A for $k=3$.

## 5. Proof of Theorem 1

Recall the statement of Theorem 1.
Theorem 1. For $k \geq 5$, let $H$ be a connected hypergraph on $n$ vertices and $m$ edges with edge size at least $k$ and with $\delta(H) \geq 1$. Then,

$$
\gamma(H) \leq \frac{n+\left\lfloor\frac{k-3}{2}\right\rfloor m}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}
$$

with equality if and only if $H \in \mathcal{D}_{k}$.
Proof. Let $H$ be a hypergraph of order $n$ and size $m$ with all edges of size at least $k$ and with $\delta(H) \geq 1$. By Theorem A, $\gamma(H) \leq\left(n+\left\lfloor\frac{k-3}{2}\right\rfloor m\right) /\lfloor 3(k-1) / 2\rfloor$. If $H \in \mathcal{D}_{k}$, then, by Observation $5(\mathrm{a}), \gamma(H)=\left(n+\left\lfloor\frac{k-3}{2}\right\rfloor m\right) /\lfloor 3(k-1) / 2\rfloor$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma(H)=\left(n+\left\lfloor\frac{k-3}{2}\right\rfloor m\right) /\lfloor 3(k-1) / 2\rfloor$. Let $H^{\prime}$ be a hypergraph obtained from $H$ by successively deleting edges of $H$ that do not contain any vertices of degree 1 in the resulting hypergraph at each stage. We note that $H^{\prime}$ is a hypergraph with all edges of size at least $k$ and with $V\left(H^{\prime}\right)=V(H)$ and $E\left(H^{\prime}\right) \subseteq E(H)$. In particular, $H^{\prime}$ has $n^{\prime}=n$ vertices and
$m^{\prime} \leq m$ edges. When $H$ is transformed to $H^{\prime}$, isolated vertices cannot arise and the domination number cannot decrease. Therefore, $\gamma(H) \leq \gamma\left(H^{\prime}\right)$, and so, by Theorem A applied to the hypergraph $H^{\prime}$, we see that

$$
\frac{n+\left\lfloor\frac{k-3}{2}\right\rfloor m}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}=\gamma(H) \leq \gamma\left(H^{\prime}\right) \leq \frac{n^{\prime}+\left\lfloor\frac{k-3}{2}\right\rfloor m^{\prime}}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor} \leq \frac{n+\left\lfloor\frac{k-3}{2}\right\rfloor m}{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}
$$

Consequently, we must have equality throughout this inequality chain. In particular, $\gamma\left(H^{\prime}\right)=\left(n^{\prime}+\left\lfloor\frac{k-3}{2}\right\rfloor m^{\prime}\right) /\lfloor 3(k-1) / 2\rfloor$ and $m^{\prime}=m$, implying that $H^{\prime}=H$. However, every edge of $H^{\prime}$ contains at least one vertex of degree 1 and hence $H^{\prime}$ is an edge-minimal connected hypergraph with all edges of size at least $k$. Therefore, $H$ is an edge-minimal hypergraph with all edges of size at least $k$, and so, by Lemma $7(\mathrm{a}), H \in \mathcal{D}_{k}$. This proves the necessity.

## 6. Proof of Theorem 2

Recall the statement of Theorem 2.
Theorem 2. Let $H$ be a connected hypergraph on $n$ vertices with edge size at least 5 and with $\delta(H) \geq 1$. Then, $\gamma(H) \leq 2 n / 9$, with equality if and only if $H \in \mathcal{H}$.

Proof. By Theorem B, $\gamma(H) \leq 2 n / 9$. If $H \in \mathcal{H}$, then, by Lemma 6, $\gamma(H)=$ $2 n / 9$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma(H)=2 n / 9$. Let $H^{\prime}$ be a hypergraph obtained from $H$ by successively deleting edges of $H$ that do not contain any vertices of degree 1 in the resulting hypergraph at each stage. We note that $H^{\prime}$ is a hypergraph with all edges of size at least 5 and with $V\left(H^{\prime}\right)=V(H)$ and $E\left(H^{\prime}\right) \subseteq E(H)$. In particular, $H^{\prime}$ has $n^{\prime}=n$ vertices and $m^{\prime} \leq m$ edges. When $H$ is transformed to $H^{\prime}$, isolated vertices cannot arise and the domination number cannot decrease. Therefore, $\gamma(H) \leq \gamma\left(H^{\prime}\right)$, and so, by Theorem B applied to the hypergraph $H^{\prime}$, we see that

$$
\frac{2 n}{9}=\gamma(H) \leq \gamma\left(H^{\prime}\right) \leq \frac{2 n^{\prime}}{9}=\frac{2 n}{9}
$$

Consequently, $\gamma\left(H^{\prime}\right)=2 n / 9$. Moreover, every edge of $H^{\prime}$ contains at least one vertex of degree 1 and hence $H^{\prime}$ is an edge-minimal hypergraph with all edges of size at least 5. By Theorem B, $\gamma(C)=2 n_{C} / 9$ for every component $C$ of $H^{\prime}$ on $n_{C}$ vertices. Applying Lemma $7(\mathrm{~b})$ to each component of $H^{\prime}$, every component of $H^{\prime}$ is isomorphic to $D_{5}$. We proceed further with two claims.

Claim I. Every vertex $v$ of degree 1 in $H^{\prime}$ is not adjacent in $H$ to any vertex from a component of $H^{\prime}$ not containing $v$.
Proof. Let $v$ be a vertex of degree 1 in $H^{\prime}$ and let $H_{v}$ denote the component of $H^{\prime}$ containing $v$. By Observation $5(\mathrm{~d})$, there is a vertex $v^{\prime}$ that dominates $V\left(H_{v}\right) \backslash\{v\}$. Let $F$ be a component of $H^{\prime}$ different from $H_{v}$. Suppose that $v$ is adjacent in $H$ to a vertex $w \in V(F)$. By Observation 5(c), there exists a $\gamma(F)$-set $S_{w}$ that contains $w$. The set $S_{w} \cup\left\{v^{\prime}\right\}$ can be extended to a dominating set $S$ of $H$ by adding to it a minimum dominating set from every component of $H^{\prime}$ different from $F$ and $H_{v}$. Since every component of $H^{\prime}$ has domination number 2, we note that $\gamma\left(H^{\prime}-V\left(F_{v}\right)\right)=\gamma\left(H^{\prime}\right)-\gamma\left(F_{v}\right)=\gamma\left(H^{\prime}\right)-2=2 n / 9-2$. Hence, $\gamma(H) \leq|S|=\gamma\left(H^{\prime}-V\left(F_{v}\right)\right)+1=2 n / 9-1=\gamma(H)-1$, a contradiction. Therefore, $v$ and $w$ are not adjacent in $H$.

Claim II. A vertex of degree 1 in $H^{\prime}$ is of degree 1 in $H$.
Proof. Let $v_{1}$ be a vertex of degree 1 in $H^{\prime}$ and let $F$ be the component of $v_{1}$ in $H^{\prime}$. Further, let $V(F)=\left\{v_{1}, v_{2}, \ldots, v_{9}\right\}$ and $E(F)=\left\{e_{1}, e_{2}, e_{3}\right\}$, where $e_{1}=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}, e_{2}=\left\{v_{4}, v_{5}, v_{6}, v_{7}, v_{8}\right\}$ and $e_{3}=\left\{v_{7}, v_{8}, v_{9}, v_{2}, v_{3}\right\}$. Let $S$ be a $\gamma\left(H^{\prime}-V(F)\right)$-set. We note that $\gamma\left(H^{\prime}-V(F)\right)=\gamma\left(H^{\prime}\right)-\gamma(F)=\gamma\left(H^{\prime}\right)-2$ $=2 n / 9-2$. Suppose $v_{1}$ is of degree at least 2 in $H$. Then there is an edge $e \in E(H) \backslash\left\{e_{1}\right\}$ that contains $v_{1}$ in $H$. By Claim I, $V(e) \subseteq V(F)$. If $v^{\prime} \in V(e)$ where $v^{\prime} \in\left\{v_{7}, v_{8}\right\}$, then $S \cup\left\{v^{\prime}\right\}$ is a dominating set of $H$, and so $\gamma(H) \leq$ $|S|+1=(2 n / 9-2)+1<2 n / 9$, a contradiction. Hence, $\left\{v_{7}, v_{8}\right\} \cap V(e)=\emptyset$.

Since $e \neq e_{1}$, we may therefore assume, renaming $v_{6}$ and $v_{9}$, if necessary, that $v_{6} \in V(e)$. If $v^{\prime} \in V(e)$ where $v^{\prime} \in\left\{v_{2}, v_{3}\right\}$, then $S \cup\left\{v^{\prime}\right\}$ is a dominating set of $H$, and so $\gamma(H) \leq|S|+1<2 n / 9$, a contradiction. Hence, $e=\left\{v_{1}, v_{4}, v_{5}, v_{6}, v_{9}\right\}$. But then $S \cup\left\{v_{4}\right\}$ is a dominating set of $H$, and so $\gamma(H) \leq|S|+1<2 n / 9$, a contradiction. Hence, $d_{H}\left(v_{1}\right)=1$.

We now return to the proof of the necessity of Theorem 2. By Claim II, the edges of $E^{\prime}=E(H) \backslash E\left(H^{\prime}\right)$ contain only vertices of degree 2 in $H^{\prime}$. Therefore since $H$ can be obtained from $H^{\prime}$ by adding to it the edges in $E^{\prime}$, we see that $H \in \mathcal{H}$ where $H^{\prime}$ is an underlying hypergraph of $H$ and where the edges of $E^{\prime}$ are the link edges.

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