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A NEIGHBORHOOD CONDITION FOR FRACTIONAL ID-$[A, B]$-FACTOR-CRITICAL GRAPHS

Sizhong Zhou, Fan Yang

School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003, P.R. China

e-mail: zsz_cumt@163.com
fanyang_just@163.com

and

Zhiren Sun

School of Mathematical Sciences
Nanjing Normal University Nanjing
Jiangsu 210046, P.R. China

e-mail: 05119@njnu.edu.cn

Abstract

Let G be a graph of order n, and let a and b be two integers with $1 \le a \le b$. Let $h : E(G) \to [0, 1]$ be a function. If $a \le \sum_{e \ni x} h(e) \le b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is fractional independent-set-deletable $[a, b]$-factor-critical (in short, fractional ID-$[a, b]$-factor-critical) if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G. In this paper, it is proved that if $n \ge \frac{(a+2b)(2a+2b-3)+1}{b}$, $\delta(G) \ge \frac{bn}{a+2b} + a$ and $|N_G(x) \cup N_G(y)| \ge \frac{(a+b)n}{a+2b}$ for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-$[a, b]$-factor-critical. Furthermore, it is shown that this result is best possible in some sense.

Keywords: graph, minimum degree, neighborhood, fractional $[a, b]$-factor, fractional ID-$[a, b]$-factor-critical graph.

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1. Introduction

The graphs considered here will be finite, undirected and without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. For any $x \in V(G)$, we denote the degree of x in G by $d_G(x)$. We write $N_G(x)$ for the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. Let S and T be two disjoint vertex subsets of G; we denote the number of edges from S to T by $e_G(S, T)$. We denote by $\delta(G)$ the minimum degree of G. For any nonempty subset S of $V(G)$, let

$$N_G(S) = \bigcup_{x \in S} N_G(x).$$

If G and H are vertex-disjoint graphs, then their join and union are denoted by $G \vee H$ and $G \cup H$, respectively.

A factor of a graph G is a spanning subgraph of G. Let a and b be two positive integers with $1 \le a \le b$. Then a factor F is an $[a, b]$-factor if $a \le d_F(x) \le b$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $a \le \sum_{e \ni x} h(e) \le b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. If $G - I$ admits a fractional $[a, b]$-factor for every independent set I of G, then we say that G is fractional ID-$[a, b]$-factor-critical [1]. A fractional ID-$[k, k]$-factor-critical graph is simply called a fractional ID-k-factor-critical graph.

Many authors have investigated factors and fractional factors in graphs; see, for instance, [2, 3, 4, 5, 6, 7, 8, 9]. Chang, Liu and Zhu [10] showed a minimum degree condition for a graph to be a fractional ID-k-factor-critical graph. Zhou, Bian and Wu [11] gave a degree condition for the existence of fractional ID-k-factor-critical graphs. Zhou [12] obtained a binding number condition for graphs to be fractional ID-k-factor-critical graphs. Zhou, Sun and Liu [1] obtained a minimum degree condition for a graph to be a fractional ID-$[a, b]$-factor-critical graph. In this paper, we proceed to study fractional ID-$[a, b]$-factor-critical graphs, and obtain a neighborhood condition for a graph to be fractional ID-$[a, b]$-factor-critical. The main result is the following theorem.

Theorem 1. *Let $1 \le a \le b$ be two integers, and let G be a graph of order n with $n \ge \frac{(a+2b)(2a+2b-3)+1}{b}$, and $\delta(G) \ge \frac{bn}{a+2b} + a$. If*

$$|N_G(x) \cup N_G(y)| \ge \frac{(a+b)n}{a+2b}$$

for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-$[a, b]$-factor-critical.

If $a = b = k$ in Theorem 1, then we obtain the following result.

Theorem 2. *Let $k \geq 1$ be an integer, and let G be a graph of order n with $n \geq 12k - 8$, and $\delta(G) \geq \frac{n}{3} + k$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{2n}{3}$$

for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-k-factor-critical.

If $k = 1$ in Theorem 2, then we get the following result.

Theorem 3. *Let G be a graph of order n with $n \geq 4$, and $\delta(G) \geq \frac{n}{3} + 1$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{2n}{3}$$

for any two nonadjacent vertices $x, y \in V(G)$, then G is fractional ID-factor-critical.

2. The Proof of Theorem 1

In order to prove Theorem 1, we rely heavily on the following lemma.

Lemma 4 [13]**.** *Let G be a graph. Then G has a fractional $[a, b]$-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

***Proof of Theorem* 1.** Let X be an independent set of G and $H = G - X$. In order to complete the proof of Theorem 1, we need only to prove that H has a fractional $[a, b]$-factor. By contradiction, suppose that H has no fractional $[a, b]$-factor. Then by Lemma 4, there exists some subset $S \subseteq V(H)$ such that

$$\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \leq -1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\}$. We first prove the following claims.

Claim 1. $|X| \leq \frac{bn}{a+2b}$.

Proof. Since $n \geq \frac{(a+2b)(2a+2b-3)+1}{b}$, the inequality holds for $|X| = 1$. In the following we may assume $|X| \geq 2$. In terms of the condition of Theorem 1, there exist $x, y \in X$ such that $|N_G(x) \cup N_G(y)| \geq \frac{(a+b)n}{a+2b}$. Since X is independent, we obtain $X \cap (N_G(x) \cup N_G(y)) = \emptyset$. Thus, we have

$$|X| + \frac{(a+b)n}{a+2b} \leq |X| + |N_G(x) \cup N_G(y)| \leq n,$$

which implies

$$|X| \leq n - \frac{(a+b)n}{a+2b} = \frac{bn}{a+2b}.$$

□

Claim 2. $\delta(H) \geq a$.

Proof. Note that $H = G - X$. Combining this with Claim 1, we obtain

$$\delta(H) \geq \delta(G) - |X| \geq \left(\frac{bn}{a+2b} + a\right) - \frac{bn}{a+2b} = a.$$

□

Claim 3. $|T| \geq b + 1$.

Proof. If $|T| \leq b$, then from Claim 2 and since $|S| + d_{H-S}(x) \geq d_H(x) \geq \delta(H)$ for each $x \in T$, we have

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \geq |T||S| + d_{H-S}(T) - a|T| \\ &= \sum\nolimits_{x \in T}(|S| + d_{H-S}(x) - a) \geq \sum\nolimits_{x \in T}(\delta(H) - a) \geq 0,\end{aligned}$$

which contradicts (1). □

Claim 4. $a|T| > b|S|$.

Proof. If $a|T| \leq b|S|$, then from (1) we obtain

$$-1 \geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| - a|T| \geq 0,$$

which is a contradiction. □

Claim 5. $|S| + |X| < \frac{(a+b)n}{a+2b}$.

Proof. According to Claim 1, Claim 4 and since $|S| + |T| + |X| \leq n$, we have

$$\begin{aligned}an &\geq a|S| + a|T| + a|X| > a|S| + b|S| + a|X| = (a+b)(|S| + |X|) - b|X| \\ &\geq (a+b)(|S| + |X|) - \frac{b^2 n}{a+2b},\end{aligned}$$

which implies

$$|S| + |X| < \frac{(a+b)n}{a+2b}.$$

□

In view of Claim 3, $T \neq \emptyset$. Define

$$h_1 = \min\{d_{H-S}(x) : x \in T\}$$

and

$$R = \{x : x \in T, d_{H-S}(x) = 0\}.$$

We write $r = |R|$ and choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T \setminus N_T[x_1] \neq \emptyset$, let

$$h_2 = \min\{d_{H-S}(x) : x \in T \setminus N_T[x_1]\}.$$

Thus, we have $0 \le h_1 \le h_2 \le a$ by the definition of T.

We shall consider various cases by the value of r and derive a contradiction in each case.

Case 1. $r \ge 2$. Obviously, there exist $x, y \in R$ such that $d_{H-S}(x) = d_{H-S}(y) = 0$ and $xy \notin E(G)$. In terms of $H = G - X$, Claim 5 and the condition of Theorem 1, we obtain

$$\begin{aligned}\frac{(a+b)n}{a+2b} &\le |N_G(x) \cup N_G(y)| \le |N_H(x) \cup N_H(y)| + |X| \\ &\le d_{H-S}(x) + d_{H-S}(y) + |S| + |X| = |S| + |X| < \frac{(a+b)n}{a+2b},\end{aligned}$$

which is a contradiction.

Case 2. $r = 1$. Clearly, $h_1 = 0$ and $|N_T[x_1]| = 1$. According to Claim 3, $r = 1$ and $|N_T[x_1]| = 1$, we have $T \setminus N_T[x_1] \neq \emptyset$ and $1 \le h_2 \le a$. Choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. It is easy to see that $x_1x_2 \notin E(G)$. According to $H = G - X$ and the condition of Theorem 1, we have

$$\begin{aligned}\frac{(a+b)n}{a+2b} &\le |N_G(x_1) \cup N_G(x_2)| \le |N_H(x_1) \cup N_H(x_2)| + |X| \\ &\le d_{H-S}(x_1) + d_{H-S}(x_2) + |S| + |X| = h_2 + |S| + |X|,\end{aligned}$$

which implies

$$|S| \ge \frac{(a+b)n}{a+2b} - h_2 - |X|. \tag{2}$$

Note that $|T \setminus N_T[x_1]| = |T| - 1$. Combining this with $|S| + |T| + |X| \le n$, (2), Claim 1, $b \ge a \ge 1$, $1 \le h_2 \le a$ and $n \ge \frac{(a+2b)(2a+2b-3)+1}{b} > \frac{(a+2b)(2a+2b-3)}{b}$, we obtain

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\ &= b|S| + d_{H-S}(N_T[x_1]) + d_{H-S}(T \setminus N_T[x_1]) - a|T|\end{aligned}$$

$$\begin{aligned}
&= b|S| + d_{H-S}(T \setminus N_T[x_1]) - a|T| \geq b|S| + h_2(|T|-1) - a|T| \\
&= b|S| - (a-h_2)|T| - h_2 \geq b|S| - (a-h_2)(n - |S| - |X|) - h_2 \\
&= (a+b-h_2)|S| - (a-h_2)n + (a-h_2)|X| - h_2 \\
&\geq (a+b-h_2)\left(\frac{(a+b)n}{a+2b} - h_2 - |X|\right) - (a-h_2)n + (a-h_2)|X| - h_2 \\
&= (a+b-h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (a-h_2)n - b|X| - h_2 \\
&\geq (a+b-h_2)\left(\frac{(a+b)n}{a+2b} - h_2\right) - (a-h_2)n - \frac{b^2n}{a+2b} - h_2 \\
&= h_2^2 + \left(\frac{bn}{a+2b} - a - b - 1\right)h_2 \\
&> h_2^2 + \left(\frac{(a+2b)(2a+2b-3)}{a+2b} - a - b - 1\right)h_2 \\
&= h_2^2 + (a+b-4)h_2 \geq h_2^2 - 2h_2 = (h_2-1)^2 - 1 \geq -1,
\end{aligned}$$

which contradicts (1).

Case 3. $r = 0$. If $h_1 = a$, then by (1) we obtain $-1 \geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + h_1|T| - a|T| = b|S| \geq 0$, which is a contradiction. Thus, we have

$$1 \leq h_1 \leq a - 1. \tag{3}$$

We now prove the following claim.

Claim 6. $T \setminus N_T[x_1] \neq \emptyset$.

Proof. Suppose that $T = N_T[x_1]$. Then from (3) we have

$$|T| = |N_T[x_1]| \leq |N_{H-S}[x_1]| = d_{H-S}(x_1) + 1 = h_1 + 1 \leq a,$$

which contradicts Claim 3. □

In view of Claim 6, there exists $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. Obviously, $x_1x_2 \notin E(G)$. According to the condition of Theorem 1, we obtain

$$\begin{aligned}
\frac{(a+b)n}{a+2b} &\leq |N_G(x_1) \cup N_G(x_2)| \leq |N_H(x_1) \cup N_H(x_2)| + |X| \\
&\leq d_{H-S}(x_1) + d_{H-S}(x_2) + |S| + |X| = h_1 + h_2 + |S| + |X|,
\end{aligned}$$

that is,

$$|S| \geq \frac{(a+b)n}{a+2b} - h_1 - h_2 - |X|. \tag{4}$$

It is easy to see that

$$|N_T[x_1]| \leq |N_{H-S}[x_1]| = d_{H-S}(x_1) + 1 = h_1 + 1. \tag{5}$$

Using $1 \leq h_1 \leq h_2 \leq a$, $|S| + |T| + |X| \leq n$, (4), (5) and Claim 1, we have

$$\begin{aligned}
\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\
&= b|S| + d_{H-S}(N_T[x_1]) + d_{H-S}(T \setminus N_T[x_1]) - a|T| \\
&\geq b|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - a|T| \\
&= b|S| - (h_2 - h_1)|N_T[x_1]| - (a - h_2)|T| \\
&\geq b|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)(n - |S| - |X|) \\
&= (a + b - h_2)|S| - (h_2 - h_1)(h_1 + 1) - (a - h_2)n + (a - h_2)|X| \\
&\geq (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_1 - h_2 - |X|\right) - (h_2 - h_1)(h_1 + 1) \\
&- (a - h_2)n + (a - h_2)|X| \\
&= (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_1 - h_2\right) - (h_2 - h_1)(h_1 + 1) \\
&- (a - h_2)n - b|X| \\
&\geq (a + b - h_2)\left(\frac{(a+b)n}{a+2b} - h_1 - h_2\right) - (h_2 - h_1)(h_1 + 1) \\
&- (a - h_2)n - \frac{b^2 n}{a+2b} \\
&= \frac{bn}{a+2b}h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1),
\end{aligned}$$

that is,

$$\delta_H(S,T) \geq \frac{bn}{a+2b}h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1). \tag{6}$$

Let $F(h_1, h_2) = \frac{bn}{a+2b}h_2 - (a + b - h_2)(h_1 + h_2) - (h_2 - h_1)(h_1 + 1)$. Thus, by (3) we have

$$\begin{aligned}
\frac{\partial F(h_1, h_2)}{\partial h_1} &= -(a + b - h_2) - (-h_1 - 1 + h_2 - h_1) = -(a + b) + 2h_1 + 1 \\
&\leq -(a + b) + 2(a - 1) + 1 \leq -1.
\end{aligned}$$

Combining this with $1 \leq h_1 \leq h_2 \leq a$, we obtain

$$F(h_1, h_2) \geq F(h_2, h_2). \tag{7}$$

In terms of (6), (7), $1 \le h_2 \le a$ and $n \ge \frac{(a+2b)(2a+2b-3)+1}{b} > \frac{(a+2b)(2a+2b-3)}{b}$, we have

$$\begin{aligned}\delta_H(S,T) &\ge F(h_1,h_2) \ge F(h_2,h_2) = \frac{bn}{a+2b}h_2 - 2(a+b-h_2)h_2 \\ &> \frac{(a+2b)(2a+2b-3)}{a+2b}h_2 - 2(a+b-h_2)h_2 \\ &= h_2(2h_2-3) \ge -1,\end{aligned}$$

which contradicts (1).

In all the cases above we obtained contradictions. Hence, H has a fractional $[a,b]$-factor, that is, G is fractional ID-$[a,b]$-factor-critical. The proof of Theorem 1 is complete. ■

3. Remarks

Remark 5. In Theorem 1, the bound in the condition

$$|N_G(x) \cup N_G(y)| \ge \frac{(a+b)n}{a+2b}$$

is sharp. We can show this by constructing a graph $G = (at)K_1 \vee (bt)K_1 \vee (bt+1)K_1$, where t is a sufficiently large positive integer. It is easy to see that $|V(G)| = n = (a+2b)t+1$ and

$$\begin{aligned}\frac{(a+b)n}{a+2b} &> |N_G(x) \cup N_G(y)| = (a+b)t = (a+b)\cdot\frac{n-1}{a+2b} \\ &= \frac{(a+b)n}{a+2b} - \frac{a+b}{a+2b} > \frac{(a+b)n}{a+2b} - 1\end{aligned}$$

for each pair of nonadjacent vertices x, y of $(bt+1)K_1 \subset G$. Set $X = (bt)K_1$. Clearly, X is an independent set of G. Put $H = G - X = (at)K_1 \vee (bt+1)K_1$, $S = (at)K_1$ and $T = (bt+1)K_1$. Then $|S| = at$, $|T| = bt+1$ and $d_{H-S}(T) = 0$. Thus, we have

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\ &= abt - a(bt+1) = -a < 0.\end{aligned}$$

In terms of Lemma 4, H has no fractional $[a,b]$-factor. Hence, G is not fractional ID-$[a,b]$-factor-critical.

Remark 6. We show that the bound on minimum degree $\delta(G) \ge \frac{bn}{a+2b} + a$ in Theorem 1 is also best possible. Consider a graph G constructed from btK_1, $(at-1)K_1$, $\frac{bt}{2}K_2$ and K_1 as follows: let $\{x_1, x_2, \ldots, x_{a-1}\} \subset (at-1)K_1$ and

$K_1 = \{u\}$, where t is a sufficiently large positive integer and bt is even. Set $V(G) = V(btK_1 \cup (at-1)K_1 \cup \frac{bt}{2}K_2 \cup \{u\})$ and $E(G) = E(btK_1 \vee (at-1)K_1 \vee \frac{bt}{2}K_2) \cup E(btK_1 \vee \{u\}) \cup \{ux_i : i = 1, 2, \ldots, a-1\}$. It is easily seen that $|N_G(x) \cup N_G(y)\}| \geq \frac{(a+b)n}{a+2b}$ for each pair of nonadjacent vertices x, y of G, $n = (a+2b)t$ and $\delta(G) = \frac{bn}{a+2b} + a - 1$. Let $X = btK_1$. It is easy to see that X is an independent set of G. Set $H = G - X$. Then $\delta(H) = d_H(u) = a - 1$. Clearly, H has no fractional $[a, b]$-factor, that is, G is not fractional ID-$[a, b]$-factor-critical.

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