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HEAVY SUBGRAPH CONDITIONS FOR LONGEST CYCLES TO BE HEAVY IN GRAPHS

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Abstract

Let G be a graph on n vertices. A vertex of G with degree at least $n/2$ is called a heavy vertex, and a cycle of G which contains all the heavy vertices of G is called a heavy cycle. In this note, we characterize graphs which contain no heavy cycles. For a given graph H, we say that G is H-*heavy* if every induced subgraph of G isomorphic to H contains two nonadjacent vertices with degree sum at least n. We find all the connected graphs S such that a 2-connected graph G being S-heavy implies any longest cycle of G is a heavy cycle.

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1. Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph on n vertices. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ and $d_H(v)$ to denote the set and the number of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. A vertex

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v is called *heavy* if $d(v) \geq n/2$, and a cycle C is called *heavy* if C contains all heavy vertices of G.

The following theorem on the existence of heavy cycles in graphs is well known.

Theorem 1 (Bollobás and Brightwell [1], Shi [4]). *Every* 2*-connected graph has a heavy cycle.*

Let G be a graph, v be a vertex, and e be an edge of G. We use $G - v$ to denote the graph obtained from G by deleting v, and $G - e$ the graph obtained from G by deleting e.

A connected graph is called *separable* if it has at least one cut vertex. By Theorem 1, if a graph G (of order at least 3) contains no heavy cycles, then it is separable. We first characterize the separable graphs which contain no heavy cycles.

Theorem 2. *Let G be a connected graph on n vertices. If G contains no heavy cycles, then G has at most two heavy vertices. Moreover,*

(1) *if G contains no heavy vertices, then G is a tree;*
(2) *if G contains only one heavy vertex, say x, then $G - x$ contains at least $n/2$ components, and each component of $G - x$ contains exactly one neighbor of x;*
(3) *if G has exactly two heavy vertices, say x and y, then $xy \in E(G)$ and xy is a cut edge of G, n is even and both components of $G - xy$ have $n/2$ vertices, and x (and y, respectively) is adjacent to every vertex in the component which contains x (y). Briefly stated, $T_1 \subseteq G \subseteq T_2$ (see Figure 1).*

We postpone the proof of Theorem 2 to Section 3.

If x and y are two vertices of a graph G, then an (x, y)*-path* is a path connecting the two vertices x and y. The *distance* between x and y, denote by $d(x, y)$, is the length of a shortest (x, y)-path in G.

Let H be a subgraph of a graph G. If H contains all edges $xy \in E(G)$ with $x, y \in V(H)$, then H is called an *induced subgraph* of G. Let X be a subset of $V(G)$. The induced subgraph of G with vertex set X is called a subgraph *induced by* X, and is denoted by $G[X]$. We use $G - X$ to denote the subgraph induced by $V(G) \setminus X$, and use the notation $G - H$ instead of $G - V(H)$.

Let G be a graph on n vertices. For a given graph H, we say that G is H*-free* if G does not contain an induced subgraph isomorphic to H. If H is an induced subgraph of G, we say that H is *heavy in* G if there are two nonadjacent vertices in $V(H)$ with degree sum at least n. A graph G is called H*-heavy* if every induced subgraph of G isomorphic to H is heavy. Note that an H-free graph is also H-heavy, and if H_1 is an induced subgraph of H_2, then an H_1-free (H_1-heavy) graph is also H_2-free (H_2-heavy).

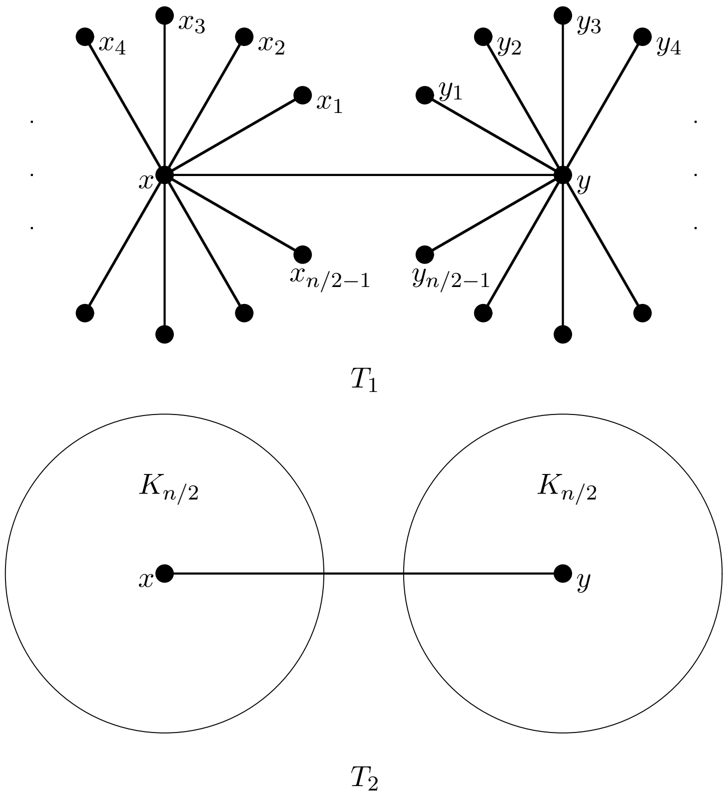

Figure 1. Extremal graphs with two heavy vertices and no heavy cycles.

In general, a longest cycle of a graph may not be a heavy cycle (see Figure 1). In this note, we mainly consider heavy subgraph conditions for longest cycles to be heavy. First, consider the following theorem.

Theorem 3 (Fan [3])**.** *Let G be a 2-connected graph. If* $\max\{d(u), d(v)\} \geq n/2$ *for every pair of vertices u, v with distance 2 in G, then G is Hamiltonian.*

This theorem implies that every 2-connected P_3-heavy graph has a Hamilton cycle, which is of course a heavy cycle. In fact we have the following theorem.

Theorem 4. *If G is a 2-connected $K_{1,4}$-heavy graph, and C is a longest cycle of G, then C is a heavy cycle of G.*

We postpone the proof of this theorem to Section 4.

Note that $K_{1,3}$ is an induced subgraph of $K_{1,4}$. So any longest cycle of a $K_{1,3}$-heavy graph is heavy. In fact we can get the following result.

Theorem 5. *Let S be a connected graph on at least 3 vertices and G be a 2-connected graph. Then G being S-free (or S-heavy) implies that every longest cycle of G is a heavy cycle, if and only if $S = P_3, K_{1,3}$ or $K_{1,4}$.*

The sufficiency of this theorem follows from Theorem 4 immediately. We will prove its necessity in Section 5.

2. Preliminaries

We first give some additional terminology and notation.

Let s, t be two integers with $s \leq t$, and x_i be a vertex of a graph for $s \leq i \leq t$. Then we use $[x_s, x_t]$ to denote the set of vertices $\{x_i : s \leq i \leq t\}$.

Let P be a path and $x, y \in V(P)$. We use $P[x, y]$ to denote the subpath of P from x to y. Let C be a cycle with a given orientation and $x, y \in V(C)$. We use $\overrightarrow{C}[x, y]$ to denote the path from x to y on $\overrightarrow{C}$, and $\overleftarrow{C}[y, x]$ to denote the path $\overrightarrow{C}[x, y]$ with the opposite direction.

Let G be a graph on n vertices and $k \geq 3$ be an integer. We call a circular sequence of vertices $C = v_1v_2 \cdots v_kv_1$ an *ore-cycle* (or shortly, *o-cycle*) of G, if for all i with $1 \leq i \leq k$, either $v_iv_{i+1} \in E(G)$ or $d(v_i)+d(v_{i+1}) \geq n$, where $v_{k+1} = v_1$. The *deficit degree* of the o-cycle C is defined by $\mathrm{def}(C) = |\{i : v_iv_{i+1} \notin E(G)$ with $1 \leq i \leq k\}|$. Thus a cycle is an o-cycle with deficit degree 0. Similarly, we can define *o-paths* of G.

Now, we prove the following lemma on o-cycles.

Lemma 6. *Let G be a graph and C be an o-cycle of G. Then there exists a cycle of G which contains all the vertices in $V(C)$.*

Proof. Assume the opposite. Let C' be an o-cycle which contains all the vertices in $V(C)$ such that $\mathrm{def}(C')$ is as small as possible. Then we have $\mathrm{def}(C') \geq 1$. Without loss of generality, we suppose that $C' = v_1v_2 \cdots v_kv_1$, where $v_1v_k \notin E(G)$ and $d(v_1) + d(v_k) \geq n$. We use P to denote the o-path $P = v_1v_2 \cdots v_k$.

If v_1 and v_k have a common neighbor in $V(G)\backslash V(P)$, denote it by x, then $C'' = Pv_kxv_1$ is an o-cycle which contains all the vertices in $V(C)$ with deficit degree smaller than $\mathrm{def}(C')$, a contradiction.

So we assume that $N_{G-P}(v_1) \cap N_{G-P}(v_k) = \emptyset$. Then we have $d_P(v_1) + d_P(v_k) \geq |V(P)|$ by $d(v_1)+d(v_k) \geq n$. Thus, there exists i with $2 \leq i \leq k-1$ such that $v_i \in N_P(v_1)$ and $v_{i-1} \in N_P(v_k)$, and then $C'' = P[v_1, v_{i-1}]v_{i-1}v_kP[v_k, v_i]v_iv_1$ is an o-cycle which contains all the vertices in $V(C)$ with deficit degree smaller than $\mathrm{def}(C')$, a contradiction. ■

Note that Lemma 6 immediately implies Theorem 1.

Let P be an (x, y)-path (or o-path) of G. If the number of vertices of P is more than that of a longest cycle of G, then, by Lemma 6, we have $xy \notin E(G)$ and $d(x) + d(y) < |V(G)|$.

In the following, we use $\overline{E}(G)$ to denote the set $\{uv : uv \in E(G) \text{ or } d(u) + d(v) \geq |V(G)|\}$.

3. Proof of Theorem 2

If G contains at least three heavy vertices, then let $X = \{x_1, x_2, \ldots, x_k\}$ be the set of heavy vertices of G, where $k \geq 3$. Thus $C = x_1 x_2 \cdots x_k x_1$ is an o-cycle. By Lemma 6, there exists a cycle containing all the vertices in X, which is a heavy cycle, a contradiction. Thus we have that G contains at most two heavy vertices.

Case 1. G contains no heavy vertices.
If G contains a cycle C, then C is a heavy cycle of G, a contradiction. Since G is connected, we have that G is a tree.

Case 2. G contains only one heavy vertex.
Let x be the heavy vertex and H be a component of $G - x$. Since G is connected, we have that $N_H(x) \neq \emptyset$. If $|N_H(x)| \geq 2$, then let x_1 and x_2 be two vertices in $N_H(x)$, and P be an (x_1, x_2)-path in H. Then $C = P x_2 x x_1$ is a cycle containing x, which is a heavy cycle, a contradiction. Thus we have $|N_H(x)| = 1$.

Since $d(x) \geq n/2$, we have that $G - x$ contains at least $n/2$ components.

Case 3. G contains exactly two heavy vertices.
Let x and y be the two heavy vertices and P be a longest (x, y)-path of G. If $|V(P)| \geq 3$, then $C = Pyx$ is an o-cycle of G. By Lemma 6, there exists a cycle containing all the vertices in $V(C)$, which is a heavy cycle, a contradiction. Thus we have that $|V(P)| = 2$, which implies that $xy \in E(G)$ and xy is a cut edge of G.

Let H_x and H_y be the two components of $G - xy$ which contain x and y, respectively. Since $d(x) \geq n/2$ and $xy' \notin E(G)$ for all $y' \in V(H_y) \backslash \{y\}$, we have that $|V(H_y)| \leq n/2$. Similarly we have that $|V(H_x)| \leq n/2$. This implies that n is even and $|V(H_x)| = |V(H_y)| = n/2$.

By $d(x) \geq n/2$ and $|V(H_x)| = n/2$, we have that $xx' \in E(G)$ for every $x' \in V(H_x) \backslash \{x\}$. Similarly, we have that $yy' \in E(G)$ for every $y' \in V(H_y) \backslash \{y\}$.

The proof is complete.

4. Proof of Theorem 4

We use n to denote the order of G, and c to denote the length of C. We give an orientation to C. Let x be a vertex in $V(G - C)$. We prove that $d(x) < n/2$.

Let H be the component of $G-C$ which contains x. Then all the neighbors of x are in $V(C)\cup V(H)$. Let $h=|V(H)|$. Note that x is not a neighbor of itself. Hence we have $d_H(x)<h$.

Claim 1. *If v_1,v_2 are two vertices in $V(C)$ such that $v_1v_2\in E(C)$, then either $xv_1\notin E(G)$ or $xv_2\notin E(G)$.*

Proof. Otherwise, $(C-v_1v_2)\cup v_1xv_2$ is a cycle longer than C, a contradiction. □

By Claim 1, we have that if P is a subpath of C, then $d_P(x)\le\lceil|V(P)|/2\rceil$. By the 2-connectedness of G, there exists a (u_0,v_0)-path (and then, a (u_0,v_0)-o-path) passing through x which is internally disjoint with C, where $u_0,v_0\in V(C)$. We choose an o-path $Q=x_{-k}x_{-k+1}\cdots x_{-1}xx_1\cdots x_l$ such that

(1) $x_{\pm1}\in N(x)$,
(2) Q is internally disjoint with C, and
(3) $|V(Q)\cap N_H(x)|$ is as large as possible, where $x_{-k}\in V(C)$ and $x_l\in V(C)$.

Claim 2. *Q contains at least half of the vertices in $N_H(x)$.*

Proof. If $d_H(x)=0$, then the assertion is obvious. So we assume that $d_H(x)\ge 1$. Suppose that $|N_H(x)\cap V(Q)|<d_H(x)/2$. Then $|N_H(x)\backslash V(Q)|\ge\lceil d_H(x)/2\rceil\ge 1$.

Claim 2.1. *For every $x'\in N_H(x)\backslash V(Q)$, $x'x_1\notin\overline{E}(G)$ and $x'x_{-1}\notin\overline{E}(G)$.*

Proof. If $x'x_1\in\overline{E}(G)$, then $Q'=Q[x_{-k},x]xx'x_1Q[x_1,x_l]$ is an o-path which contains more vertices in $N_H(x)$ than Q, a contradiction. Thus we have $x'x_1\notin\overline{E}(G)$. The second assertion is symmetric. □

Claim 2.2. *$x_{-1}x_1\in\overline{E}(G)$.*

Proof. Suppose that $x_{-1}x_1\notin\overline{E}(G)$. Let x',x'' be any pair of vertices in $N_H(x)\backslash V(Q)$. By Claim 2.1, we have that $x'x_{\pm1}\notin\overline{E}(G)$ and $x''x_{\pm1}\notin\overline{E}(G)$. Since G is a $K_{1,4}$-heavy graph, we have that $x'x''\in\overline{E}(G)$.

By the 2-connectedness of G, there is a path from $N_H(x)\backslash V(Q)$ to $V(C)\cup V(Q)$ not passing through x. Let $R'=y_1y_2\cdots y_r$ be such a path, where $y_1\in N_H(x)\backslash V(Q)$ and $y_r\in V(C)\cup V(Q)\backslash\{x\}$. Let R be an o-path from x to y_1 passing through all the vertices in $N_H(x)\backslash V(Q)$ (recall that, for any $x',x''\in N_H(x)\backslash V(Q)$, we have $x'x''\in\overline{E}(G)$).

If $y_r\in V(C)\backslash\{x_{-k},x_l\}$, then $Q'=Q[x_{-k},x]RR'$ is an o-path which contains at least half of the vertices in $N_H(x)$, a contradiction.

If $y_r\in V(Q[x_1,x_l])$, then $Q'=Q[x_{-k},x]RR'Q[y_r,x_l]$ is an o-path which contains at least half of the vertices in $N_H(x)$, a contradiction.

The case $y_r\in V(Q[x_{-k},x_{-1}])$ follows by symmetry.

Thus the claim holds. □

Now, we choose an o-path $R = xx'_1x'_2 \cdots x'_r$ which is internally disjoint with $C \cup Q$, where $x'_r \in V(C) \cup V(Q)\backslash\{x\}$ such that

(1) $x'_1 \in N(x)$, and

(2) $|V(R) \cap (N_H(x)\backslash V(Q))|$ is as large as possible.

Claim 2.3. *R contains at least half of the vertices in $N_H(x)\backslash V(Q)$.*

Proof. Note that $d_{H-Q}(x) \geq 1$. It is easy to see that $x'_1 \in N_H(x)\backslash V(Q)$. By Claim 2.1, we have that $x'_1x_1 \notin \overline{E}(G)$.

Suppose that $|V(R)\cap(N_H(x)\backslash V(Q))| < d_{H-Q}(x)/2$. Let $N_H(x)\backslash V(Q)\backslash V(R) = \{x''_1, x''_2, \ldots, x''_s\}$, where $s \geq \lceil d_{H-Q}(x)/2\rceil$.

For every vertex $x''_i \in N_H(x)\backslash V(Q)\backslash V(R)$, by Claim 2.1, we have that $x''_ix_1 \notin \overline{E}(G)$. Symmetrically, we can prove that $x''_ix'_1 \notin \overline{E}(G)$.

For any pair of vertices $x''_i, x''_j \in N_H(x)\backslash V(Q)\backslash V(R)$, we have that $x''_ix_1 \notin \overline{E}(G)$, $x''_ix'_1 \notin \overline{E}(G)$, $x''_jx_1 \notin \overline{E}(G)$, $x''_jx'_1 \notin \overline{E}(G)$ and $x'_1x_1 \notin \overline{E}(G)$. Since G is $K_{1,4}$-heavy, we have that $x''_ix''_j \in \overline{E}(G)$.

By the 2-connectedness of G, there is a path from $N_H(x)\backslash V(Q)\backslash V(R)$ to $V(C) \cup V(Q)$ not passing through x. Let $T' = y_1y_2 \cdots y_t$ be such a path, where $y_1 \in N_H(x)\backslash V(Q)\backslash V(R)$ and $y_t \in V(C) \cup V(Q)\backslash\{x\}$. Let T be an o-path from x to y_1 passing through all the vertices in $N_H(x)\backslash V(Q)\backslash V(R)$. Then $R' = TT'$ is an o-path from x to $V(C)\cup V(Q)\backslash\{x\}$ which contains at least half of the vertices in $N_H(x)\backslash V(Q)$, a contradiction. $\square$

By Claim 2.3, we have that R contains at least one quarter of the vertices in $N_H(x)$.

Claim 2.4. $x'_r \in V(C)\backslash\{x_{-k}, x_l\}$.

Proof. Assume the opposite. Without loss of generality, we assume that $x'_r \in [x_1, x_l]$.

If $x'_r = x_1$, then $Q' = Q[x_{-k}, x]RQ[x_1, x_l]$ is an o-path which contains more vertices in $N_H(x)$ than Q, a contradiction.

If $x'_r = x_i$, where $2 \leq i \leq l$, then let x_j be the last vertex in $[x_1, x_{i-1}]$ such that $x_j \in N(x)$. Then $Q' = Q[x_{-k}, x_{-1}]x_{-1}x_1Q[x_1, x_j]x_jxRQ[x'_r, x_l]$ is an o-path which contains more vertices in $N_H(x)$ than Q, a contradiction.

Thus we have $x'_r \in V(C)\backslash\{x_{-k}, x_l\}$. $\square$

Now we finish the proof of Claim 2. If $Q[x, x_l]$ contains fewer than one quarter of the vertices in $N_H(x)$, then $Q' = Q[x_{-k}, x]R$ is an o-path which contains more vertices in $N_H(x)$ than Q, a contradiction. This implies that $Q[x, x_l]$ contains at least one quarter of the vertices in $N_H(x)$. Similarly, we have that $Q[x_{-k}, x]$ contains at least one quarter of the vertices in $N_H(x)$. Thus Q contains at least half of the vertices in $N_H(x)$, a contradiction. $\square$

By Claim 2, we have that $k+l-2 \geq d_H(x)/2$.

Let $u_0 = x_{-k} \in V(C)$ and $v_0 = x_l \in V(C)$. We assume that the length of $\overrightarrow{C}[v_0, u_0]$ is r_1+1 and length of $\overrightarrow{C}[u_0, v_0]$ is r_2+1, where $r_1+r_2+2 = c$. We use $\overrightarrow{C} = v_0 v_1 v_2 \cdots v_{r_1} u_0 v_{-r_2} v_{-r_2+1} \cdots v_{-1} v_0$ to denote C with the given orientation, and $\overleftarrow{C} = u_0 u_1 u_2 \cdots u_{r_1} v_0 u_{-r_2} u_{-r_2+1} \cdots u_{-1} u_0$ to denote C with the opposite direction, where $v_i = u_{r_1+1-i}$ and $v_{-j} = u_{-r_2-1+j}$.

Claim 3. *$r_1 \geq k+l-1$, and for every vertex $v_s \in [v_1, v_l]$, $xv_s \notin E(G)$, and for every vertex $u_t \in [u_1, u_k]$, $xu_t \notin E(G)$.*

Proof. Note that Q contains $k+l-1$ vertices in $V(H)$. If $r_1 < k+l-1$, then $C' = Q\overleftarrow{C}[v_0, u_0]$ is an o-cycle longer than C. By Lemma 6, there exists a cycle which contains all the vertices in $V(C')$, a contradiction. Thus, we have $r_1 \geq k+l-1$.

If $xv_s \in E(G)$, where $v_s \in [v_1, v_l]$, then $C' = \overrightarrow{C}[v_s, v_0]Q[v_0, x]xv_s$ is an o-cycle which contains all the vertices in $(V(C)\backslash[v_1, v_{s-1}]) \cup V(Q[x, x_{l-1}])$, and $|V(C')| > c$, a contradiction.

If $xu_t \in E(G)$, where $u_t \in [u_1, u_k]$, then we can prove the result similarly. □

Similarly, we can prove the following claim.

Claim 4. *$r_2 \geq k+l-1$, and for every vertex $v_{-s} \in [v_{-l}, v_{-1}]$, $xv_{-s} \notin E(G)$, and for every vertex $u_{-t} \in [u_{-k}, u_{-1}]$, $xu_{-t} \notin E(G)$.*

Let $d_1 = d_{\overrightarrow{C}[v_1, u_1]}(x)$ and $d_2 = d_{\overleftarrow{C}[v_{-1}, u_{-1}]}(x)$. Then $d_C(x) \leq d_1 + d_2 + 2$.

Claim 5. *$d_1 \leq (r_1 - (k+l) + 1)/2$, $d_2 \leq (r_2 - (k+l) + 1)/2$.*

Proof. If $r_1 = k+l-1$, then by Claim 3, we have $d_1 = 0$. So we assume that $r_1 \geq k+l$. By Claim 3, we have that $d_1 = d_{\overrightarrow{C}[v_{l+1}, u_{k+1}]}(x)$. By Claim 1, we have that $d_1 \leq \lceil (r_1 - (k+l))/2 \rceil \leq (r_1 - (k+l) + 1)/2$.

The second assertion is symmetric. □

By Claim 5, we have that

$$d_C(x) \leq d_1 + d_2 + 2 \leq (r_1 + r_2 + 2 - 2(k+l))/2 + 2 = c/2 - (k+l-2).$$

Note that $k+l-2 \geq d_H(x)/2$, we have $d_C(x) \leq (c - d_H(x))/2$. Thus $d(x) = d_C(x) + d_H(x) \leq (c + d_H(x))/2 < (c+h)/2 \leq n/2$.

The proof is complete.

5. Proof of the necessity of Theorem 5

Note that an S-free graph is also S-heavy. Thus we only need to prove that a longest cycle of a 2-connected S-free graph is not necessarily a heavy cycle for $S \neq P_3, K_{1,3}$ and $K_{1,4}$.

First consider the following fact: if a connected graph S on at least 3 vertices is not $P_3, K_{1,3}$ or $K_{1,4}$, then S must contain K_3, P_4, C_4 or $K_{1,5}$ as an induced subgraph. Thus we only need to show that not every longest cycle in a K_3, P_4, C_4 or $K_{1,5}$-free graph is heavy.

We construct three graphs G_1, G_2 and G_3 (see Figures 2, 3 and 4).

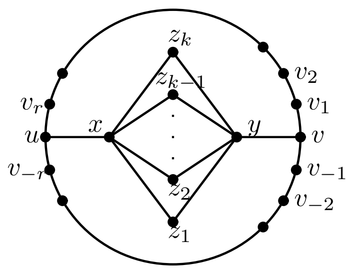

Figure 2. Graph G_1 ($r \geq 4$ and $k \geq 2r + 2$).

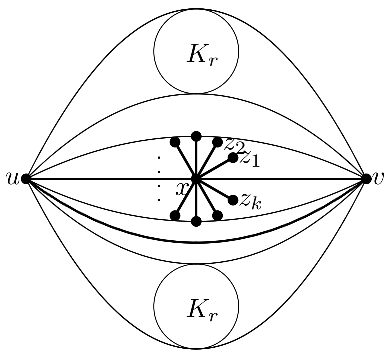

Figure 3. Graph G_2 ($r \geq 4$ and $k \geq 2r - 1$).

Remark 7. In the graph G_2, the subgraph $G_2[\{x\} \cup [z_1, z_k]]$ is a star $K_{1,k}$, and u and v are adjacent to all the vertices in the $K_{1,k}$ and the two K_r's (note that $uv \in E(G_2)$). In the graph G_3, x and y are adjacent to all the vertices in the three K_k's.

Note that G_1 is K_3-free, G_2 is P_4 and C_4-free, and G_3 is $K_{1,5}$-free, and the longest cycles of the three graphs are all not heavy. Thus the necessity of the theorem holds.

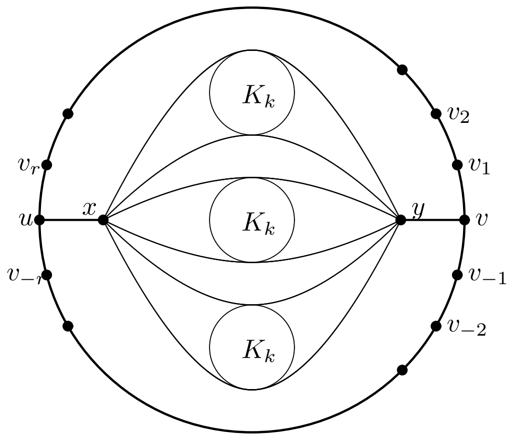

Figure 4. Graph G_3 ($r \geq 11$ and $(2r+2)/3 \leq k \leq r-3$).

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