# A NEIGHBORHOOD CONDITION FOR FRACTIONAL ID- $[A, B]$-FACTOR-CRITICAL GRAPHS 

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#### Abstract

Let $G$ be a graph of order $n$, and let $a$ and $b$ be two integers with $1 \leq a \leq$ $b$. Let $h: E(G) \rightarrow[0,1]$ be a function. If $a \leq \sum_{e \ni x} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G\left[F_{h}\right]$ a fractional $[a, b]$-factor of $G$ with indicator function $h$, where $F_{h}=\{e \in E(G): h(e)>0\}$. A graph $G$ is fractional independent-set-deletable $[a, b]$-factor-critical (in short, fractional ID-[ $a, b]$ -factor-critical) if $G-I$ has a fractional $[a, b]$-factor for every independent set $I$ of $G$. In this paper, it is proved that if $n \geq \frac{(a+2 b)(2 a+2 b-3)+1}{b}, \delta(G) \geq$ $\frac{b n}{a+2 b}+a$ and $\left|N_{G}(x) \cup N_{G}(y)\right| \geq \frac{(a+b) n}{a+2 b}$ for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID-[a,b]-factor-critical. Furthermore, it is shown that this result is best possible in some sense.

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## 1. Introduction

The graphs considered here will be finite, undirected and without loops or multiple edges. Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$, respectively. For any $x \in V(G)$, we denote the degree of $x$ in $G$ by $d_{G}(x)$. We write $N_{G}(x)$ for the set of vertices adjacent to $x$ in $G$, and $N_{G}[x]$ for $N_{G}(x) \cup\{x\}$. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$, and $G-S=G[V(G) \backslash S]$. Let $S$ and $T$ be two disjoint vertex subsets of $G$; we denote the number of edges from $S$ to $T$ by $e_{G}(S, T)$. We denote by $\delta(G)$ the minimum degree of $G$. For any nonempty subset $S$ of $V(G)$, let

$$
N_{G}(S)=\bigcup_{x \in S} N_{G}(x)
$$

If $G$ and $H$ are vertex-disjoint graphs, then their join and union are denoted by $G \vee H$ and $G \cup H$, respectively.

A factor of a graph $G$ is a spanning subgraph of $G$. Let $a$ and $b$ be two positive integers with $1 \leq a \leq b$. Then a factor $F$ is an $[a, b]$-factor if $a \leq d_{F}(x) \leq b$ for each $x \in V(G)$. Let $h: E(G) \rightarrow[0,1]$ be a function. If $a \leq \sum_{e \ni x} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G\left[F_{h}\right]$ a fractional $[a, b]$-factor of $G$ with indicator function $h$, where $F_{h}=\{e \in E(G): h(e)>0\}$. If $G-I$ admits a fractional $[a, b]$-factor for every independent set $I$ of $G$, then we say that $G$ is fractional ID-[ $a, b]$-factor-critical [1]. A fractional ID- $[k, k]$-factor-critical graph is simply called a fractional ID- $k$-factor-critical graph.

Many authors have investigated factors and fractional factors in graphs; see, for instance, $[2,3,4,5,6,7,8,9]$. Chang, Liu and Zhu [10] showed a minimum degree condition for a graph to be a fractional ID- $k$-factor-critical graph. Zhou, Bian and Wu [11] gave a degree condition for the existence of fractional ID- $k$-factorcritical graphs. Zhou [12] obtained a binding number condition for graphs to be fractional ID- $k$-factor-critical graphs. Zhou, Sun and Liu [1] obtained a minimum degree condition for a graph to be a fractional ID- $[a, b]$-factor-critical graph. In this paper, we proceed to study fractional ID- $[a, b]$-factor-critical graphs, and obtain a neighborhood condition for a graph to be fractional ID- $[a, b]$-factor-critical. The main result is the following theorem.

Theorem 1. Let $1 \leq a \leq b$ be two integers, and let $G$ be a graph of order $n$ with $n \geq \frac{(a+2 b)(2 a+2 b-3)+1}{b}$, and $\delta(G) \geq \frac{b n}{a+2 b}+a$. If

$$
\left|N_{G}(x) \cup N_{G}(y)\right| \geq \frac{(a+b) n}{a+2 b}
$$

for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID- $[a, b]$ -factor-critical.

If $a=b=k$ in Theorem 1 , then we obtain the following result.
Theorem 2. Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$ with $n \geq 12 k-8$, and $\delta(G) \geq \frac{n}{3}+k$. If

$$
\left|N_{G}(x) \cup N_{G}(y)\right| \geq \frac{2 n}{3}
$$

for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID- $k$-factorcritical.

If $k=1$ in Theorem 2, then we get the following result.
Theorem 3. Let $G$ be a graph of order $n$ with $n \geq 4$, and $\delta(G) \geq \frac{n}{3}+1$. If

$$
\left|N_{G}(x) \cup N_{G}(y)\right| \geq \frac{2 n}{3}
$$

for any two nonadjacent vertices $x, y \in V(G)$, then $G$ is fractional ID-factorcritical.

## 2. The Proof of Theorem 1

In order to prove Theorem 1, we rely heavily on the following lemma.
Lemma 4 [13]. Let $G$ be a graph. Then $G$ has a fractional $[a, b]$-factor if and only if for every subset $S$ of $V(G)$,

$$
\delta_{G}(S, T)=b|S|+d_{G-S}(T)-a|T| \geq 0
$$

where $T=\left\{x: x \in V(G) \backslash S, d_{G-S}(x) \leq a\right\}$ and $d_{G-S}(T)=\sum_{x \in T} d_{G-S}(x)$.
Proof of Theorem 1. Let $X$ be an independent set of $G$ and $H=G-X$. In order to complete the proof of Theorem 1, we need only to prove that $H$ has a fractional $[a, b]$-factor. By contradiction, suppose that $H$ has no fractional [ $a, b]$-factor. Then by Lemma 4, there exists some subset $S \subseteq V(H)$ such that

$$
\begin{equation*}
\delta_{H}(S, T)=b|S|+d_{H-S}(T)-a|T| \leq-1 \tag{1}
\end{equation*}
$$

where $T=\left\{x: x \in V(H) \backslash S, d_{H-S}(x) \leq a\right\}$. We first prove the following claims.
Claim 1. $|X| \leq \frac{b n}{a+2 b}$.
Proof. Since $n \geq \frac{(a+2 b)(2 a+2 b-3)+1}{b}$, the inequality holds for $|X|=1$. In the following we may assume $|X| \geq 2$. In terms of the condition of Theorem 1 , there
exist $x, y \in X$ such that $\left|N_{G}(x) \cup N_{G}(y)\right| \geq \frac{(a+b) n}{a+2 b}$. Since $X$ is independent, we obtain $X \cap\left(N_{G}(x) \cup N_{G}(y)\right)=\emptyset$. Thus, we have

$$
|X|+\frac{(a+b) n}{a+2 b} \leq|X|+\left|N_{G}(x) \cup N_{G}(y)\right| \leq n
$$

which implies

$$
|X| \leq n-\frac{(a+b) n}{a+2 b}=\frac{b n}{a+2 b}
$$

Claim 2. $\delta(H) \geq a$.
Proof. Note that $H=G-X$. Combining this with Claim 1, we obtain

$$
\delta(H) \geq \delta(G)-|X| \geq\left(\frac{b n}{a+2 b}+a\right)-\frac{b n}{a+2 b}=a
$$

Claim 3. $|T| \geq b+1$.
Proof. If $|T| \leq b$, then from Claim 2 and since $|S|+d_{H-S}(x) \geq d_{H}(x) \geq \delta(H)$ for each $x \in T$, we have

$$
\begin{aligned}
\delta_{H}(S, T) & =b|S|+d_{H-S}(T)-a|T| \geq|T||S|+d_{H-S}(T)-a|T| \\
& =\sum_{x \in T}\left(|S|+d_{H-S}(x)-a\right) \geq \sum_{x \in T}(\delta(H)-a) \geq 0
\end{aligned}
$$

which contradicts (1).
Claim 4. $a|T|>b|S|$.
Proof. If $a|T| \leq b|S|$, then from (1) we obtain

$$
-1 \geq \delta_{H}(S, T)=b|S|+d_{H-S}(T)-a|T| \geq b|S|-a|T| \geq 0
$$

which is a contradiction.
Claim 5. $|S|+|X|<\frac{(a+b) n}{a+2 b}$.
Proof. According to Claim 1, Claim 4 and since $|S|+|T|+|X| \leq n$, we have

$$
\begin{aligned}
a n & \geq a|S|+a|T|+a|X|>a|S|+b|S|+a|X|=(a+b)(|S|+|X|)-b|X| \\
& \geq(a+b)(|S|+|X|)-\frac{b^{2} n}{a+2 b}
\end{aligned}
$$

which implies

$$
|S|+|X|<\frac{(a+b) n}{a+2 b}
$$

In view of Claim $3, T \neq \emptyset$. Define

$$
h_{1}=\min \left\{d_{H-S}(x): x \in T\right\}
$$

and

$$
R=\left\{x: x \in T, d_{H-S}(x)=0\right\}
$$

We write $r=|R|$ and choose $x_{1} \in T$ such that $d_{H-S}\left(x_{1}\right)=h_{1}$. If $T \backslash N_{T}\left[x_{1}\right]$ $\neq \emptyset$, let

$$
h_{2}=\min \left\{d_{H-S}(x): x \in T \backslash N_{T}\left[x_{1}\right]\right\}
$$

Thus, we have $0 \leq h_{1} \leq h_{2} \leq a$ by the definition of $T$.
We shall consider various cases by the value of $r$ and derive a contradiction in each case.

Case 1. $r \geq 2$. Obviously, there exist $x, y \in R$ such that $d_{H-S}(x)=d_{H-S}(y)$ $=0$ and $x y \notin E(G)$. In terms of $H=G-X$, Claim 5 and the condition of Theorem 1, we obtain

$$
\begin{aligned}
\frac{(a+b) n}{a+2 b} & \leq\left|N_{G}(x) \cup N_{G}(y)\right| \leq\left|N_{H}(x) \cup N_{H}(y)\right|+|X| \\
& \leq d_{H-S}(x)+d_{H-S}(y)+|S|+|X|=|S|+|X|<\frac{(a+b) n}{a+2 b}
\end{aligned}
$$

which is a contradiction.
Case 2. $r=1$. Clearly, $h_{1}=0$ and $\left|N_{T}\left[x_{1}\right]\right|=1$. According to Claim 3, $r=1$ and $\left|N_{T}\left[x_{1}\right]\right|=1$, we have $T \backslash N_{T}\left[x_{1}\right] \neq \emptyset$ and $1 \leq h_{2} \leq a$. Choose $x_{2} \in T \backslash N_{T}\left[x_{1}\right]$ such that $d_{H-S}\left(x_{2}\right)=h_{2}$. It is easy to see that $x_{1} x_{2} \notin E(G)$. According to $H=G-X$ and the condition of Theorem 1, we have

$$
\begin{aligned}
\frac{(a+b) n}{a+2 b} & \leq\left|N_{G}\left(x_{1}\right) \cup N_{G}\left(x_{2}\right)\right| \leq\left|N_{H}\left(x_{1}\right) \cup N_{H}\left(x_{2}\right)\right|+|X| \\
& \leq d_{H-S}\left(x_{1}\right)+d_{H-S}\left(x_{2}\right)+|S|+|X|=h_{2}+|S|+|X|
\end{aligned}
$$

which implies

$$
\begin{equation*}
|S| \geq \frac{(a+b) n}{a+2 b}-h_{2}-|X| \tag{2}
\end{equation*}
$$

Note that $\left|T \backslash N_{T}\left[x_{1}\right]\right|=|T|-1$. Combining this with $|S|+|T|+|X| \leq n$, (2), Claim $1, b \geq a \geq 1,1 \leq h_{2} \leq a$ and $n \geq \frac{(a+2 b)(2 a+2 b-3)+1}{b}>\frac{(a+2 b)(2 a+2 b-3)}{b}$, we obtain

$$
\begin{aligned}
\delta_{H}(S, T) & =b|S|+d_{H-S}(T)-a|T| \\
& =b|S|+d_{H-S}\left(N_{T}\left[x_{1}\right]\right)+d_{H-S}\left(T \backslash N_{T}\left[x_{1}\right]\right)-a|T|
\end{aligned}
$$

$$
\begin{aligned}
& =b|S|+d_{H-S}\left(T \backslash N_{T}\left[x_{1}\right]\right)-a|T| \geq b|S|+h_{2}(|T|-1)-a|T| \\
& =b|S|-\left(a-h_{2}\right)|T|-h_{2} \geq b|S|-\left(a-h_{2}\right)(n-|S|-|X|)-h_{2} \\
& =\left(a+b-h_{2}\right)|S|-\left(a-h_{2}\right) n+\left(a-h_{2}\right)|X|-h_{2} \\
& \geq\left(a+b-h_{2}\right)\left(\frac{(a+b) n}{a+2 b}-h_{2}-|X|\right)-\left(a-h_{2}\right) n+\left(a-h_{2}\right)|X|-h_{2} \\
& =\left(a+b-h_{2}\right)\left(\frac{(a+b) n}{a+2 b}-h_{2}\right)-\left(a-h_{2}\right) n-b|X|-h_{2} \\
& \geq\left(a+b-h_{2}\right)\left(\frac{(a+b) n}{a+2 b}-h_{2}\right)-\left(a-h_{2}\right) n-\frac{b^{2} n}{a+2 b}-h_{2} \\
& =h_{2}^{2}+\left(\frac{b n}{a+2 b}-a-b-1\right) h_{2} \\
& >h_{2}^{2}+\left(\frac{(a+2 b)(2 a+2 b-3)}{a+2 b}-a-b-1\right) h_{2} \\
& =h_{2}^{2}+(a+b-4) h_{2} \geq h_{2}^{2}-2 h_{2}=\left(h_{2}-1\right)^{2}-1 \geq-1
\end{aligned}
$$

which contradicts (1).
Case 3. $r=0$. If $h_{1}=a$, then by (1) we obtain $-1 \geq \delta_{H}(S, T)=b|S|+$ $d_{H-S}(T)-a|T| \geq b|S|+h_{1}|T|-a|T|=b|S| \geq 0$, which is a contradiction. Thus, we have

$$
\begin{equation*}
1 \leq h_{1} \leq a-1 \tag{3}
\end{equation*}
$$

We now prove the following claim.
Claim 6. $T \backslash N_{T}\left[x_{1}\right] \neq \emptyset$.
Proof. Suppose that $T=N_{T}\left[x_{1}\right]$. Then from (3) we have

$$
|T|=\left|N_{T}\left[x_{1}\right]\right| \leq\left|N_{H-S}\left[x_{1}\right]\right|=d_{H-S}\left(x_{1}\right)+1=h_{1}+1 \leq a
$$

which contradicts Claim 3.
In view of Claim 6, there exists $x_{2} \in T \backslash N_{T}\left[x_{1}\right]$ such that $d_{H-S}\left(x_{2}\right)=h_{2}$. Obviously, $x_{1} x_{2} \notin E(G)$. According to the condition of Theorem 1, we obtain

$$
\begin{aligned}
\frac{(a+b) n}{a+2 b} & \leq\left|N_{G}\left(x_{1}\right) \cup N_{G}\left(x_{2}\right)\right| \leq\left|N_{H}\left(x_{1}\right) \cup N_{H}\left(x_{2}\right)\right|+|X| \\
& \leq d_{H-S}\left(x_{1}\right)+d_{H-S}\left(x_{2}\right)+|S|+|X|=h_{1}+h_{2}+|S|+|X|
\end{aligned}
$$

that is,

$$
\begin{equation*}
|S| \geq \frac{(a+b) n}{a+2 b}-h_{1}-h_{2}-|X| \tag{4}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
\left|N_{T}\left[x_{1}\right]\right| \leq\left|N_{H-S}\left[x_{1}\right]\right|=d_{H-S}\left(x_{1}\right)+1=h_{1}+1 . \tag{5}
\end{equation*}
$$

Using $1 \leq h_{1} \leq h_{2} \leq a,|S|+|T|+|X| \leq n$, (4), (5) and Claim 1, we have

$$
\begin{aligned}
\delta_{H}(S, T) & =b|S|+d_{H-S}(T)-a|T| \\
& =b|S|+d_{H-S}\left(N_{T}\left[x_{1}\right]\right)+d_{H-S}\left(T \backslash N_{T}\left[x_{1}\right]\right)-a|T| \\
& \geq b|S|+h_{1}\left|N_{T}\left[x_{1}\right]\right|+h_{2}\left(|T|-\left|N_{T}\left[x_{1}\right]\right|\right)-a|T| \\
& =b|S|-\left(h_{2}-h_{1}\right)\left|N_{T}\left[x_{1}\right]\right|-\left(a-h_{2}\right)|T| \\
& \geq b|S|-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right)-\left(a-h_{2}\right)(n-|S|-|X|) \\
& =\left(a+b-h_{2}\right)|S|-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right)-\left(a-h_{2}\right) n+\left(a-h_{2}\right)|X| \\
& \geq\left(a+b-h_{2}\right)\left(\frac{(a+b) n}{a+2 b}-h_{1}-h_{2}-|X|\right)-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right) \\
& -\left(a-h_{2}\right) n+\left(a-h_{2}\right)|X| \\
& =\left(a+b-h_{2}\right)\left(\frac{(a+b) n}{a+2 b}-h_{1}-h_{2}\right)-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right) \\
& -\left(a-h_{2}\right) n-b|X| \\
& \geq\left(a+b-h_{2}\right)\left(\frac{(a+b) n}{a+2 b}-h_{1}-h_{2}\right)-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right) \\
& -\left(a-h_{2}\right) n-\frac{b^{2} n}{a+2 b} \\
& =\frac{b n}{a+2 b} h_{2}-\left(a+b-h_{2}\right)\left(h_{1}+h_{2}\right)-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right),
\end{aligned}
$$

that is,
(6) $\quad \delta_{H}(S, T) \geq \frac{b n}{a+2 b} h_{2}-\left(a+b-h_{2}\right)\left(h_{1}+h_{2}\right)-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right)$.

Let $F\left(h_{1}, h_{2}\right)=\frac{b n}{a+2 b} h_{2}-\left(a+b-h_{2}\right)\left(h_{1}+h_{2}\right)-\left(h_{2}-h_{1}\right)\left(h_{1}+1\right)$. Thus, by
(3) we have

$$
\begin{aligned}
\frac{\partial F\left(h_{1}, h_{2}\right)}{\partial h_{1}} & =-\left(a+b-h_{2}\right)-\left(-h_{1}-1+h_{2}-h_{1}\right)=-(a+b)+2 h_{1}+1 \\
& \leq-(a+b)+2(a-1)+1 \leq-1
\end{aligned}
$$

Combining this with $1 \leq h_{1} \leq h_{2} \leq a$, we obtain

$$
\begin{equation*}
F\left(h_{1}, h_{2}\right) \geq F\left(h_{2}, h_{2}\right) \tag{7}
\end{equation*}
$$

In terms of $(6),(7), 1 \leq h_{2} \leq a$ and $n \geq \frac{(a+2 b)(2 a+2 b-3)+1}{b}>\frac{(a+2 b)(2 a+2 b-3)}{b}$, we have

$$
\begin{aligned}
\delta_{H}(S, T) & \geq F\left(h_{1}, h_{2}\right) \geq F\left(h_{2}, h_{2}\right)=\frac{b n}{a+2 b} h_{2}-2\left(a+b-h_{2}\right) h_{2} \\
& >\frac{(a+2 b)(2 a+2 b-3)}{a+2 b} h_{2}-2\left(a+b-h_{2}\right) h_{2} \\
& =h_{2}\left(2 h_{2}-3\right) \geq-1
\end{aligned}
$$

which contradicts (1).
In all the cases above we obtained contradictions. Hence, $H$ has a fractional $[a, b]$-factor, that is, $G$ is fractional ID-[ $a, b]$-factor-critical. The proof of Theorem 1 is complete.

## 3. REmarks

Remark 5. In Theorem 1, the bound in the condition

$$
\left|N_{G}(x) \cup N_{G}(y)\right| \geq \frac{(a+b) n}{a+2 b}
$$

is sharp. We can show this by constructing a graph $G=(a t) K_{1} \vee(b t) K_{1} \vee$ $(b t+1) K_{1}$, where $t$ is a sufficiently large positive integer. It is easy to see that $|V(G)|=n=(a+2 b) t+1$ and

$$
\begin{aligned}
\frac{(a+b) n}{a+2 b} & >\left|N_{G}(x) \cup N_{G}(y)\right|=(a+b) t=(a+b) \cdot \frac{n-1}{a+2 b} \\
& =\frac{(a+b) n}{a+2 b}-\frac{a+b}{a+2 b}>\frac{(a+b) n}{a+2 b}-1
\end{aligned}
$$

for each pair of nonadjacent vertices $x, y$ of $(b t+1) K_{1} \subset G$. Set $X=(b t) K_{1}$. Clearly, $X$ is an independent set of $G$. Put $H=G-X=(a t) K_{1} \vee(b t+1) K_{1}$, $S=(a t) K_{1}$ and $T=(b t+1) K_{1}$. Then $|S|=a t,|T|=b t+1$ and $d_{H-S}(T)=0$. Thus, we have

$$
\begin{aligned}
\delta_{H}(S, T) & =b|S|+d_{H-S}(T)-a|T| \\
& =a b t-a(b t+1)=-a<0
\end{aligned}
$$

In terms of Lemma 4, $H$ has no fractional $[a, b]$-factor. Hence, $G$ is not fractional ID- $[a, b]$-factor-critical.

Remark 6. We show that the bound on minimum degree $\delta(G) \geq \frac{b n}{a+2 b}+a$ in Theorem 1 is also best possible. Consider a graph $G$ constructed from $b t K_{1}$, $(a t-1) K_{1}, \frac{b t}{2} K_{2}$ and $K_{1}$ as follows: let $\left\{x_{1}, x_{2}, \ldots, x_{a-1}\right\} \subset(a t-1) K_{1}$ and
$K_{1}=\{u\}$, where $t$ is a sufficiently large positive integer and $b t$ is even. Set $V(G)=V\left(b t K_{1} \cup(a t-1) K_{1} \cup \frac{b t}{2} K_{2} \cup\{u\}\right)$ and $E(G)=E\left(b t K_{1} \vee(a t-1) K_{1} \vee \frac{b t}{2} K_{2}\right)$ $\cup E\left(b t K_{1} \vee\{u\}\right) \cup\left\{u x_{i}: i=1,2, \ldots, a-1\right\}$. It is easily seen that $\mid N_{G}(x) \cup$ $\left.N_{G}(y)\right\} \left\lvert\, \geq \frac{(a+b) n}{a+2 b}\right.$ for each pair of nonadjacent vertices $x, y$ of $G, n=(a+2 b) t$ and $\delta(G)=\frac{b n}{a+2 b}+a-1$. Let $X=b t K_{1}$. It is easy to see that $X$ is an independent set of $G$. Set $H=G-X$. Then $\delta(H)=d_{H}(u)=a-1$. Clearly, $H$ has no fractional $[a, b]$-factor, that is, $G$ is not fractional ID- $[a, b]$-factor-critical.

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